Microscopic calculation of the nonlinear current fluctuations of a metallic resistor: The problem of heating in perturbation theory

A.-M. Tremblay* and B. Patton†
Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

P. C. Martin
Physics Department and Division of Applied Sciences, Harvard University, Cambridge, Massachusetts 02138

P. F. Maldague‡
IBM Thomas J. Watson Research Laboratories, Yorktown Heights, New York 10598

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The effect of a constant electric field $\vec{E}$ on the low-temperature Johnson noise of a metallic resistor is calculated by nonequilibrium diagrammatic perturbation theory. The expansion, in powers of E^2, is shown to diverge, and the divergence is associated with accumulating Joule heat. An insight into how to treat a steady state properly when there is heating is provided by the Boltzmann equation. A vertex function that corresponds to a steady-state solution of this equation is identified. With this vertex function it is possible to evaluate not only one-particle operators, for which the Boltzmann equation is sufficient, but also troublesome contributions to the current fluctuations and other two-body observables for which a many-body approach is required. The calculated corrections to the current correlations agree qualitatively, but not quantitatively, with simple phenomenological agruments. In particular, it is not possible to describe the leading effects of the electric field solely in terms of a variable local temperature. The model calculation illustrates and sheds some light on several problems that only appear when nonlinear deviations from thermal equilibrium must be considered: the complicated effects to which heat generation leads, the identification and treatment of these effects in perturbation theory, and the difficulties these effects pose for hypothetical generalizations of the fluctuation dissipation theorem. Some possible connections between the calculation and $1/f$ noise are also briefly explored. In particular, recent experiments that may confirm the conclusion of this calculation, that systems in which there is only impurity scattering do not exhibit $1/f$ noise, are noted.

I. INTRODUCTION

Ever since the work of Bernard and Callen[1] theorists have sought in vain for an explanation of nonequilibrium current or voltage fluctuations that was as deep and general as the explanation of equilibrium fluctuations in terms of the fluctuation-dissipation theorem. To nonlinear order there probably is no correspondingly general explanation; the details of specific models play a major role.

In this paper we report on a first-principle microscopic calculation of the nonlinear response and fluctuations for a specific model. To make the necessarily lengthy calculations as short and easy as possible we study a system of metallic electrons that interact only with dilute, static, and isotropically scattering impurities. This can be a reasonable model for a metallic resistor at low temperatures. We calculate the properties of this system using a method developed some time ago.[2,3] The method has been widely applied but it has not, to our knowledge, previously been used to evaluate nonlinear fluctuations.

One motivation for performing these calculations was our desire to understand the intriguing experimental phenomenon[4] ($1/f$ noise) that nonequilibrium current fluctuations are supposed to exhibit. Our system does not appear to exhibit this phenomenon. This result seems to agree with recent experimental findings of Hooge and Vandamme[5] on dirty systems. These authors argue that their negative results show that the noise must be due to phonons. For reasons discussed in the Conclusion, we consider this finding suggestive but not compelling. In any event, we agree with these authors that since most systems exhibit $1/f$ noise, a counterexample may well be useful in unravelling the riddle.

Although the calculation does not explain $1/f$ noise and is not necessarily generalizable, it does provide some new, interesting, and unexpected insights. For example, the reader may be surprised, as we were, to discover that perturbation theory yields divergent current fluctuations for the model. In retrospect this divergence can be simply understood on physical grounds. In thermal equilibrium, for frequencies ω_0 much smaller than both the collision frequency and the average thermal energy, the spectrum of current fluctuations is given by[6]

$$\int_{-\infty}^{\infty} e^{i\omega_0 t}\langle I(0)I(t)\rangle\, dt = \frac{2kT}{R}\,; \tag{1.1}$$

R is the resistance, k Boltzmann's constant, and T the absolute temperature. We are interested in current fluctuations about a nonzero average current driven by an applied constant electric field. If we treat the entire electric field as an external perturbation, we *cannot* avoid the fact that, to second order in this field, the temperature of the system will rise (Joule heating) during a typical measurement. Indeed, if the resistance is temperature independent, we should expect to replace the expression $2kT/R$ by

$$(2k/R)[T(t_0)+(t-t_0)dT(t_0)/dt]\,.$$

It is the second term that produces the divergence. Our calculation shows how heat generation appears in the lowest-nonlinear-order perturbation theory, suggests how it must occur in higher orders of perturbation theory, and indicates why it can be neglected in linear response and what difficulties it poses for generalized fluctuation-dissipation theorems.

To obtain a finite result at long times, we must explicitly cool the system. We must include the interaction with the environment that compensates the Joule heating and allows it to reach a steady state. The actual cooling mechanism is neither general nor universal. We might expect to be able to balance locally the heat generated by the electric field and the thermal energy flux produced by a thermal gradient. Although this idea is qualitatively correct, we shall find that the cooling mechanism in a true nonequilibrium stationary state and a thermal gradient are not precisely identical.

By eliminating the shortcomings of perturbation theory and allowing the heat to flow away, we obtain, with certain restrictions, finite nonequilibrium current fluctuations. Our field-dependent corrections to Johnson noise do not resolve the $1/f$ mystery. At best, they eliminate some theories and offer some clues. We discuss these matters briefly in the Conclusion.

Our discussion is organized as follows: In Sec. II, using the Boltzmann equation, we study the problem of heating and cooling for simple one-body observables. In Sec. III we show, guided by the results of Sec. II, that after the divergences have been removed meaningful results can be obtained to leading order. In Sec. IV we summarize the results and in Sec. V we comment on $1/f$ noise, how to go beyond our calculation, and nonequilibrium perturbation theory in general.

II. JOULE HEAT FOR ONE-BODY OBSERVABLES: STATIONARY DISTRIBUTION FUNCTION

Although it is unavoidable, Joule heat appears in a slightly obscure manner when we calculate current fluctuations. To achieve a clearer understanding let us first discuss the effects of heating on one-body observables using the Boltzmann equation.

The results we shall establish will guide us in interpreting and evaluating the infinite vertex corrections that arise in calculating current fluctuations and shed light on some general aspects of heat generation in perturbation theory.

The Boltzmann equation corresponding to our model can be written in the form

$$\frac{\partial f}{\partial t}+e\vec{\mathrm{E}}\cdot\vec{\nabla}_p f+\vec{\mathrm{v}}\cdot\vec{\nabla}_x f=-\frac{f-\bar{f}}{\tau}\,, \tag{2.1}$$

where f is a distribution function which depends on the momentum ($\vec{\mathrm{p}}=m\vec{\mathrm{v}}$) and the position $\vec{\mathrm{x}}$, τ is an energy-independent scattering time, and

$$\bar{f}(p)\equiv\int\frac{d\hat{p}}{4\pi}f(\vec{\mathrm{p}})\,. \tag{2.2}$$

Assuming the solution can be expanded in powers of the applied electric field, we write

$$f=f^{(0)}+f^{(1)}+f^{(2)}+\cdots\,, \tag{2.3}$$

with

$$f^{(0)}\equiv\{1/[e^{\beta(p^2/2m-\mu)}+1]\}\,. \tag{2.4}$$

In Eq. (2.4) we have introduced the conventional notations μ for the chemical potential and β for $1/kT$. Substituting Eq. (2.3) into Eq. (2.1), we find in a uniform system, for times much longer than τ, that

$$f^{(1)}=-\tau e\vec{\mathrm{E}}\cdot\vec{\nabla}_p f^{(0)}\,, \tag{2.5}$$

$$\bar{f}^{(2)}=\int\frac{d\hat{p}}{4\pi}\tau(e\vec{\mathrm{E}}\cdot\vec{\nabla}_p)^2 f^{(0)}t\,. \tag{2.6}$$

Equations (2.3)–(2.6), the "standard" perturbation solution[7] of the Boltzmann equation, should be contrasted with Huberman's[7] "long-time expansion." The "standard" solution is valid only for systems that remain close to equilibrium ($\delta T/T\ll 1$).

The second-order change in the energy density can be calculated from

$$\delta\mathcal{E}\equiv 2N(0)\Omega\int_{-\infty}^{\infty}d\epsilon\,\epsilon\bar{f}^{(2)}\,, \tag{2.7}$$

where Ω is the volume of the system, $\epsilon=p^2/2m-\mu$ and $N(0)$ is the single-spin density of states for a system whose Fermi wave vector is k_F (we take $\hbar=1$)

$$N(0)=mk_F/2\pi^2\,. \tag{2.8}$$

Substituting Eq. (2.6) in Eq. (2.7), we find that

$$\delta\mathcal{E}=(V^2/R)t\,, \tag{2.9}$$

where R, the resistance of the system,[8] can be

expressed in terms of L, the length along which the voltage $V=EL$ is applied, and A, the cross-sectional area, as

$$R=mL/ne^2\tau A\ ; \tag{2.10}$$

n is the density

$$n=k_F^3/3\pi^2\ . \tag{2.11}$$

Clearly, Eq. (2.9) represents Joule heating. Since most experiments are performed in steady-state configurations, we seek stationary solutions of the Boltzmann equation. Intuitively, we expect to be able to apply a "thermal gradient" to the system that will take advantage of the electronic thermal conductivity to carry away the heat generated by the electric field and lead it to a nonequilibrium steady state. Let us obtain the stationary solution of Eq. (2.1) in the presence of an electric field and a "thermal" gradient. Expanding in powers of the electric field and allowing $f^{(2)}$ to be position dependent, we obtain from the Boltzmann equation (2.1)

$$f^{(1)}=-\tau e\vec{\mathrm{E}}\cdot\vec{\nabla}_p f^{(0)}\ , \tag{2.12}$$

$$i\vec{\mathrm{q}}\cdot\vec{\mathrm{v}}f^{(2)}+\tau^{-1}[f^{(2)}-\bar{f}^{(2)}]=\tau(e\vec{\mathrm{E}}\cdot\vec{\nabla}_p)^2 f^{(0)}\ , \tag{2.13}$$

where $\vec{\mathrm{q}}$ is the wave vector in the direction of the gradient which we arbitrarily choose to be perpendicular to the electric field.

Equation (2.13) can be solved by expanding in spherical harmonics and keeping only the first two terms.[9] Substituting the solution in Eq. (2.7), we find that

$$\delta\mathcal{E}=2N(0)(eE/q)^2\Omega\ , \tag{2.14}$$

which is identical to the result that can be heuristically derived from

$$\delta\mathcal{E}=C_p\delta T_q \tag{2.15}$$

and

$$V^2/R=\kappa q^2\delta T_q\Omega\ , \tag{2.16}$$

where C_p is the specific heat for our model,[10]

$$C_p=\Omega(\pi^2/2)(kT/E_F)nk\ , \tag{2.17}$$

and κ is the thermal conductivity,[11]

$$\kappa=\frac{C_pD}{\Omega}=\frac{n\tau}{m}\frac{\pi^2}{3}k^2T=\frac{\pi^2}{3}\frac{k^2}{e^2}T\sigma\ . \tag{2.18}$$

The last equality has been written in terms of the electrical conductivity σ to recover the Wiedemann-Franz law.

Equation (2.16) can be obtained by the following arguments: Suppose that we include a thermal gradient to remove the heat generated by an electric field. The change in the local energy density produced by the electric field and thermal gradient can be computed classically. For a uniform current[12] ($\vec{\nabla}\cdot\vec{\mathrm{j}}=0$) we have

$$\frac{1}{\Omega}\frac{d\mathcal{E}}{dt}=\frac{1}{\Omega}\frac{V^2}{R}+\frac{\partial\kappa}{\partial T}(\vec{\nabla}T)^2-T\frac{\partial Q}{\partial T}(\vec{\nabla}T)\cdot\vec{\mathrm{j}}+\kappa\nabla^2T\ , \tag{2.19}$$

where Q is the thermopower and $\vec{\mathrm{j}}$ the current density. In the configuration where current and temperature gradient are perpendicular, we have

$$\vec{\nabla}T\cdot\vec{\mathrm{j}}=0\ . \tag{2.20}$$

δT, the difference between the maximum and minimum temperatures across the sample, satisfies the condition

$$\delta T\ll T\ . \tag{2.21}$$

The inequality

$$|\kappa\nabla^2T|\gg\frac{d\kappa}{dT}(\vec{\nabla}T)^2 \tag{2.22}$$

follows because

$$\frac{d\kappa}{dT}(\vec{\nabla}T)^2\sim\kappa\frac{\delta T}{L_\perp^2}\left(\frac{\delta T}{T}\right) \tag{2.23}$$

and

$$\kappa\nabla^2T\sim\kappa\delta T/L_\perp^2\ , \tag{2.24}$$

where $L_\perp$ is the length of the sample along the thermal gradient. Equation (2.19) therefore reduces to

$$\frac{1}{\Omega}\frac{d\mathcal{E}}{dt}=\frac{1}{\Omega}\frac{V^2}{R}+\kappa\nabla^2T\ . \tag{2.25}$$

Thus the energy density will be time independent if

$$V^2/\Omega R=-\kappa\nabla^2T\ . \tag{2.26}$$

In Fourier space, this equation is identical with Eq. (2.16).

It should be emphasized that the solution to Eq. (2.13) is *not exactly* like the solution we would obtain by applying a simple spatially dependent temperature to the system and requiring that heat be carried away by the thermal gradient. We elaborate on this point in the Appendix.

The second-order effect of the electric field on *any* one-body observable can be calculated from

$$\langle A\rangle=2N(0)\Omega\int_{-\infty}^{\infty}d\epsilon\,\bar{f}^{(2)}\Gamma^A(\epsilon)\ , \tag{2.27}$$

where $\Gamma^A(\epsilon)$ depends on the observable, e.g., for the energy density $\Gamma^A(\epsilon)=\epsilon$. Note that every one-body observable will have a time-independent value when the solution to Eq. (2.13) is used in Eq. (2.27). In the case of the energy, that value is the

one expected from simple macroscopic arguments.

Although the two solutions to the Boltzmann equation we have presented correspond to different physical situations, the values of $f^{(1)}$ [see Eqs. (2.5) and (2.12)] are the same. Since the resistance of the system, for example, can be determined by calculating the nonequilibrium current from $f^{(1)}$, the value of the resistance determined in a steady-state experiment [corresponding to Eq. (2.12)] or in an experiment that lasts for a "short time" [corresponding to Eq. (2.5)] are the same.[13] Here, "short time" means

$$\tau \ll t \ll t_H , \tag{2.28}$$

$$t_H \sim \pi^2 \tau (kT/eEl)^2 , \tag{2.29}$$

where t_H is the heating time of the system that can be determined from

$$\frac{\delta T}{T} \sim 1 \sim \frac{V^2}{TC_p R} t_H . \tag{2.30}$$

These results suggest the reason why, in linear-response theory, we need not examine how the heat is dissipated when we make realistic calculations. However it is done, as long as the heat is dissipated the linear response is the same.

By contrast, a quantity that depends on $f^{(2)}$ will behave quite differently in experiments described by Eq. (2.6) and Eq. (2.13). Joule heat is one obvious quantity that depends on $f^{(2)}$. A less obvious quantity is the current fluctuation to lowest nonlinear order (E^2) in the applied field.

The Boltzmann equation can only be used to calculate *one-body* observables. To determine the current fluctuations, a *two-body* observable, we must use a microscopic theory. As we noted in the Introduction a straightforward application of microscopic perturbation theory leads to infinities reflecting Joule heating. These infinities in the current fluctuations can be identified with a certain part of a diagram. We can find how to include the "cooling" mechanism that leads to a steady state by establishing a formal equivalence between the equation for the troublesome part of that diagram (a vertex equation), and the perturbative solutions to the Boltzmann equation discussed here.

III. NONEQUILIBRIUM CURRENT FLUCTUATIONS

A. Model

1. Hamiltonian

The resistor model[8] described by the Boltzmann equation (2.1) consists of free electrons in a positive background scattering off random one-body impurity potentials. The Hamiltonian is given by

$$H = H_0 + H_1 , \tag{3.1}$$

$$H_0 = \sum_{\vec{p},s} \epsilon_{\vec{p}} C^{\dagger}_{\vec{p}s} C_{\vec{p}s} , \tag{3.2}$$

$$H_1 = \sum_{\vec{p},\vec{q},s} \sum_{\vec{x}_i}{}' u(-\vec{q}) e^{i\vec{q}\cdot\vec{x}_i} C^{\dagger}_{\vec{p}s} C_{\vec{p}+\vec{q}s} , \tag{3.3}$$

where $u(\vec{q})$ is the spatial Fourier transform of the scattering potential, $\vec{x}_i$ are the positions of the random impurities, $\epsilon_p = (p^2/2m) - \mu$, and $C_{\vec{p}s}$ destroys a particle in a state of momentum $\vec{p}$ and spin s. The field operator Ψ_s is given by

$$\Psi_s(\vec{x}, t) = \sum_{\vec{p}} C_{\vec{p}s}(t) e^{i\vec{p}\cdot\vec{x}} . \tag{3.4}$$

In the dilute limit (an impurity concentration n_c of, say, less than 1%), it is possible to determine the properties of this model[8] by averaging over the positions of the impurities. Indeed, once the averaging is carried out, the problem is equivalent to solving a field theory with an effective interaction Hamiltonian given, in the interaction representation, by

$$H_1 = \frac{n_c}{2} \sum_{\vec{p},\vec{p}',\vec{q}} \sum_{ss'} C^{\dagger}_{\vec{p}'s'}(t') C_{\vec{p}'-\vec{q}s'}(t') \times [-i|u(\vec{q})|^2] C^{\dagger}_{\vec{p}s}(t) C_{\vec{p}+\vec{q}s}(t) . \tag{3.5}$$

We must also neglect all diagrams that can be disconnected by cutting an arbitrary number of interaction lines. We shall assume that the scattering potential (a) has no bound states, and (b) is so short ranged that it leads only to s-wave scattering of the electrons at the Fermi surface. Note that we do not need to assume that the Born approximation is valid.[14,15] When it is not, we may set

$$\begin{aligned} &|u(\vec{q})|^2 \to |t(\vec{q})|^2 \quad \text{(beyond Born approximation)} , \\ &|t(\vec{q})|^2 \to |t(0)|^2 \quad (s\text{-wave scattering}) , \\ &|u|^2 \equiv |t(0)|^2 , \end{aligned} \tag{3.6}$$

where $|t(\vec{q})|^2$ is the t matrix for scattering off the potential.

Many of our results are independent of assumptions (a) and (b).[8] The Coulomb interaction between the electrons can also be neglected since we shall be concerned with space-independent perturbations which do not alter the density and therefore do not require us to take the long-range Coulomb interaction into account immediately. Finally, this model is realistic only when the temperature is low enough so that electron-phonon collisions are less important than impurity scattering. For simplicity, and in line with this restriction we shall take the electrons to be a highly

degenerate Fermi gas.

We shall need the following propagators[16,17]:

$$G^{>}(1,1') = -i\langle\Psi(1)\Psi^{\dagger}(1')\rangle\,, \tag{3.7}$$

$$G^{<}(1,1') = i\langle\Psi^{\dagger}(1')\Psi(1)\rangle\,, \tag{3.8}$$

$$G^{R}(1,1') = -i[\langle\Psi(1)\Psi^{\dagger}(1')\rangle + \langle\Psi^{\dagger}(1')\Psi(1)\rangle]\Theta(t_1 - t_{1'})\,, \tag{3.9}$$

$$G^{A}(1,1') = i[\langle\Psi(1)\Psi^{\dagger}(1')\rangle + \langle\Psi^{\dagger}(1')\Psi(1)\rangle]\Theta(t_{1'} - t_1)\,, \tag{3.10}$$

where 1 and 1′ stand for the space-time coordinates and $\Theta(t)$ equals one for $t>0$ and zero for $t<0$.

2. *Thermal equilibrium solution of the model*

In the limit

$$k_F l \gg 1\,, \tag{3.11}$$

where l is the mean free path [defined in Eq. (3.13)], we can neglect all overlapping diagrams. The solution for the electron propagator[8] equation (Fig. 1) is then, in units in which $\hbar = 1$,

$$G^{R}(\vec{k},\omega) = \int d^3x\,dt\,e^{-i\vec{k}\cdot\vec{x}+i\omega t}G^{R}(\vec{x},t;0,0)$$

$$= \frac{1}{\omega - \epsilon_k + i/2\tau} = [G^{A}(\vec{k},\omega)]^{*}. \tag{3.12}$$

The collision time τ and the mean free path l are defined by

$$\tau^{-1} \equiv 2\pi N(0) n_c |u|^2\,; \quad l \equiv k_F\tau/m \equiv v_F\tau\,. \tag{3.13}$$

We now make the assumption of thermal equilibrium by using the boundary condition[18] valid in this case

$$G^{<}(\vec{k},\omega) = -e^{-\beta\omega}G^{>}(\vec{k},\omega)\,. \tag{3.14}$$

From Eqs. (3.12), (3.7)–(3.10), and (3.14), we obtain

$$i[G^{R}(\vec{k},\omega) - G^{A}(\vec{k},\omega)] = A(\vec{k},\omega) = \frac{1/\tau}{(\omega-\epsilon_k)^2 + (1/2\tau)^2}\,, \tag{3.15}$$

$$G^{<}(\vec{k},\omega) = i f(\omega) A(\vec{k},\omega)\,, \tag{3.16}$$

$$G^{>}(\vec{k},\omega) = -i[1 - f(\omega)]A(\vec{k},\omega)\,, \tag{3.17}$$

where

$$f(\omega) = (e^{\beta\omega}+1)^{-1}\,. \tag{3.18}$$

This completes the solution of the problem in thermal equilibrium.

3. *Coupling to the external electromagnetic field*

We wish to determine the effect of an electric field described by the time-dependent transverse ($\vec{\nabla}\cdot\vec{A} = 0$) vector potential in the interaction Hamiltonian

$$H_{\text{int}} = -\frac{1}{c}\int j_i^{p}(\vec{x},t)A_i(\vec{x},t)\,d^3x - \frac{e}{2mc^2}\int \rho_e(\vec{x},t)A_i(\vec{x},t)A_i(\vec{x},t)\,d^3x\,. \tag{3.19}$$

The index i refers to the vector Cartesian components (with a summation convention). With the notation of Schrieffer,[19] we have

$$j_i(\vec{x},t) \equiv \frac{-e}{2mi}\sum_s \{\Psi_s^{\dagger}(\vec{x},t)\nabla_i\Psi_s(\vec{x},t) - [\nabla_i\Psi_s^{\dagger}(\vec{x},t)]\Psi_s(\vec{x},t)\} + \frac{e}{mc}\rho_e(\vec{x},t)A_i(\vec{x},t)$$

$$\equiv j_i^{p}(\vec{x},t) + \frac{e}{mc}\rho_e(\vec{x},t)A_i(\vec{x},t)\,. \tag{3.20}$$

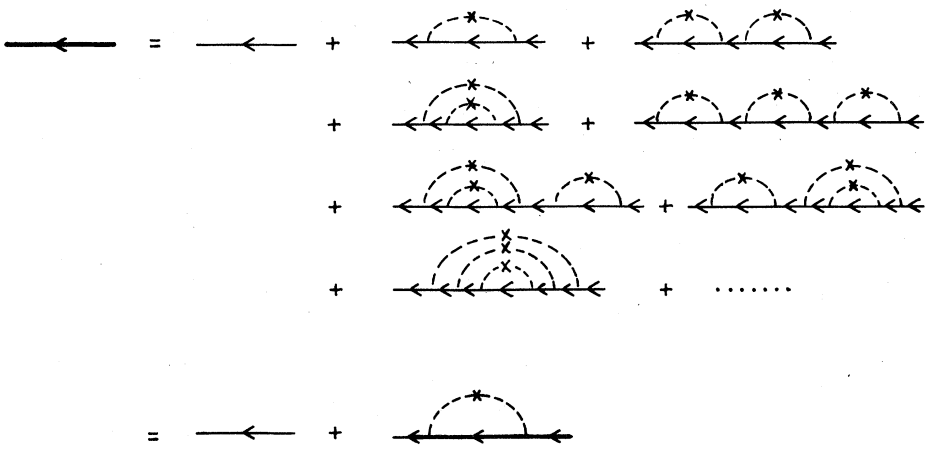

FIG. 1. Propagator equation including all nonoverlapping self-energy diagrams. The thick solid line is the "fully dressed" electron propagator with the arrow pointing from the $\Psi^{\dagger}$ coordinate toward the $\Psi^{\dagger}$ coordinate.

The electric charge density is defined by

$$\rho_e(\vec{x},t) \equiv -e\sum_s \Psi_s^\dagger(\vec{x},t)\Psi_s(\vec{x},t)\,. \tag{3.21}$$

We have described the electric field by a vector potential and not a scalar potential for technical convenience.

B. Method of calculation

Let us define the propagator[17]

$$iG_0(1,1') = \langle T_c \Psi(1)\Psi^\dagger(1')\rangle_0\,, \tag{3.22}$$

where T_c represents the time-ordering operator along the Kadanoff-Baym-Langreth (KBL) contour. The usual rules of perturbation theory[8,19,20] apply to nonlinear response if propagators are time ordered along the KBL contour.

When perturbation theory[21] yields, for example,

$$\delta G(t,t') = \bar{A}\int_c d\tau\, G_0(t,\tau)G_0(\tau,t')U(\tau), \tag{3.23}$$

where $\bar{A}$ is a constant, $U(\tau)$ an interaction potential, and C the KBL contour, rules for recasting this expression in terms of the usual propagators are useful. For this type of "series" multiplication we have

$$\delta G^{>}(t,t') = \bar{A}\int_{-\infty}^{\infty} d\tau\,[G_0^{>}(t,\tau)G_0^{A}(\tau,t') + G_0^{R}(t,\tau)G_0^{>}(\tau,t')]U(\tau), \tag{3.24}$$

$$\delta G^{R(A)}(t,t') = \bar{A}\int_{-\infty}^{\infty} d\tau\, G^{R(A)}(t,\tau) \times G^{R(A)}(\tau,t')U(\tau)\,. \tag{3.25}$$

These results can be generalized. Suppose that (symbolically), we have

$$\delta G = A' G_0 G_0 G_0 \cdots G_0\,. \tag{3.26}$$

We may then write

$$\delta G^{>} = A'[G_0^{>}G_0^{A}G_0^{A}\cdots G_0^{A} + G_0^{R}G_0^{>}G_0^{A}\cdots G_0^{A} + G_0^{R}G_0^{R}G_0^{>}\cdots G_0^{A} + \cdots + G_0^{R}G_0^{R}G_0^{R}\cdots G_0^{>}]\,, \tag{3.27}$$

$$\delta G^{R(A)} = A' G_0^{R(A)}G_0^{R(A)}G_0^{R(A)}\cdots G_0^{R(A)}\,. \tag{3.28}$$

The other rule we shall employ concerns "parallel" multiplication. If

$$F(t,t') \equiv G_0(t,t')G_0(t',t) \tag{3.29}$$

(no integration implied) then we may write

$$F^{>}(t,t') = G_0^{>}(t,t')G_0^{<}(t',t)\,. \tag{3.30}$$

Equivalent rules may be obtained by the analytical continuation procedures of Gor'kov and Eliashberg[22] or several other methods.[23,24]

C. Current autocorrelation function in thermal equilibrium: Johnson noise

Although the phenomenon of Johnson noise does not depend on the specific nature of the resistor, it is instructive to see how, without the fluctuation-dissipation theorem, Johnson noise is predicted by the nonequilibrium formalism for our model. The quantity measured in a Johnson noise experiment can ordinarily be expressed in terms of[25]

$$\langle j^i_{-\omega_0} j^j_{\omega_0}\rangle \equiv \int_{-\infty}^{\infty} e^{i\omega_0(t'-t)}\langle j_i^p(\vec{x},t)j_j^p(\vec{x}',t')\rangle \times dt\,dt'\,d^3x\,d^3x'\,. \tag{3.31}$$

The bracketed quantity on the right-hand side can be regarded as a time-ordered product along a Kadanoff-Baym-Langreth[3] or Keldysh[2] contour with t always later than t'. Since the time ordering requires that t be later than t' we can write Eq. (3.31) as

$$\langle j^i_{-\omega_0} j^j_{\omega_0}\rangle = 2\int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\frac{e^2k_ik_j}{m^2} \times G^{>}(\vec{k},\omega)G^{<}(\vec{k},\omega+\omega_0)\Omega\mathcal{T}\,. \tag{3.32}$$

The total time of the experiment $\mathcal{T}$ and the volume of the system Ω enter because the Green's functions depend only on the time- and space-coordinate difference. The expression (3.3) corresponds to the diagram[26] of Fig. 2. Because the scattering is isotropic and no momentum is transferred at the current vertex, all of the vertex corrections vanish (i.e., loops of the type illustrated in Fig. 3 vanish after integration over the direction of $\vec{k}$).

Assuming particle-hole symmetry, we first perform the ϵ integration[27] and then the ω integration, obtaining[28]

$$\langle j^i_{-\omega_0} j^j_{\omega_0}\rangle = \delta_{ij}\frac{2ne^2}{m}\frac{\tau}{1+\omega_0^2\tau^2}\frac{\omega_0}{e^{\beta\omega_0}-1}\Omega\mathcal{T}. \tag{3.33}$$

From linear response theory[8] we know that the resistance R of this model is given by Eq. (2.10). From the definition (3.31) we have

$$\langle j^i_{-\omega_0} j^j_{\omega_0}\rangle = L^2\int dt\,dt'\,e^{i\omega_0(t'-t)}\langle I^i(t)I^j(t')\rangle\,, \tag{3.34}$$

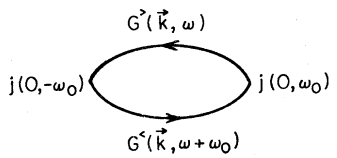

FIG. 2. Diagram for the current fluctuations. The solid lines are dressed propagators. The vertex corrections have not been included because they vanish; see text.

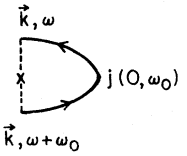

FIG. 3. Example of vertex correction that vanishes for our model.

where I is the macroscopically measured current. In the limit $\omega_0 \ll 1/\tau$, we deduce from Eqs. (3.34), (2.10), and (3.33)

$$\int dt\, e^{i\omega_0(t'-t)}\langle I(t)I(t')\rangle = \mathcal{T}^{-1}\int dt\, dt'\, e^{i\omega_0(t'-t)}\langle I(t)I(t')\rangle$$
$$= 2R^{-1}[\omega_0/(e^{\beta\omega_0}-1)] \quad (\omega_0\tau \ll 1)$$
$$\sim 2kT/R \quad (\omega_0\tau \ll 1, \beta\omega_0 \ll 1). \tag{3.35}$$

D. Electric-field-dependent corrections to Johnson noise

In this section, we outline our calculation, emphasizing the technical difficulties to which we referred earlier. In particular, we examine the divergent terms that arise in perturbation theory, and which will be treated more carefully in the following section.

We want to find the current autocorrelation function to second order in the vector potential $\vec{A}$ for the interaction Hamiltonian in Eq. (3.19). If we take the time integrals along the KBL contour, we can proceed exactly as in the equilibrium case.[8,20] Using Eqs. (3.19) and (3.20) and letting T_c inside the angular brackets denote a time-ordered product along the KBL contour, we immediately obtain

$$\langle T_c\, j_i(x)j_j(x')\rangle_{\text{2nd order}} = \left(\frac{e}{mc}\right)^2\langle T_c\rho_e(x)\rho_e(x')\rangle A_i(x)A_j(x') + \frac{i}{c}\frac{e}{mc}\int_c d^4\bar{x}\,\langle T_c\rho_e(x)j_j^p(x')j_k^p(\bar{x})\rangle A_i(x)A_k(\bar{x})$$
$$+\frac{i}{c}\frac{e}{mc}\int_c d^4\bar{x}\,\langle T_c\, j_i^p(x)\rho_e(x')j_k^p(\bar{x})\rangle A_j(x')A_k(\bar{x}) + \frac{ie}{2mc^2}\int_c d^4\bar{x}\,\langle T_c\, j_i^p(x)j_j^p(x')\rho_e(\bar{x})\rangle A_k(\bar{x})A_k(\bar{x})$$
$$+\left(\frac{i}{c}\right)^2\frac{1}{2!}\int_c d^4\bar{x}\, d^4\bar{\bar{x}}\,\langle T_c\, j_i^p(x)j_j^p(x')j_k^p(\bar{x})j_l^p(\bar{\bar{x}})\rangle A_k(\bar{x})A_l(\bar{\bar{x}}). \tag{3.36}$$

Here x stands for the four coordinates (spatial and temporal). The time integral extends along the KBL contour and the spatial integrals over the entire volume of the infinite sample. The first term in Eq. (3.36) comes from the diamagnetic contribution to j_i and j_j, the second and third ones come from a first-order expansion of the S matrix and a diamagnetic term (respectively that of j_i and that of j_j), the fourth term comes from the last term of Eq. (3.19) in first-order perturbation theory, and the last term comes from the first term of Eq. (3.19) in second-order perturbation theory.

Since the external field is real, we choose

$$A_{\text{ext}}(t) = Ae^{-i\nu t} + A^*e^{i\nu t}, \tag{3.37}$$

$$E_{\text{ext}}(t) = -\frac{1}{c}\frac{\partial A_{\text{ext}}(t)}{\partial t} = \frac{i\nu}{c}(Ae^{-i\nu t} - A^*e^{i\nu t}). \tag{3.38}$$

The Fourier transform of Eq. (3.37) is

$$\int e^{-i\vec{q}\cdot\vec{x}}e^{i\omega t}A_{\text{ext}}(t)\, d^3x\, dt$$
$$= \delta^3(\vec{q})[A\delta(\omega-\nu) + A^*\delta(\omega+\nu)](2\pi)^4. \tag{3.39}$$

If we substitute Eq. (3.37) in Eq. (3.36) we find terms proportional to A^2, A^{*2}, and AA^*. We shall calculate the latter terms which give the desired time-independent contribution to the correlation function.[29]

As in the case of Johnson noise we shall be interested in

$$\langle j^i_{-\omega_0}j^j_{\omega_0}\rangle_{\text{ne}} \equiv \int_{-\infty}^{\infty} e^{i\omega_0(t'-t)}\langle j^i(\vec{x},t)j^j(\vec{x},t')\rangle_{\text{ne}} \times dt\, dt'\, d^3x\, d^3x'. \tag{3.40}$$

From Eq. (3.36) we can enumerate the diagrams we need to evaluate Eq. (3.40). They are sketched in Fig. 4. It is understood that each of these graphs yields all of the "time orders" the rules of Sec. III B require. As noted previously, only the graphs proportional to AA^* have been kept. Parts (a)–(d) of Fig. 4 come, respectively, from the first, second, third, and fourth terms of Eq. (3.36) while parts (e)–(h) all come from the last term of Eq. (3.36); parts (i) and (j) define the shaded areas that appear in the graphs and represent vertex corrections. The various other symbols are defined in the figure caption.

As they did in Eq. (3.32), the corrections to a single current vertex [Fig. 4(i)] vanish for isotropic scattering when $q = 0$. However, in general, the corrections of Fig. 4(j) do not.

Gauge invariance imposes many constraints on the calculation. In particular, the vertex corrections in Fig. 4(i) (see, for example, Ref. 30)

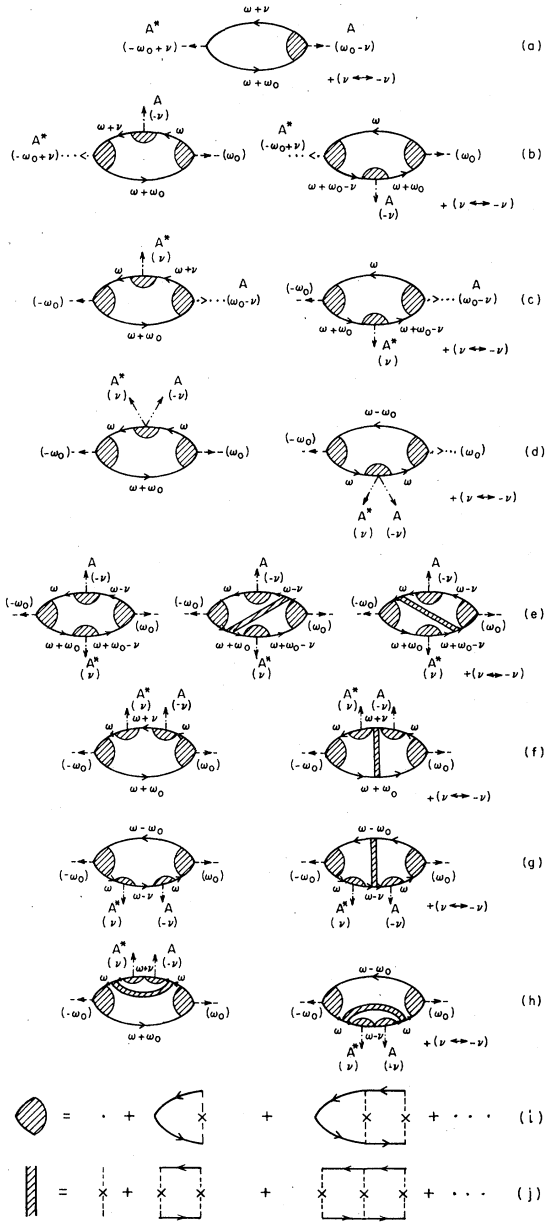

FIG. 4. Diagrams for the current autocorrelation function to second order in the applied field. The shaded areas are defined by Fig. 4(h) and 4(i). All diagrams must be time ordered with the rules of Sec. III B. Only the frequency labels have been marked on the propagators. The frequency in parentheses is the frequency that comes out at a vertex (momentum never comes out of the vertices in these figures).

- `--→--` represents a $\vec{j}^{p}$ vertex [see Eq. (3.20)];
- `·> · ·` represents a $\rho_e \vec{A}$ vertex [see Eq. (3.20)];
- `—·→` represents a $\vec{j}^{p}\cdot\vec{A}$ vertex [see Eq. (3.19)];
- `<·:·→` represents a $\rho_e \vec{A}^2$ vertex [see Eq. (3.19)].

Note that for each diagram there is another one with $(\nu \rightarrow -\nu)$. Additional diagrams obtained by this simple variable change must be included in the final result. Note that the vertex corrections in Fig. 4(i) vanish for a $\vec{j}^{p}$ or $\vec{j}^{p}\cdot\vec{A}$ vertex.

as well as those of Fig. 4(j)[31] must be included when the propagator in Fig. 2 is employed. Also, since there can be no response to a pure longitudinal vector potential at zero frequency, and since the response must be even in ν because of the $\nu \rightarrow -\nu$ symmetry, the leading contribution for the diagrams of Fig. 4 will be proportional to ν^2.

We proceed to evaluate the diagrams of Fig. 4. The only two-vertex diagram we must evaluate is shown in Fig. 4(a). When the vertex in Fig. 4(i) is inserted for the shaded area, Fig. 4(a) clearly represents a density-density correlation function and vanishes at $q=0$.[32]

The three-vertex diagrams are those of Figs. 4(b), 4(c), and 4(d). For a system with particle-hole symmetry these diagrams cancel[33] in pairs. We can prove this fact by using the identities

$$G^{R}(\omega,\epsilon) = -G^{A}(-\omega,-\epsilon)\,, \tag{3.41}$$

$$G^{>}(\omega,\epsilon) = -G^{<}(-\omega,-\epsilon)\,, \tag{3.42}$$

and changing a few integration variables. The same method can also be used to show that each four-vertex diagram is exactly equal to its "upside down" counterpart.

Before evaluating the finite corrections that the diagrams in Figs. 4(e), 4(f), and 4(g) yield, let us consider the troublesome diagrams in Fig. 4(h). As in the other diagrams in Fig. 4, the vertex corrections in Fig. 4(i) vanish here. It is the vertex corrections of Fig. 4(j) that lead to problems. Figure 5 shows schematically all nonvanishing time orderings of the diagram obtained by inserting the first term in Fig. 4(j) into the first diagram in Fig. 4(h). (Each of the remaining time orderings demanded by the rules in Sec. III B has a loop that consists entirely of advanced or entirely of retarded propagators and vanishes). Using the identity

$$G^{R}(\omega,\epsilon) = G^{A}(\omega,\epsilon)^{*}\,, \tag{3.43}$$

and the fact that the product of $G^{>}$ and $G^{<}$ is real, we see that the diagram in Fig. 5(b) is real and that the diagrams in Figs. 5(a) and 5(c) are complex conjugates. The sum of all these diagrams is real and its value is

$$I'_{h_1} = \mathcal{T}\Omega\,\frac{2e^4}{c^2m^4}\,\frac{N(0)}{\tau}\,\frac{k_F^4}{9}\,4\left(\frac{\omega_0-\nu}{e^{\beta(\omega_0-\nu)}-1}-\frac{\omega_0}{e^{\beta\omega_0}-1}\right)$$

$$\times\,\frac{1}{\nu^2+1/\tau^2}\,\frac{AA^{*}}{\omega_0^2+1/\tau^2}\,. \tag{3.44}$$

To include the remaining vertex corrections in Fig. 4(j), we multiply the result of Eq. (3.44) by the vertex shown in Fig. 6. Although we are interested in the case $\omega_1=\omega_2=\omega$, for the moment

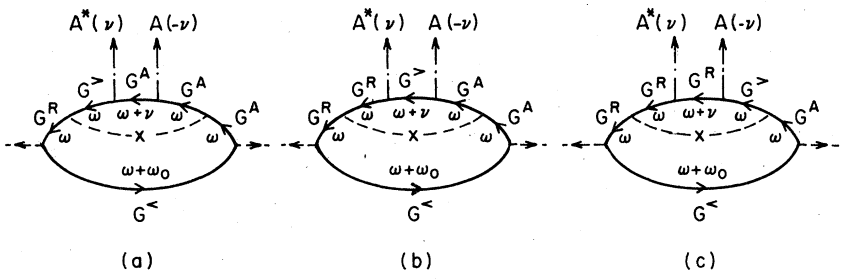

FIG. 5. Only nonvanishing contributions to one of the diagrams in Fig. 4(h).

let the frequencies ω_1 and ω_2 remain arbitrary. We can write the solution to the vertex equation

$$\Gamma = 1 + n_c|u|^2 N(0) \int d\epsilon \frac{1}{\omega_1 - \epsilon - i/2\tau} \frac{1}{\omega_2 - \epsilon + i/2\tau} \Gamma , \tag{3.45}$$

using (3.13), in the form

$$\Gamma = (\omega_1 - \omega_2 - i/\tau)/(\omega_1 - \omega_2) . \tag{3.46}$$

It is infinite when $\omega_1 = \omega_2 = \omega$! Before discussing this infinity further let us calculate the remaining diagrams in Fig. 4(h) formally. For this purpose, we add the $(\nu \rightarrow -\nu)$ term to Eq. (3.44), multiply the result by Γ, and double it to account for the "upside down" diagrams. The result is

$$I_h = \Omega \mathcal{T} \frac{2e^4}{c^2 m^4} \frac{N(0)}{\tau} \frac{k_F^4}{9} 8 \left(\frac{\omega_0 - \nu}{e^{\beta(\omega_0 - \nu)} - 1} + \frac{\omega_0 + \nu}{e^{\beta(\omega_0 + \nu)} - 1} - \frac{2\omega_0}{e^{\beta\omega_0} - 1} \right) \frac{AA^* \Gamma}{(\nu^2 + 1/\tau^2)(\omega_0^2 + 1/\tau^2)} . \tag{3.47}$$

We shall come back to this equation.

For the diagrams of Figs. 4(e), 4(f), and 4(g), we obtain after a calculation whose tediousness cannot be overemphasized

$$\langle j_{-\omega_0} j_{\omega_0} \rangle = \mathcal{T}\Omega \left(\frac{1}{9} + \frac{1}{5} \right) k_F^4 \frac{2e^4}{c^2 m^4} \frac{N(0)}{\tau} \times 4 \left(\frac{(\omega_0 - \nu)}{e^{\beta(\omega_0 - \nu)} - 1} - \frac{\omega_0}{e^{\beta\omega_0} - 1} \right) \times \frac{1}{\nu^2 + 1/\tau^2} \frac{1}{\omega_0^2 + 1/\tau^2} AA^* + (\nu \rightarrow -\nu) . \tag{3.48}$$

Note that (a) this correlation function, as well as Eq. (3.47), vanish when $\tau = \infty$. To verify that this behavior is correct, observe that with noninteracting propagators the calculation would contain products of the form $G^>(\omega, \epsilon) G^<(\omega + \omega_0, \epsilon)$ which vanish due to the δ functions in the spectral weights. (b) Equations (3.47) and (3.48) also give a vanishing contribution when $\beta = \infty$ $(T = 0)$ or $\omega_0 = \infty$ as expected. (c) In the limit of a constant electric field $(\nu \rightarrow 0)$, we have

$$B(\omega_0) \equiv \lim_{\beta\nu \rightarrow 0} \frac{6}{\beta\nu^2} \left[\frac{\omega_0 - \nu}{e^{\beta(\omega_0 - \nu)} - 1} + \frac{\omega_0 + \nu}{e^{\beta(\omega_0 + \nu)} - 1} - \frac{2\omega_0}{e^{\beta\omega_0} - 1} \right] = 3(\operatorname{csch}^2 \tfrac{1}{2}\beta\omega_0)(\tfrac{1}{2}\beta\omega_0 \coth \tfrac{1}{2}\beta\omega_0 - 1) . \tag{3.49}$$

To relate Eq. (3.48) to directly measured quantities, let us consider a special case. In the experimentally relevant regime, $\beta\omega_0 \ll 1$ and $\beta\nu \ll 1$, the approximation

$$\lim_{\beta\omega_0 \rightarrow 0} B(\omega_0) = 1 \tag{3.50}$$

is valid. In terms of the constant external electric field E,[34]

$$2\nu^2 AA^* = c^2 E^2 \tag{3.51}$$

in the limit $\nu\tau \ll 1$, $\omega_0\tau \ll 1$, Eq. (3.48) reduces to

$$\langle j_{-\omega_0} j_{\omega_0} \rangle \sim \mathcal{T}\Omega \left(\frac{1}{9} + \frac{1}{5} \right) \frac{e^4}{m^4} k_F^4 N(0) \frac{4\beta}{6} \tau^3 E^2 . \tag{3.52}$$

Using Eqs. (2.10), (2.11), and (3.34), we can write Eq. (3.52) as

$$\int d(t' - t) e^{i\omega_0(t' - t)} \langle I(t) I(t') \rangle_{\text{ne}} \equiv \delta \langle II \rangle_{\omega_0} = \frac{1}{2} \left(\frac{1}{9} + \frac{1}{5} \right) \left(\frac{eEl}{kT} \right)^2 \left(\frac{2kT}{R} \right) = 0.156 \left(\frac{l}{L} \right)^2 \left(\frac{eV}{kT} \right)^2 \left(\frac{2kT}{R} \right) , \tag{3.53}$$

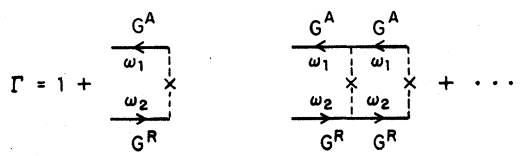

FIG. 6. Nonvanishing contributions to the vertex corrections in Fig. 4(h).

where L is the length of the sample in the direction of the electric field and $V = EL$ is the applied external voltage.

Disregarding the infinite contribution from Eq. (3.47), the total current autocorrelation function including Johnson noise is given by Eqs. (3.35) and (3.53) as

$$\langle II\rangle_{\omega_0} = \frac{2kT}{R}\left[1 + 0.156\left(\frac{eEl}{kT}\right)^2 + \cdots\right], \tag{3.54}$$

the dots standing for higher-order terms in a power series in $(eEl/kT)^2$. This parameter can be simply interpreted as the ratio of the energy gained between two collisions to the average thermal energy kT: it must be small for the term calculated in this section to be the dominant correction to Johnson noise.[35] Note also that in thermal equilibrium and when $\omega_0\tau \ll 1$, the current and voltage autocorrelation functions satisfy to a good approximation the relation

$$\langle VV\rangle_{\omega_0} = R^2\langle II\rangle_{\omega_0}. \tag{3.55}$$

This relation is *not* valid for Eq. (3.54) however. When nonlinear fluctuations are important the nonlinear response cannot be neglected. Indeed, the nonlinear fluctuations and the nonlinear response are both related to various analytic pieces of the same imaginary frequency four-particle vertex.[1,22]

The significance of the infinite vertex corrections can be understood by considering how Johnson noise is modified when the Joule heat produced by the weak currents does not escape. After some time, the thermal equilibrium noise should increase by an amount

$$\delta\langle II\rangle_{\omega_0} = (2/R)k\delta T, \tag{3.56}$$

with δT, the small temperature increase, determined by

$$\delta T = \frac{dT}{d\mathcal{E}}\frac{d\mathcal{E}}{dt}\delta t = \frac{1}{C_p}\frac{V^2}{R}\delta t, \tag{3.57}$$

where C_p is the specific heat. (For sufficiently small V^2, δT will be small even after a relatively long time.) Using Eq. (2.17), the free-electron specific heat appropriate for our impurity model, we can calculate δT in Eq. (3.57) and substitute in Eq. (3.56) obtaining

$$\delta\langle II\rangle_{\omega_0} = \frac{2kT}{R}\left(\frac{eEl}{kT}\right)^2\frac{1}{\pi^2}\left(\frac{\delta t}{\tau}\right). \tag{3.58}$$

If we combine Eqs. (3.45), (3.47), (3.49), (3.50), (3.51), (2.10), (2.11), and (3.34), we can express the contribution of the infinite diagrams in Fig. 4(h) in the form

$$\delta\langle II\rangle_{\omega_0} = \frac{2kT}{R}\left(\frac{eEl}{kT}\right)^2\frac{\Gamma}{9}. \tag{3.59}$$

Clearly then, with[36]

$$\Gamma = \frac{9}{\pi^2}\frac{\delta t}{\tau} \sim \frac{\delta t}{\tau} \tag{3.60}$$

the infinite diagrams correspond to the modification of Johnson noise that results when the temperature continues to increase linearly with time forever. In the calculation of this section we assumed implicitly that the experiment lasted for an infinite time when we took the Fourier transform in Eq. (3.40). The plausibility of the identification in Eq. (3.60) may perhaps be enhanced by rewriting the vertex (3.45) in the limit $\omega_1 = \omega_2 = \omega$ as

$$\Gamma = 1 + \Gamma = 1 + 1 + 1 + 1 + 1 + \cdots \tag{3.61}$$

and arguing that if Γ counts the number of collisions that occur during the time of the experiment, then it is indeed approximately equal to $\delta t/\tau$ where δt is the time of the experiment (infinite in Sec. IV), and τ is the average time between collisions.

Naturally, Eq. (3.58) is not valid when δt is large since the specific heat Eq. (2.17) is a function of temperature and thus will change with time. [See also Eqs. (5.5) and (5.6) and accompanying remarks.] We shall see in Sec. IV how the infinite quantities can be avoided and the heuristic arguments given above put on a firmer basis.

E. Equivalence between the Joule divergent vertex equation and a Boltzmann equation for the one-body distribution function: Calculation of heating effects in the current fluctuations

To relate the calculation of heating effects on one-body and two-body correlation functions, it is helpful to calculate the second-order effect of the electric field on the energy density. We shall find that this calculation has infinite vertex corrections similar to those we encountered when we calculated the current fluctuations.

We take for the energy density operator[37]

$$\mathcal{E}(x) = \sum_s \left[\left(-\frac{\vec{\nabla}}{i} + \frac{e\vec{A}}{c}\right)\Psi_s^\dagger(x)\frac{1}{2m}\left(\frac{\vec{\nabla}}{i} + \frac{e\vec{A}}{c}\right)\Psi_s(x) - \left(\mu - \sum_i u(\vec{x} - \vec{x}_i)\right)\Psi_s^\dagger(x)\Psi_s(x)\right], \tag{3.62}$$

where s is a spin index and $\vec{x}_i$ is the position of the impurities interacting with the conduction electrons through the potential $u(\vec{x} - \vec{x}_i)$.

To calculate the total energy *change* produced by an electric field (described as usual, by a vector potential), we must evaluate the graphs in Fig. 7, which correspond to

$$\mathcal{E}(\vec{q} = 0, \nu' = 0) = \int_{-\infty}^{\infty}\mathcal{E}(\vec{x}, t)\, d^3x\, dt. \tag{3.63}$$

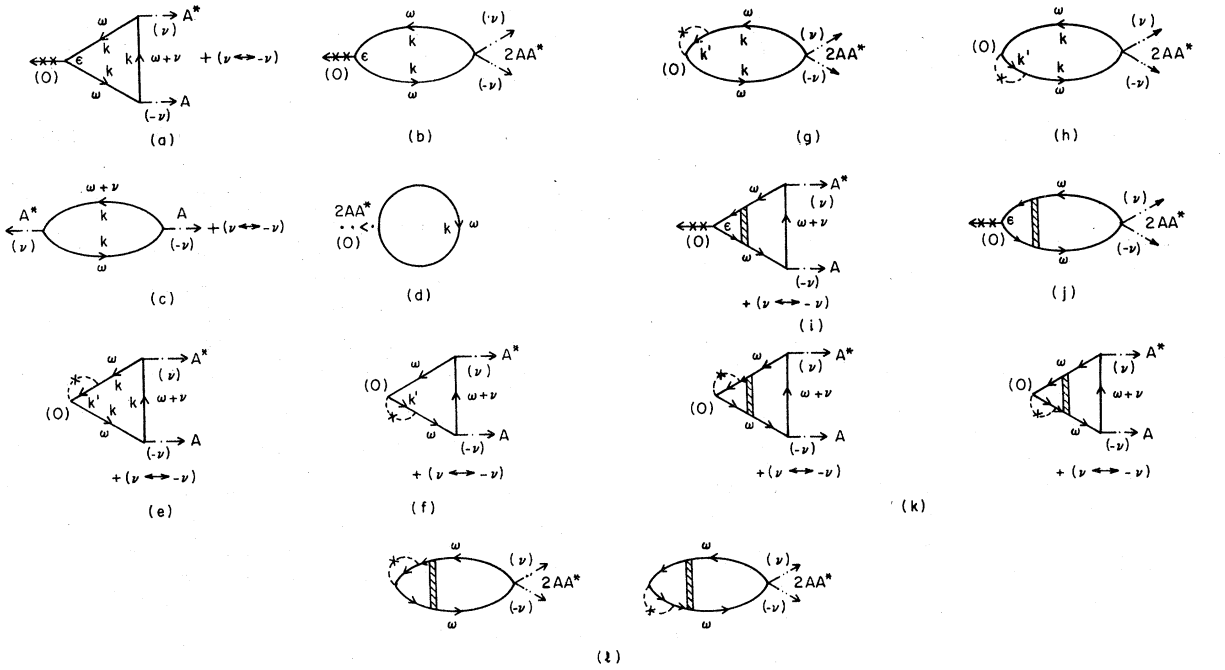

FIG. 7. Diagrams for the change in total energy caused by an applied external electric field. $\rightarrow\!\!\times\!\!\rightarrow$ represents a kinetic energy vertex (ϵ); $\overset{\cdots\times\cdots}{\longrightarrow}$ represents a potential energy vertex. The potential energy vertices are associated with a factor $n_c|u|^2(i)(-i)\int d^3k'\,[G(k',\omega)]/(2\pi)^3$. The frequency ω is unchanged at the vertex. The other symbols are defined in Fig. 4. The shaded areas represent vertex corrections.

Note that we can evaluate the diagrams as if we were calculating the time average of a quantity independent of time, i.e., no frequency emerges from the $\mathcal{E}$ vertex. We know, on the other hand, that the total energy must be a linear function of time when a constant electric field is applied. That is why the infinite-vertex corrections (proportional to the "infinite" time of the experiment) enter. Let us show this fact more explicitly.

First, note that the three-vertex diagrams in Fig. 7 (in contrast to those in Fig. 4) may contribute; the additional factor of $(\epsilon = k^2/2m - \mu)$ associated with the kinetic energy vertex changes the parity of the integrands.

It is tempting to infer from this argument that the diagrams with two corners vanish. However, this conclusion is fallacious. With the additional factor of ϵ from the energy vertex, the integrals extend far inside the Fermi surface and particle-hole symmetry is violated.

Because the $\Psi^\dagger$ associated with an energy vertex is always later on the KBL contour than the corresponding Ψ, the rules of Sec. III B for "series" multiplication can be used to enumerate the various time orders.

The most interesting graphs, those that give rise to Joule heat, are illustrated in Figs. 7(i)–7(l). Writing the contribution of Fig. 7(j) in the form[38]

$$\mathcal{E}_j = -2i^2\left(\frac{ie}{2mc^2}2AA^*\right)\int\frac{d^4k}{(2\pi)^4}(-e)GG\frac{dk_x}{dk_x}\times\int\frac{d^3k'}{(2\pi)^3}\epsilon' GG\Gamma\mathfrak{A}\mathcal{T}\,, \tag{3.64}$$

we can show, by performing an integration by parts in the first integral, that $\mathcal{E}_j$ equals minus the $\nu\to 0$ limit of the expression corresponding to Fig. 7(i).[39] Adding both terms we find that the only nonvanishing time orders are those illustrated in Fig. 8.

Evaluating them, we obtain[40]

$$\mathcal{E}_i + \mathcal{E}_j = \frac{2AA^*}{c^2}\frac{e^2k_F^2}{3m^2}\,2N(0)\frac{\nu^2}{\nu^2+1/\tau^2}\mathfrak{A}\mathcal{T}\Gamma \tag{3.65}$$

which, with the help of Eqs. (2.10), (2.11), and (3.51), can be written in the form

$$\mathcal{E}_i + \mathcal{E}_j = (V^2/R)\tau\mathcal{T}\Gamma\,. \tag{3.66}$$

Note that the vertex Γ appearing in Eqs. (3.65) and (3.66) is the complex conjugate of the one appearing in Fig. 6 with $\omega_1 = \omega_2 = \omega$.

The diagrams in Fig. 7(k) and 7(l) cancel. The leading contribution to the diagrams of Fig. 7

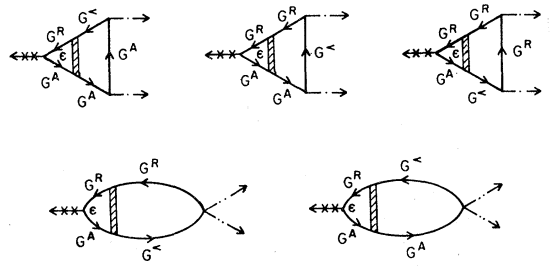

FIG. 8. Nonvanishing contributions to the diagrams of Figs. 7(i) and 7(j).

is thus given by Eq. (3.66)

$$\mathcal{E} \sim (V^2/R)\mathcal{T}\tau\Gamma\,. \tag{3.67}$$

We may determine the value of the vertex heuristically by noting that the overall factor of $\mathcal{T}$ in Eq. (3.66) arises from the assumed time independence of $\mathcal{E}$. Consistent with this assumption, we may write

$$\mathcal{E}(\vec{q}=0,\nu'=0)/\mathcal{T} = \mathcal{E}(\vec{q}=0,\mathcal{T}) = (V^2/R)\tau\Gamma\,. \tag{3.68}$$

But it is well known that for a resistor

$$\mathcal{E}(t) = (V^2/R)t\,. \tag{3.69}$$

We see that Eqs. (3.68) and (3.69) agree if

$$\Gamma = \mathcal{T}/\tau\,, \tag{3.70}$$

in qualitative agreement with Eq. (3.60).[41] Thus, even when the nonlinear response to a harmonic perturbation seems to occur at zero frequency (and be time independent), observable quantities (e.g., the energy in this section and the current autocorrelation function in Sec. III D), can depend on the total time of the experiment! The similarity between Eqs. (3.70) and (3.60) suggests that the divergence that occurs in two-body and one-body correlation functions has the same origin. Indeed, using the vector potential

$$A'(t) = \Theta(t)[Ae^{-i\nu t} + A^* e^{i\nu t}] - \Theta(t)[A + A^*] \tag{3.71}$$

corresponding to the electric field

$$E'(t) = \Theta(t)E(t) \tag{3.72}$$

[where $E(t)$ is given by Eq. (3.38)], we can show that the vertex equation we obtain is equivalent to Eq. (2.6) for the perturbed distribution function. Since we are interested in stationary rather than time-dependent solutions, we present instead a proof of the equivalence between the vertex equation with a finite momentum flowing through and Eq. (2.13) for the distribution function. If we define

$$\tilde{f}^{(2)}(\omega, \hat{k}\cdot\vec{q}, q) \equiv -\tilde{\Gamma}(q,\omega)/(\vec{v}_F\cdot\vec{q} - i/\tau)\,, \tag{3.73}$$

then the analytical expression corresponding to

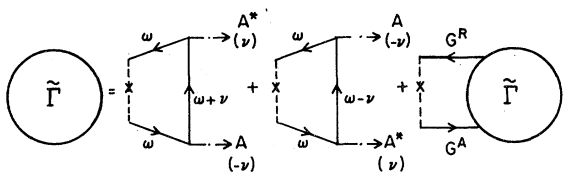

FIG. 9. Diagrammatic representation of the equation for the vertex defined in Eq. (3.73). The time orders of the inhomogeneous terms are the same as those of the corresponding part of Fig. 5.

Figs. 7(i) and 7(j), with a net outgoing momentum $\vec{q}$ at the energy vertex, can be written in the form

$$\mathcal{E}(\vec{q},0) = 2N(0)\int d\omega\,\omega\bar{\tilde{f}}^{(2)}\mathcal{T}\Omega\,. \tag{3.74}$$

In the limit $\nu\tau \ll 1$ the vertex $\tilde{\Gamma}$ then obeys the equation (see Fig. 9)

$$\tilde{\Gamma}(q,\omega) = \left(\frac{e^2v_F^2}{3}\,i\tau\frac{\partial^2 f}{\partial\omega^2}E^2 + O(ql)^2\right) + \frac{i}{\tau}\int\frac{d\hat{k}}{4\pi}\,\frac{\tilde{\Gamma}(q,\omega)}{-\vec{v}_F\cdot\vec{q}+i/\tau}\,. \tag{3.75}$$

From this equation and the definition (3.73), we deduce that

$$i\vec{q}\cdot\vec{v}_F\tilde{f}^{(2)} + \tau^{-1}(\tilde{f}^{(2)} - \bar{\tilde{f}}^{(2)}) = \tau\frac{e^2v_F^2}{3}\,\frac{\partial^2 f}{\partial\omega^2}E^2\,, \tag{3.76}$$

which should be compared with Eq. (2.13). *The vertex equation (3.75) with a nonzero momentum transfer $\vec{q}$ is equivalent to the Boltzmann equation (2.13).* The solution of Eq. (3.75) to leading order in $(ql)^2$ is

$$\tilde{\Gamma}(q,\omega) = \frac{i}{\tau}\left(\frac{eE}{q}\right)^2\frac{\partial^2 f}{\partial\omega^2}[1+O(ql)^2]\,. \tag{3.77}$$

The vertex that was infinite before is now finite and it can be used to calculate the Joule divergent diagrams. For this purpose it is useful to observe that the diagram element that appears in Fig. 10 has the value

$$\Gamma_{RA}^{jj} = 2\pi N(0)if(\omega+\omega_0)\frac{e^2k_F^2}{3m^2} \times\frac{2}{\omega_0^2+1/\tau^2}[1+O(ql)^2]\,. \tag{3.78}$$

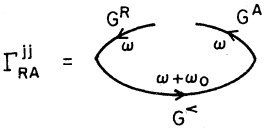

FIG. 10. The first diagram in Fig. 4(h) can be calculated by multiplying the piece of diagram illustrated here by the vertex of Fig. 9 and various constant factors. All the integrals except the one over ω have been performed in the result quoted for this diagram in Eq. (3.78).

Using Eqs. (3.77) and (3.78), we obtain from all the diagrams in Fig. 4(h)

$$\delta\langle II\rangle_{\omega_0}=\frac{2kT}{R}\left[\frac{\pi^2}{9}\frac{\delta T_q}{T}+O\left(\frac{eEl}{kT}\right)^2+\cdots\right]\times\frac{B(\omega_0)}{(\omega_0\tau)^2+1}\,, \tag{3.79}$$

where

$$\frac{\delta T_q}{T}=\left(\frac{eE}{qkT}\right)^2\frac{3}{\pi^2} \tag{3.80}$$

is the temperature increase consistent with Eq. (2.16).

The difference between a pure thermal gradient and this stationary solution for the vertex equation (distribution function) is discussed in the Appendix.

The terms of order $(ql)^2$ in Eqs. (3.77) and (3.78), give rise, in Eq. (3.79), to $(eEl/kT)^2$ corrections, smaller than the $\delta T_q/T$ terms. We have not evaluated these corrections. Indeed, they give rise to a technical problem we have not resolved. It appears that the value of these terms depends upon how the momentum q is distributed among the current vertices. For example, if we calculate the terms of order $(ql)^2$ in Eq. (3.78) for

$$\langle j(\tfrac{1}{2}\vec{q}(1+\alpha),t)j(\tfrac{1}{2}\vec{q}(1-\alpha),t')\rangle\,,$$

α being an arbitrary real number, we find, in the limit $\omega_0\tau\ll 1$

$$O(ql)^2=2\pi N(0)\frac{e^2k_F^2}{m^2}if(\omega+\omega_0)\times[(v_Fq)^2/15]\tau^4[(-7-\alpha^2)/2]\,. \tag{3.81}$$

An unambiguous result should not depend on α. A similar difficulty appears to plague Eq. (3.77). If we introduce a parameter β to describe the division of the momentum between the current vertices exhibited in Fig. 9, the value of the terms of order $(ql)^2$ depends on β.

Finally, let us observe that (a) the change in the current correlation function of a uniform system is a factor of $\pi^2/9$ larger than we would guess by inserting the temperature change calculated from Joule heating [see Sec. III D and Eqs. (3.57) and (3.60)]. This conclusion agrees with Huberman's[7] observation that a uniform electric field does not simply raise the temperature of a degenerate electron gas. (b) The vertex in Fig. 9 describes the change in the one-particle distribution function produced by the electric field. This identification may be substantiated diagrammatically by noting that in equilibrium the current correlation function is given by the combination $G^>G^<$ shown in Fig. 2, or since

$$G^>=-i[1-f(\omega)](1/\tau)[G^R(\vec{k},\omega)G^A(\vec{k},\omega)]\,, \tag{3.82}$$

by the product of $(-i)[1-f(\omega)]\tau^{-1}$ and the graphical elements shown in Fig. 10. The nonequilibrium current fluctuations contribution in Fig. 4(h) is obtained from the same elements with Fig. 9 replacing $(-i)[1-f(\omega)]\tau^{-1}$.

IV. SUMMARY OF THE RESULTS FOR THE CURRENT FLUCTUATIONS

We have ascribed to Joule heating the divergent addition to the current fluctuations we found by perturbation theory to second order in the electric field. More specifically we have shown that the diverging vertex reflects the secular growth in time of the E^2 correction to the one-body distribution function when there is no "cooling" mechanism. A finite, time-independent solution of the vertex equation (or equivalently the Boltzmann equation) can be obtained by including an additional "self-consistent" cooling gradient transverse[42] to the electric field. With this stationary vertex, we obtain finite current fluctuations.

The stationary distribution function cannot be described precisely by a spatially dependent local temperature [see Eq. (3.79) and the Appendix]. As a result, the shifted values of most observables *cannot* be obtained from simple macroscopic arguments and the usual definition of temperature. The energy density is a notable exception.

Our final result for the steady-state current-current correlation function of an electron gas with impurities (which is a good model for a metallic resistor at low temperatures) is

$$\int_{-\infty}^{\infty}e^{i\omega_0(t'-t)}\langle I(t)I(t')\rangle d(t'-t)=\frac{2kT}{R}\left[1+\frac{\pi^2}{9}\frac{\delta T_q}{T}+O\left(\frac{eEl}{kT}\right)^2+\cdots\right]\frac{B(\omega_0)}{(\omega_0\tau)^2+1}\,, \tag{4.1}$$

where $B(\omega_0)$ incorporates the quantum corrections [see Eq. (3.49)]

$$\lim_{\beta\omega_0\to 0}B(\omega_0)=1\,,\qquad \lim_{\beta\omega_0\to\infty}B(\omega_0)=0\,. \tag{4.2}$$

In Eq. (4.1), δT_q, the amplitude of the Fourier component of the "temperature gradient" in the system, satisfies Eq. (2.16), T is the temperature, R is the resistance [Eq. (2.10)], k Boltz-

mann's constant, l the mean free path [Eq. (3.13)], and e the electron charge. The first term of Eq. (4.1) is the equilibrium result (Johnson noise), the second term, of order $\delta T_q/T$, represents the fact that the "temperature" of the system is on the average slightly higher when an electric field is present [Eq. (3.79)], and the third term represents corrections that have been calculated in Sec. III D [see Eq. (3.53)]. These corrections involve the expansion parameter $(eEl/kT)^2$, the square of the ratio of the energy gained between collisions to the average thermal energy.[43] As noted in Sec. III E, additional ambiguous corrections of the same order arise from the diagram of Fig. 4(h) after it is made finite. These ambiguous corrections *may* reflect the fact that the corrections to the boundary dependent hydrodynamic terms are themselves sensitive to the boundaries. However, by setting $q \sim L_\perp^{-1}$ in Eq. (3.80) we find that

$$\frac{\delta T}{T} \sim \frac{3}{\pi^2}\left(\frac{L_\perp}{l}\right)^2\left(\frac{eEl}{kT}\right)^2, \tag{4.3}$$

which indicates that the ambiguous corrections are probably unobservable. We shall therefore not study them further.

Sufficient conditions for the validity of Eq. (4.1) are listed below. They fall into several categories. Presumably several of the conditions are not necessary.

A. Validity of perturbation theory

(i) The expansion parameters $(eEl/kT)^2$ and $\delta T_q/T$ must be small. The latter expansion parameter allowed us to use the "standard" perturbative solution of the Boltzmann equation for the infinite vertex [see the remark below Eq. (2.6) and Eqs. (2.13), (3.73), and (3.75)]. Note that Eq. (4.1) describes the nonlinear *current* fluctuations along the applied constant electric field. When the corrections to Johnson noise are important, the distinction between corrections to *voltage* and *current* fluctuations is also important; they are not, in general, simply related [see remark following Eq. (3.55)].

B. Validity of the impurity scattering model

(ii) Impurity scattering must be dominant. Typically this condition holds at low temperatures where phonon scattering is negligible (see Sec. III A).

(iii) The impurities must be dilute. This condition on their concentration may be expressed in the form $k_F l \gg 1$, where k_F is the Fermi wave vector and l the mean free path [see Eq. (3.11)]. Literally, our model is restricted to impurity scattering potentials which have no bound state and are very short ranged [see Eq. (3.6)]. The condition on the range enters through the assumption of s-wave, isotropic scattering. If the scattering is anisotropic, vertex corrections which account for forward scattering will alter the value of τ_{tr}, the collision transport time. However, in terms of τ_{tr} our conclusions are probably unchanged.[8]

C. Validity of hydrodynamic assumptions

(iv) The conditions for the Boltzmann equation should be satisfied. In particular, the sample must be large enough for the inequality $L_\perp/l \gg 1$ to hold [see remark before Eq. (3.77)] $(L_\perp \sim q^{-1})$.

(v) The frequency may have to be larger than the rate at which the system exchanges energy with the substrate or other parts of the apparatus. Since, in general, these couplings involve long relaxation times, they can only affect correlations at very low frequencies. If $L_\perp$ is the shortest sample dimension, it is inconceivable that those couplings could modify our conclusions at frequencies higher than the thermalization frequency $(l/L_\perp)^2/3\tau$.

Some time ago, Hooge and Hoppenbrouwers[44] reported observing high-frequency corrections to Johnson noise that scaled with the square of the applied voltage. Since δT_q in Eq. (4.1) is related to the electric field via Eq. (4.3), we surmise that they were simply looking at the first term of Eq. (4.1) (heating). (See Note added in proof.) The more recent experiments of the Eindhoven group[5] mentioned in the Introduction and in Sec. V may be more relevant to our calculation.

V. CONCLUSION

In this final section we comment on the relation between our calculation and $1/f$ noise, indicate how to extend our calculation, and reflect on what we have learned about nonequilibrium perturbation theory.

A. Could our model exhibit $1/f$ noise?

Our calculation does not give $1/f$ noise. This result may be confirmed by recent experiments[5] which show that as impurity scattering becomes dominant, the amplitude of $1/f$ noise decreases. However, the systems investigated experimentally are Ohmic metal semiconductor point contacts

whose modeling by free electrons scattering off impurities is not entirely unambiguous. If our model does not apply to the systems studied by Hooge and Vandamme[5] and exhibits $1/f$ noise, should we have found it?

It might be argued that $1/f$ noise can only occur in a regime in which perturbation theory fails. After all, although $1/f$ noise is observed to be proportional to E^2, for sufficiently small frequencies it must become larger than the (frequency-independent) term of zero order in the electric field (Johnson noise). In addition, the term of order E^2 is supposed to be larger than higher-order terms since no dependence upon E^4, E^6, etc. has been reported. We do not believe that these arguments make a calculation to order E^2 senseless. If our model contained $1/f$ noise, we believe the lowest-order term in perturbation theory should at least show an anomaly in the frequency dependence. This does not mean that it would not be valuable to treat our model without making expansions in the electric field.[45-47] We doubt, however, that when $\delta T_q/T < 1$ the results would differ qualitatively from those we have found.

Some investigations of the stationarity of the $1/f$ phenomenon have suggested that $1/f$ fluctuations imply that there is no stationary state. Not all experiments agree.[48] Since our calculation does not yield $1/f$ noise, there is little reason for expecting it to give additional credence to nonstationarity. Indeed, at least in perturbation theory, there is little doubt that our model has a stationary state.

Moreover, the absence of $1/f$ noise cannot be ascribed either to the omission of terms inversely proportional to the volume, or to a disregard of varying collision times. A slightly different model with a distribution of scattering times would not behave differently.

1. *Factor of 1/N*

Apart from the term of order $\delta T_q/T$, the current fluctuations we have obtained in Eq. (4.1) are proportional to $1/N$, where N is the number of charge carriers. This factor is present in the phenomenological description of $1/f$ noise[4] and might be expected from general statistical arguments. In our calculation, if we suppose that the nonlinear resistance is given by

$$R(E) = R\left[1 + b\left(\frac{eEl}{kT}\right)^2 + \cdots\right], \tag{5.1}$$

where b is a numerical factor, then, apart from numerical factors, we can write

$$\delta\langle VV\rangle_{\omega_0} \sim RkT\left(\frac{eEl}{kT}\right)^2. \tag{5.2}$$

Using $V = EL, \Omega = AL$ and Eq. (2.10) we find that

$$\delta\langle VV\rangle_{\omega_0} \sim \frac{1}{N}\left(\frac{ml^2}{\tau kT}\right) V^2, \tag{5.3}$$

where $N = n\Omega$ is the number of charge carriers.

2. *Matthiessen's rule*

We cannot obtain $1/f$ noise from our model by including many decay times for the electrons because our system obeys Matthiessen's rule.[49] This rule, which can be deduced from the propagator equation in the dilute limit $k_F l \gg 1$ implies, for example, that if there are two types of scattering impurities described by the lifetimes τ_1 and τ_2 our result applies if we set $\tau^{-1} = \tau_1^{-1} + \tau_2^{-1}$. This composition law should be contrasted with the addition of the current autocorrelation functions due to τ_1 alone and τ_2 alone. The latter composition law, which is sometimes used to explain $1/f$ noise,[50] may be valid for distinguishable, independent sources of noise, but it is certainly *not* valid for impurity scattering in metal films, even in the "classical limit" $\beta\omega_0 \ll 1$.

To summarize, we do not believe that our model or slight variants of it exhibit $1/f$ noise but our arguments, which are based on perturbation theory, are not rigorous. We cannot exclude the possibility that $1/f$ noise arises entirely from nonanalytical contributions that perturbation theory omits.

B. Physical processes neglected in our calculation which could lead to $1/f$ noise

The only microscopic time in our model is the scattering time τ. Longer times could be included, for example, by taking into account the substrate-sample interaction more realistically[51] and/or including the effect of hydrodynamic modes.

Since our calculation was performed for the bulk,[43] it does not account for all couplings to hydrodynamic modes that can conceivably occur.[52] However, it should be noted that our model cannot give rise to $1/f$ noise via the mechanism of Voss and Clarke.[53] Our model has a temperature-independent resistance and their proposal is based on the temperature dependence of the resistance. Despite this fact, our model does involve a coupling to the energy density.[54] Although the resistance is temperature independent, the current fluctuations are not. Note also that the first term in Eq. (4.1) is size dependent and thus in some sense is sensitive to the manner in which Joule heat is dissipated. This agrees indirectly with

some experiments.[55]

Obviously many other processes and more sophisticated models could be investigated. For example, we might consider phonons[45,46] that disappear at the boundary of the system, liberating Joule heat to the surroundings. Whether or not such models would be qualitatively different from ours is a matter on which we can only speculate.

C. Nonequilibrium perturbation theory

Our calculation shows that the manifestations of heating must be understood before realistic calculations on many-body systems beyond linear order can be performed. It also sheds some light on the general structure of high-order many-body response theory.

1. Adiabatic switching

We have found that, in nonlinear order, when an external perturbation is applied for a finite time, secular terms that describe heating grow. Thus if the perturbation is applied adiabatically, divergences can occur. The secular terms that occur when the field is adiabatically applied can be eliminated by treating the transfer of "heat" to the outside, simultaneously and "self-consistently." In the case we studied, this self-consistency is achieved by including gradient terms. Stationarity could also be achieved by introducing phonons which can disappear at the boundaries or are constrained to remain in equilibrium.[22,46] Once a satisfactory "cooling" mechanism has been included a calculation of the effect of applying an external (electric) field adiabatically will not diverge.

2. Fluctuation-dissipation theorems

The basic idea behind recent efforts[56-58] to derive generalizations of the fluctuation-dissipation theorem resembles the idea behind the work of Bernard and Callen[1] even though the methods that have been employed have evolved considerably.[56] The present work illustrates some of the difficulties that are bound to plague such efforts.

Even if our calculations were performed using "equilibrium" diagrams so that what happens in the nonequilibrium state is in some remote sense associated with the behavior of the equilibrium state, we wish to emphasize two difficulties (which have not to our knowledge been appreciated previously) and which must plague any attempt to relate the two-current correlation function for steady-state nonequilibrium experiments and the four-current equilibrium correlation function: (a) Some of the "time-orders" of the four-current correlation functions that are needed to calculate the two-current nonequilibrium correlation function cannot really be measured in the equilibrium state since they diverge as $q \to 0$. (b) Any generalized fluctuation-dissipation theorem must reflect the restrictions implied by the fact that the nonequilibrium state depends, to nonlinear order, on the way the system is cooled.[59]

It is important to realize that, in our calculation, the equilibrium density matrix played the role of an initial condition: it cannot act as a thermostat preventing the system from heating.

The fact that in our model, when no cooling is present, the current fluctuations are divergent to order E^2 while the resistance is finite[60] to order E^3 further illustrates the difference between response and fluctuations to nonlinear order.

3. Structure of the perturbation expansion: dynamics and heating

The successive terms of a perturbation expansion in powers of an applied external field certainly represent for any physical system (e.g., an atom) corrections to the *dynamics* of the system induced by the field. For a many-body system, however, there are also corrections to the *statistics* of the system that occur naturally as terms that depend on powers of the applied field, multiplied by the time. Loosely speaking, we can say that for statistical systems, there is a new expansion parameter in the theory given by $\delta T/T$, the ratio of the "temperature" change induced by the field to the starting equilibrium temperature.

The results derived in Ref. 60 suggest strongly that these terms of order $\delta T/T$ occur as follows: If the temperature T appears in the response of an observable to a given order in powers of the applied external field, then an infinitesimal rise in the temperature caused by the field will manifest itself only in the next-higher-order terms. This is consistent with the behavior of the corrections to the current fluctuations and to the resistance in our model.[60]

We also surmise that if the temperature change in the system is appreciable, it must manifest itself through higher-and-higher-order terms in the perturbation expansion. (Note that Joule poles are rigorously excluded only in the zero- and linear-order terms of the expansion.) This should be clear from the fact that Eq. (3.58) for the nonequilibrium current fluctuations yields, roughly speaking,

$$\delta\langle II\rangle_{\omega_0} \sim \frac{2kT}{R}\left(\frac{eEl}{kT}\right)^2 \frac{1}{\pi^2}\left(\frac{\delta t}{\tau}\right), \tag{5.4}$$

where δt is the length of time ($\pm\tau$) during which the field has acted. Equation (5.4) cannot be valid for arbitrarily large δt since we know that

$$\delta\langle H\rangle_{\omega_0} \sim (2k/R)(T_f - T)\,, \tag{5.5}$$

where T_f can be obtained by integrating Eq. (3.57), using the value (2.17) for the specific heat:

$$\delta\langle H\rangle_{\omega_0} \sim \frac{2kT}{R}\left\{\left[1+\frac{2}{\pi^2}\left(\frac{eEl}{kT}\right)^2\frac{\delta t}{\tau}\right]^{1/2}-1\right\}. \tag{5.6}$$

Clearly, Eq. (5.6) reduces to Eq. (5.4) when δt is small enough. We expect that some of the higher-order terms in the perturbation expansion should correspond to the terms obtained from expanding the square root in Eq. (5.6). If this is true, it suggests that the perturbation series will break down for a δt (or a temperature rise) given by the radius of convergence of the Taylor series for the square root: $[(T_f - T)/T \sim 0.4]$. (This is clearly only a rough estimate. Other correlation functions will in general have a functional dependence on the temperature very different from that of the current fluctuations.) This also suggests that a perturbation theory to order E^2 might not be meaningful if the heating effect of the electric field makes the temperature grow to an effective value $T' \gg T$. In such a case, many higher-order terms in the perturbation expansion must be taken into account and it is not obvious that these would only lead to an effective temperature T'.[7]

We summarize the most important remarks of this section by writing the perturbation expansion for the current response to an applied field schematically

$$I = \frac{V}{R} + \left(AV^3 + B\frac{\partial R^{-1}}{\partial T}\delta T V\right) + O(V^5) + \cdots . \tag{5.7}$$

V is the applied external potential and V/R the usual linear response. The first term in the parentheses (AV^3) represents the change in the dynamics of the system induced by the field; the second term, the change in the temperature (statistics). The magnitude is also of order V^3 since the leading term in δT is of order V^2.

4. *Justifying linear-response theory*

There has been some criticism of linear-response theory. Equation (5.7) clearly shows that the complete theory does involve terms which, as a function of time, rapidly become larger than the linear-order terms. For example, we can estimate that at room temperature, the second term in the parentheses of Eq. (5.7) becomes dominant after less than 1 sec, for an applied external field of 2.5×10^{-3} V/cm, a mean free path of order 10^{-6} cm and a collision time of order 10^{-14} sec [see Eq. (2.29)].

One second is much larger than the duration time of most experiments. However, even for steady-state experiments that last a long time, the use of linear-response theory can be justified by allowing for the cooling mechanism that exactly cancels the secular heating terms in Eq. (5.7). If the cancellation were not exact the system could not be in a steady state. Our calculation gives an example of how such a cancellation arises. Equation (5.7) then shows that as long as the field is small enough so the terms of the form AV^3 in Eq. (5.7) are negligible only the linear terms remain. Often the cooling mechanism leads to additional linear effects but these are usually easily treated. In the case of a thermal gradient, for example, the cooling mechanism would introduce a thermoelectric term in Eq. (5.7).[60]

Note added in proof. A generalization of this hypothesis was verified by T. G. M. Kleinpenning [Physica B 84, 353 (1976)].

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APPENDIX

In this Appendix, we show that the stationary distribution function found from Eq. (2.13) or (3.77) and (3.73) cannot be exactly described by a local temperature. When the steady state can be described by a Fermi distribution function characterized by a local, spatially varying temperature we find that for times much larger than τ, but much less than the thermalization time, the perturbative solution to the Boltzmann equation satisfies

$$f^{(1)} = -\tau\vec{\mathbf{v}}\cdot\vec{\nabla}_x f^{(0)} \tag{A1}$$

$$\bar{f}^{(2)} = \int\frac{d\hat{p}}{4\pi}\tau(\vec{\mathbf{v}}\cdot\vec{\nabla}_x)^2 f^{(0)} t\,. \tag{A2}$$

Substituting Eq. (A2) into Eq. (2.7) we obtain

$$\delta\mathcal{E}_{\text{heat}} = \left(\kappa\nabla^2 T + \frac{\partial\kappa}{\partial T}(\nabla T)^2\right)t\,, \tag{A3}$$

where κ is given by Eq. (2.18). This supports the interpretation of the solution Eq. (A2) in terms of the classical notion of a temperature gradient.[61]

On the other hand, in the presence of an electric field, Eq. (2.6) gave us

$$\delta\mathcal{E}_{\text{Joule}} = (V^2/R)t \ . \tag{A4}$$

We might prefer to set up a stationary state by adjusting the temperature field so that the resultant total energy is time independent

$$\delta\mathcal{E} = \delta\mathcal{E}_{\text{Joule}} + \delta\mathcal{E}_{\text{heat}} = 0 \ . \tag{A5}$$

From Eqs. (A3) and (A4) we see that when Eq. (2.21) is satisfied

$$\frac{\delta\mathcal{E}_{\text{Joule}}}{\delta\mathcal{E}_{\text{heat}}} = \frac{V^2}{R}\,\frac{1}{\kappa\nabla^2 T} = -1 \ . \tag{A6}$$

This result should be compared with Eq. (2.26). Equation (A6) determines the temperature field and thus the solution (A2). Using this solution and Eqs. (2.6) and (2.27) we find for an arbitrary observable, to leading order in kT/E_F,

$$\frac{\langle A\rangle_{\text{Joule}}}{\langle A\rangle_{\text{heat}}} = \left(\frac{eE}{q}\right)^2 \frac{T}{\delta T}\left(\int_{-\infty}^{\infty} d\epsilon\, \frac{\partial^2 f}{\partial \epsilon^2}\,\Gamma^A(\epsilon)\right) \times\left(\int_{-\infty}^{\infty} d\epsilon\, \epsilon\,\frac{\partial f}{\partial \epsilon}\,\Gamma^A(\epsilon)\right)^{-1} . \tag{A7}$$

This ratio differs in general from one observable to another.[62] Consequently, when δT is fixed by Eq. (A6), for most observables we will obtain

$$\langle A\rangle_{\text{Joule}} + \langle A\rangle_{\text{heat}} = Ct \neq 0 \ , \tag{A8}$$

where C is a time-independent constant. This proves that a distribution function described by a local temperature cannot lead to a stationary solution in the presence of an electric field.

The correct stationary solution Eqs. (3.73) and (3.77) is energy conserving, and thus can be interpreted in terms of Eq. (2.26) but its detailed energy and momentum dependence is different from the simple ansatz corresponding to a Fermi distribution with a spatially dependent temperature. All of this is in agreement with the results of Huberman[7] quoted at the end of Sec. III E.

Note that Eq. (A7) can also be derived from diagrams by taking the ratio of the responses of a one-body observable to an electric field (to second order) and to a temperature gradient.[61] Since the observable on which the "electric" and "thermal" vertices are "closed" is arbitrary, it is not surprising that the effect of both perturbations will not be the same unless their energy and momentum dependence is the same.[63] Thus, in this context, Eq. (A7) also says that the energy and momentum dependence of a "simple" temperature perturbation cannot exactly cancel the effect of an electric field perturbation to second order.

Finally note that although a stationary one-particle distribution function can make all the observables in Eq. (2.27) time independent, we know of no proof that the two-particle correlation functions must be stationary whenever the one-particle distribution function is. This is nevertheless what we have found for the special case considered here.

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*New address: LASSP, Clark Hall, Cornell University, Ithaca, N. Y. 14853.
†New address: Physics Dept., Ohio State University, Columbus, Ohio 43210.
‡New address: Physics Dept., Brandeis University, Waltham, Mass. 02154.

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[22]L. P. Gor'kov and G. M. Eliashberg, Zh. Eksp. Teor. Fiz. 54, 612 (1968) [Sov. Phys. JETP 27, 328 (1968)].

[23]See Ref. 24, Chap. I for more details.

[24]A.-M. Tremblay, Ph.D. thesis (Massachusetts Institute of Technology, 1978) (unpublished).

[25]F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965), pp. 583–594.

[26]Diagrams composed of two distinct pieces, each of which is connected to a different current measurement, correspond to $\langle j(t')\rangle\langle j(t)\rangle$, and thus will be neglected. Only the linked diagrams contribute to the current *fluctuations*.

[27]This would not be allowed if we were calculating the retarded or advanced correlation functions (Ref. 8, pp. 312 and 341). The subtractions, needed to perform the ϵ integration first, cancel in Eq. (3.33) because the correlation function is proportional to the difference between the retarded and advanced functions and the subtractions needed for these are identical. See Ref. 24, Chap. IV, Appendix B for more details.

[28]I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965), p. 306, integral 3.315.2.

[29]See Ref. 24, Chap. IV, Appendix D.

[30]See Ref. 19, pp. 224–240.

[31]See Ref. 24, Chap. IV, Appendix C.

[32]D. Forster, *Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions* (Benjamin, Reading, Mass., 1975), Eq. (4.44a); J. M. Luttinger, Phys. Rev. 135, A1505 (1964).

[33]This cancellation is analogous to Furry's theorem. See Ref. 24, Chap. IV, Sec. 4.

[34]The effective constant applied electric field should equal the average of the square of the low-frequency ($\nu \to 0$) field. This can be made more precise; see Ref. 24, Chap. IV, Appendix D.

[35]The same parameter appears in Huberman's (Ref. 7) long-time expansion for the Boltzmann equation.

[36]We shall comment later on the factor of $9/\pi^2$.

[37]$\mathcal{E}(\chi) = \sum_s [\Psi_s^\dagger(x)(i\partial/\partial t)\Psi_s(x) + \text{H.c.}]$ can also be used in principle but great care must be exercised. See Ref. 24, Chap. IV, Appendix E.

[38]This expression is written in terms of the KBL propagators defined in Eq. (3.22).

[39]In other words, there is a Ward identity (cf. Ref. 19) relating both types of diagrams.

[40]The integral involving ϵ in the numerator can be evaluated by contour integration only after performing a few manipulations to recast it in a form well behaved at infinity.

[41]The identification in Eq. (3.70) contains an ambiguous factor of 2 which depends on the heuristic arguments.

[42]The direction of the heat flow should not affect the first term of Eq. (4.1). However, if the gradient was in the direction of the current measurement, it could be coupled to the hydrodynamic modes. See the following footnote.

[43]By analogy with Sec. III E we should also allow for a finite momentum transfer q in the finite graphs of Sec. III D. However, unless the coupling to the hydrodynamic modes transverse to the current measurement is singular, this will only produce corrections of order $(ql)^2(eEl/kT)^2$.

[44]F. N. Hooge and A. M. H. Hoppenbrouwers, Physica 45, 386 (1969).

[45]Some steps towards a nonperturbative approach have apparently been taken by Sh. M. Kogan and A. Ya. Shul'man, Zh. Eksp. Teor. Fiz. 56, 862 (1969) [Sov. Phys. JETP 29, 467 (1969)] and by the authors in Refs. 46 and 47. We were unaware of Refs. 45–47 when the calculation was performed

[46]S. V. Gantsevich, V. L. Gurevich, and R. Katilyus, Zh. Eksp. Teor. Fiz. 57, 503 (1969) [Sov. Phys. JETP 30, 276 (1970)].

[47]S. S. Rozhkov and P. M. Tomchuk, Zh. Eksp. Teor. Fiz. 72, 248 (1977) [Sov. Phys. JETP 45, 130 (1977)].

[48]M. Stoisiek and D. Wolf, J. Appl. Phys. 47, 362 (1976); W. J. Moore, J. Appl. Phys. 45, 1896 (1974); R. A. Dell, M. Epstein, and C. R. Kannewurf, J. Appl. Phys. 44, 472 (1973); F. N. Hooge and A. M. H. Hoppenbrouwers, Physica 42, 33 (1969); J. J. Brophy, Phys. Rev. 116, 827 (1968).

[49]See Ref. 10, p. 323.

[50]For a theoretical review with a good bibliography see A. van der Ziel, in *Proceedings of the Symposium on 1/f Fluctuations Tokyo, July 1977*, edited by H. Fukuyo and T. Musha (Tokyo Institute of Technology, Nagatsuta Midori-ku, Yokohama, 1977) and also Ref. 5. See also the introductory remarks of P. H. Handel, in *1975 NATO Advanced Study Institute on Linear and Non-Linear Electron Transport in Solids*, Antwerp, Belgium, edited by J. T. Devreese and V. E. van Doren (Plenum, New York, 1976), p. 515.

[51]See S. H. Liu, Phys. Rev. B 16, 4218 (1977) for a recent theoretical attempt to incorporate substrate-sample interactions in the temperature fluctuation model of Voss and Clarke (Ref. 53). J. Clarke and Y. Hsiang, Phys. Rev. Lett. 34, 1217 (1975) found experimental evidence for the importance of the substrate in some cases. See also T. G. M. Kleinpenning Physica B 84, 353 (1976).

[52]The density dependence of the resistance, for example, could produce a coupling to the density mode.

[53]R. F. Voss and J. Clarke, Phys. Rev. B 13, 556 (1976).

[54]For the calculation of Fig. 4(h) in Sec. III E we needed q finite (cf. Ref. 43).

[55]J. Clarke and M. B. Ketchen, Phys. Rev. B 17, 114 (1978). Recent experiments by J. W. Eberhard and P. M. Horn [Phys. Rev. B 18, 6681 (1978)] suggest that the substrate is important for at least one of the "two types" (type A) of $1/f$ noise they identified. However, these authors, as well as many others [for example, F. N. Hooge, Phys. Lett. A 29, 139 (1969)] suggest that, at least in some systems, the surfaces are unimportant. See also T. G. M. Kleinpenning, Physica B 84, 353 (1976).

[56]G. N. Bochkov and Yu. E. Kuzovlev, Zh. Eksp. Teor. Fiz. 72, 238 (1977) [Sov. Phys. JETP 45, 125 (1977)].

[57]R. L. Stratonovich, Zh. Eksp. Teor. Fiz. 58, 1612 (1970) [Sov. Phys. JETP 31, 864 (1970)].

[58]G. F. Efremov, Zh. Eksp. Teor. Fiz. 55, 2322 (1969) [Sov. Phys. JETP 28, 1232 (1969)].

[59]While the Joule divergent diagrams are greatly influenced by the cooling mechanism, the finite diagrams in Fig. 4 are not (cf. Ref. 43).

[60]See Ref. 24, Chap. IV, Appendix F.

[61]We also note that the results that can be computed with Eq. (A2) are exactly the same as those obtained diagrammatically using the prescription of L. P. Kadanoff and P. C. Martin, Ann. Phys. 24, 419 (1963), Appendix C, to define the interaction Hamiltonian corresponding to a temperature gradient. The latter method can also be extended to two-body observables (Ref. 24, Chap. IV, Appendix E).

[62]A similar expression can be derived for multiparticle observables.

[63]At times larger than the thermalization time, the result $\delta\mathcal{E} = C_p \delta T$ for a temperature perturbation is obviously different from Eq. (A4) for an electric field perturbation. We are concerned here with a cancellation between electric and thermal perturbations at "short times" when Eq. (A2) is valid.