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Saturation effects and stimulated scattering in two-standing mode operation of a lasing system with stationary atoms

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The simultaneous oscillation of two standing waves in a solid-state laser is considered. An exact solution in terms of continued-fraction expansion is derived. This method enables one to consider high-intensity fields. Two distinct regimes are treated: a weak-coupling regime and a strong-coupling regime, depending on the intensities of the two modes. The effects of spatial inhomogeneity are stressed in the two different ranges of operation. The gain function turns out to be the sum of two parts: the first one includes gain saturation and the second one derives from the direct coupling of the two modes. The dispersionlike behavior of this latter term is discussed as a general result of nonlinear interactions. The amplification of a small-amplitude mode in presence of a strong one is interpreted in terms of a hyper-Rayleigh effect.

I. INTRODUCTION

Laser light is often made up of several modes of the resonant cavity with frequencies close to the lasing transition frequency, while their spread is limited by the atomic line shape. The competition between cavity modes is an intrinsic cause of instability in the output light of lasers spiking behavior in solid-state lasers,[1] or free-running operation in gas lasers,[2] or it may even cause, in other favorable cases, a stable operation in which all the modes are locked together.[3] The competition itself is quite different, depending on whether the atomic (or molecular) line of gain is homogeneously broadened or not.[4]

The recent progress in the semiclassical theory of lasers allows one to treat high-intensity fields in laser cavities through a suitable formalism that gives convergent expansions of the atomic variables involved in the interaction, however large the intensities of the modal fields are. Therefore we are in a position to have a general view of the interaction phenomena which arise in multimode operation in lasers. As a matter of fact, the computational difficulty increases as soon as the number of the simultaneously oscillating modes is raising. In the present paper we deal with a two-mode case in detail showing that the main features of the multimode operation can be deduced anyway. Most of these phenomena are well known and are given a detailed description; others have received less attention. In this simple situation, when two standing electromagnetic modes of arbitrary intensities oscillate simultaneously in a cavity in which an inverted medium with homogeneous gain line supplies the necessary power to sustain oscillations against the losses, the saturated gain of the two modes is independent of their relative phase. Therefore we are not concerned with any locking mechanism of the frequencies of the modes. However we make a distinction between two kinds of energy exchange among the modes: one which is always phase independent, even in multimode operation, the other which is basically a hyper-Rayleigh effect,[5] depending on the relative phases when three or more modes oscillate. This latter crossing term accounts for a phase-dependent energy exchange or phase-locking mechanism. The assumed homogeneity of the line of gain greatly simplifies the problem. Actually inhomogeneities can appear even in a stationary atomic array, both for diffusive effects[6] of for local strain effects which can alter the atomic transition frequency.[7] However both these inhomogeneity effects cause no trouble in our treatment, since each active atom "sees" two frequencies of the electromagnetic (em) field, whatever its transition frequency is. The problem becomes complicated when the inhomogeneity is introduced by the motion of atoms relative to the laboratory frame, where the modal frequencies are determined by the optical cavity. In this case, owing to the stationary character of the modes, an atom, with an axial velocity v, sees the two frequencies ω_1 and ω_2 split by Doppler effect: $\omega_1 \pm Kv$ and $\omega_2 \pm Kv$. Therefore a two-frequency problem becomes a four-frequency one. It appears that the problem of two standing modes interacting with a homogeneous medium which we describe in this paper, is analogous to the case of two running modes interacting with a Doppler-broadened medium. This latter situation has been described in great detail,[8-10] because of its relevance in saturation spectroscopy experiments.[11] Therefore it is pertinent to compare our results with those obtained in the quoted papers. In particular Haroche and Hartmann[9] have dealt with various effects that arise when a weak, nonsaturating beam interacts with a Doppler-broadened medium saturated by a strong oppositely-running wave. Our results have a fairly simple interpretation which parallels their discussion. The complete solution of our problem is given in terms of a continued fraction expansion. This expansion was applied to microwave spectroscopy problems by Autler and Townes,[12] and first applied to the semiclassical theory of lasers by Stenholm and Lamb.[13] Since then it received great attention by a number of authors,[14-20] who treated several problems connected with the interaction between a classical em field and a two-level atomic system. Recently it has been extended to treat multimode operation phenomena.[21]

One of the advantages of this expansion is that it gives convergent results for arbitrarily high-field intensities. Its lowest-order approximation makes it clear the features of the rate equations approach in the previous perturbative treatment.[22] Furthermore, the intrinsic limitations of the perturbation expansion are avoided and all the effects that arise in strong-signal operation can be described. The analysis has been limited to the gain interaction because the two-mode operation is phase independent and any frequency-pushing or -pulling effect has small influence. However these effects could easily be evaluated in terms of the in phase component of the polarization of the medium whose expression is given in Sec. III.

The paper is organized as follows: in Sec. II we briefly derive the basic formulas for the two mode operation; the inversion population density and the polarization of the medium are given in terms of a continued fraction expansion. This expression of the gain experienced by each mode is also derived and its physical significance is stressed out. In Sec. III the results obtained by numerical computation in the small saturation regime are discussed and a few graphs are reported in order to show the main phenomena which occur in that regime. Section IV is devoted to the discussion of the large saturation regime. When this regime is achieved, the beating of the two modes gives rise to the oscillation of the inversion population density and this in turn is responsible for a parametric coupling of the modes. Although this regime could be hardly achieved in a solid state laser, its features are relevant to the phase-locking mechanism in any kind of laser. Finally in Sec. V a comparison with previous results is made, stressing out the analogies of interpretation of only seemingly different problems.

II. FOURIER-EXPANSION ANALYSIS

The em field in the cavity is assumed to be made up of two longitudinal standing modes, whose electric field component is:

$$\vec{E}(\vec{r},t)=\vec{E}_1(\vec{r},t)\cos(\omega_1 t+\phi_1)\sin k_1 z + \vec{E}_2(\vec{r},t)\cos(\omega_2 t+\phi_2)\sin k_2 z\,. \tag{1}$$

The resonance condition $\sin(k_{1,2}L)=0$, where L is the cavity length, determines the possible values of $k_{1,2}=N_{1,2}\pi/L$, where $N_{1,2}$ are very large integers. The polarization of the electric field is assumed to be transverse to the z axis (longitudinal axis) of the optical cavity so that the x axis may be chosen to be along the field vectors. Furthermore, the two modes are assumed to be plane waves, so that the field amplitude depends only on z. In the stationary regime, the two field amplitudes are constant in space and in time, and accordingly are taken to be parameters in our treatment. The em field is coupled through an electric-dipole interaction with a stationary atomic system, whose levels $|a\rangle$ and $|b\rangle$ are separated by an energy $E=\hbar\omega=E_a-E_b$, which is close to the cavity resonant frequencies ω_1 and ω_2. Accordingly, the atomic medium is described as a two-level system, and its density matrix $\rho(z,t)$ obeys the equation of motion

$$\begin{aligned}\dot\rho_{aa}&=\lambda_a(z,t)-\gamma_a\rho_{aa}+iV(z,t)(\rho_{ab}-\rho_{ba})\,,\\ \dot\rho_{bb}&=\lambda_b(z,t)-\gamma_b\rho_{bb}+f\gamma_a\rho_{aa}-iV(z,t)(\rho_{ab}-\rho_{ba})\,,\\ \dot\rho_{ab}&=-\gamma_{ab}\rho_{ab}-i\omega\rho_{ab}+iV(z,t)(\rho_{aa}-\rho_{bb})\,,\\ \dot\rho_{ba}&=\dot\rho^*_{ab}\,.\end{aligned} \tag{2}$$

In these equations, λ_a and λ_b represent the source terms for the atomic-level population, and may be supposed slowly varying in a time in which the atomic variables vary appreciably. All the losses of the two levels a and b are lumped in the terms $\gamma_a\rho_{aa}$ and $\gamma_b\rho_{bb}$, respectively. If the lower-level b is the ground level of the system, $\gamma_b\rho_{bb}$ can account for the optical pumping to an upper level, and the branching ratio $f\simeq 1$. The coupling with em field is given by the term $V(z,t)$ which reads

$$V(z,t)=-(\vec{\mathcal{P}}/\hbar)\cdot\vec{E}(z,t)=-(\mathcal{P}_x/\hbar)E(z,t)\,, \tag{3}$$

where P is the dipole moment of the resonant atomic transition. The two frequencies of the em field are close to the (optical) atomic-transition frequency, therefore we may neglect the counter-rotating components in the coupling term of Eqs. (2). Let us put

$$\rho_{ab}=\rho_1 e^{-i\phi^+ t}, \tag{4}$$

where

$$\phi^+=\nu^+ t+\phi', \tag{5}$$

ν^+ is an optical frequency, and ϕ' is a phase term which is specified later on. We neglect all the components of ρ_1 which vary as $\exp(\pm i2\nu^+ t)$. This approximation is a very good one at optical frequencies and is referred to as the rotating-wave approximation (RWA). Substituting Eq. (4) for ρ_{ab} and ρ_{ba}, in Eqs. (2), we get

$$\begin{aligned}\mathring{\rho}_{aa}&=\lambda_a-\gamma_a\rho_{aa}-i[\rho_1\mathcal{E}(z,t)-\rho_1^*\mathcal{E}^*(z,t)],\\ \mathring{\rho}_{bb}&=\lambda_b-\gamma_b\rho_{bb}+f\gamma_a\rho_{aa}+i[\rho_1\mathcal{E}(z,t)-\rho_1^*\mathcal{E}^*(z,t)],\\ \mathring{\rho}_1&=-\gamma_{ab}\rho_1+i(\nu^++\phi'-\omega)\rho_1-i(\rho_{aa}-\rho_{bb})\mathcal{E}^*(z,t),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathcal{E}(z,t)=&\beta_1\sin(k_1z)\exp[i(\omega_1t+\phi_1-\phi^+)]\\ &+\beta_2\sin(k_2z)\exp[i(\omega_2t+\phi_2-\phi^+)]\end{aligned} \tag{7}$$

and

$$\beta_j=\mathcal{P}E_j/2\hbar\quad(j=1,2). \tag{8}$$

Because the atoms are stationary, the variable z is a parameter in ρ, i.e., Eqs. (6) give the time behavior of those atoms at position z, interacting with a field having amplitudes $E_1\sin(k_1z)$ and $E_2\sin(k_2z)$, respectively. This spatial dependence gives rise to phenomena which are discussed in the next sections. With a suitable choice of $\phi^+(t)$, we can give the function $\mathcal{E}(z,t)$ a simple time dependence. Let us put

$$\phi^+\equiv\tfrac{1}{2}(\omega_1+\omega_2)t+\tfrac{1}{2}(\phi_1+\phi_2). \tag{9}$$

Then

$$\mathcal{E}(z,t)=\beta_1\sin(k_1z)e^{i\phi^-}+\beta_2\sin(k_2z)e^{-i\phi^-}, \tag{10}$$

where

$$\phi^-=\tfrac{1}{2}(\omega_1-\omega_2)t+\tfrac{1}{2}(\phi_1-\phi_2). \tag{11}$$

A stationary solution of system (6), with the substitution (10), is readily obtainable if we allow $\dot{\phi}^\pm$ and E_1, E_2 to be constant in time. Then the four elements of the density matrix may be expanded in terms of a Fourier expansion, just because the coupling function $\mathcal{E}(z,t)$ is periodic in time, with a fundamental frequency given by $\dot{\phi}^-$. However, these Fourier expansions are still a good "physical" solution if the amplitudes E_1 and E_2, as well as the tuned frequency $\dot{\phi}^+$, vary little in a period $1/\dot{\phi}^-$. These solutions are therefore valid when the cavity modes have sharp resonances, i.e., when the modes decay in a time much longer than the inverse of their beating frequency. Otherwise, each mode loses its physical significance, and the description of the laser phenomena is better given in the time domain. Let us introduce the Fourier expansions:

$$\rho_{aa}(z,t)=\sum_m r_m^{(a)}(z)e^{im\phi^-}, \tag{12a}$$

$$\rho_{bb}(z,t)=\sum_m r_m^{(b)}(z)e^{im\phi^-}, \tag{12b}$$

$$\rho_1(z,t)=\sum_m p_m(z)e^{im\phi^-}. \tag{12c}$$

If we substitute Eqs. (12) in system (6), we find an algebric system for the Fourier components $r_m^{(a)}$, $r_m^{(b)}$, and p_m:

$$\begin{aligned}(im\dot{\phi}^-+\gamma_a)r_m^{(a)}=&\lambda_a\delta_{0,m}-i[\tilde{\beta}_1(p_{m-1}-p^*_{-m-1})\\ &+\tilde{\beta}_2(p_{m+1}-p^*_{-m+1})],\end{aligned} \tag{13a}$$

$$\begin{aligned}(im\dot{\phi}^-+\gamma_b)r_m^{(b)}=&\lambda_b\delta_{0,m}+\Gamma r_m^{(a)}+i[\tilde{\beta}_1(p_{m-1}-p^*_{-m-1})\\ &+\tilde{\beta}_2(p_{m+1}-p^*_{-m+1})],\end{aligned} \tag{13b}$$

$$\begin{aligned}&[i(m\dot{\phi}^--\phi^++\omega)+\gamma_{ab}]p_m\\ &\quad=-i[\tilde{\beta}_1(r_{m+1}^{(a)}-r_{m+1}^{(b)})+\tilde{\beta}_2(r_{m-1}^{(a)}-r_{m-1}^{(b)})],\end{aligned} \tag{13c}$$

where

$$\tilde{\beta}_{1,2}=\beta_{1,2}\sin(k_{1,2}z). \tag{14}$$

Eqs. (13) can be reduced to a single recurrence equation for the levels' population difference

$$n(z,t)=\rho_{aa}(z,t)-\rho_{bb}(z,t), \tag{15}$$

whose Fourier coefficients are given by

$$n_m(z)=r_m^{(a)}(z)-r_m^{(b)}(z). \tag{16}$$

Then the recurrence equation for n_m reads:

$$\begin{aligned}n_m=&N_0\delta_{0,m}-(\gamma_a+\gamma_b+2im\dot{\phi}^--\Gamma)/[(\tilde{\gamma}_{ab}+im\dot{\phi}^-)^2-\gamma^2]\\ &\times\{n_m[(L_{m-1}+L^*_{-m-1})\tilde{\beta}_1^2+(L_{m+1}+L^*_{-m+1})\tilde{\beta}_2^2]\\ &+n_{m+2}(L_{m+1}+L^*_{-m-1})\tilde{\beta}_1\tilde{\beta}_2\\ &+n_{m-2}(L_{m-1}+L^*_{-m+1})\tilde{\beta}_1\tilde{\beta}_2\},\end{aligned} \tag{17}$$

where

$$\tilde{\gamma}_{ab}=\tfrac{1}{2}(\gamma_a+\gamma_b), \tag{18a}$$

$$\gamma=\tfrac{1}{2}(\gamma_a-\gamma_b), \tag{18b}$$

and

$$L_m=[im\dot{\phi}^-+\gamma_{ab}-i(\dot{\phi}^+-\omega)]^{-1}, \tag{19}$$

$$N_0=\frac{\lambda_a}{\gamma_a}\left(1-\frac{\Gamma}{\gamma_b}\right)-\frac{\lambda_b}{\gamma_b} \tag{20}$$

is the stationary population difference in absence of any em field in the cavity. A simple physical interpretation of the recurrence relation (17) can be given: the Fourier coefficients n_m vanish for $m \neq 0$, if β_1 (or β_2) is zero, i.e., when there is only one mode oscillating in the cavity. In this case, no time modulation occurs in the population difference $n(z,t)$, and the steady value of $n(z)$ is readily obtained

$$n_0 = N_0 \left(1 + \frac{2\tilde{\gamma}_{ab} - \Gamma}{\gamma_a \gamma_b} \frac{2\gamma_{ab}}{\gamma_{ab}^2 + (\omega_1 - \omega)^2} \tilde{\beta}_1^2\right)^{-1}. \tag{21}$$

The steady state (21) is reached in a time $\tau = 1/\gamma'$,

$$\gamma'^{-1} = \gamma_a^{-1} + \gamma_b^{-1} = 2\tilde{\gamma}_{ab}/\gamma_a\gamma_b .$$

The validity of Eq. (21) is restricted to the cases in which the em field amplitudes and phases are constant or vary little in the time τ. In two mode operation, the product $\tilde{\beta}_1\tilde{\beta}_2$ does not vanish and a temporal dependence of the population difference arises. If the modes amplitudes do not vary appreciably in a time $1/\gamma'$, then a stationary regime is reached in which the population difference oscillates in time. The period of the oscillation can be readily found from Eqs. (12) and (17). Only even Fourier components of $n(z,t)$ appear in the expansion (12a) and (12b), [see Eq. (17): the odd components of n, namely n_{2m+1} form a homogeneous system whose unique solution $n_{2m+1} = 0$]. Therefore the period T of $n(z,t)$ is given by

$$T = 2\pi/2\dot{\phi}^- = 2\pi[\omega_1 - \omega_2 + (\dot{\phi}_1 - \dot{\phi}_2)]^{-1}. \tag{22}$$

As a matter of fact the difference $\dot{\phi}_1 - \dot{\phi}_2$ is much smaller than the frequency spacing of the modes. Hence we shall ignore this difference in the following. The rate equation approximation (REA) in two-mode operation (stationary regime) applies when we can ignore the time dependence of $n(z,t)$, i.e., when the coefficient n_0 is much greater than $|n_2|$. This condition yields

$$\frac{\gamma_a\gamma_b}{\tilde{\gamma}_{ab} - \frac{1}{2}\Gamma} + 4\gamma_{ab}\left(\frac{\tilde{\beta}_1^2}{\gamma_{ab}^2 + (\omega_1 - \omega)^2} + \frac{\tilde{\beta}_2^2}{\gamma_{ab}^2 + (\omega_2 - \omega)^2}\right) \gg 4\tilde{\beta}_1\tilde{\beta}_2 \left| \frac{\gamma_{ab} - \frac{1}{2}i\delta}{(\gamma_{ab} - \frac{1}{2}i\delta)^2 + [\frac{1}{2}(\omega_1 + \omega_2) - \omega]^2} \right| , \tag{23}$$

with

$$\delta = \omega_1 - \omega_2 . \tag{24}$$

Therefore in strong intensity regime the REA does not apply even in a stationary condition.

The solution of Eq. (17) can be expressed in terms of a continued fraction.[13,14] If we denote

$$\Omega_m = (\gamma_a + \gamma_b + im\delta - \Gamma)[(\tilde{\gamma}_{ab} + \tfrac{1}{2}im\delta)^2 - \gamma^2]^{-1}, \tag{25}$$

$$A_m = \Omega_m \tilde{\beta}_1\tilde{\beta}_2(L_{m+1} + L^*_{-m-1}), \tag{26a}$$

$$B_m = \Omega_m[(L_{m-1} + L^*_{-m-1})\tilde{\beta}_1^2 + (L_{m+1} + L^*_{-m+1})\tilde{\beta}_2^2], \tag{26b}$$

$$C_m = \Omega_m \tilde{\beta}_1\tilde{\beta}_2(L_{m-1} + L^*_{-m+1}), \tag{26c}$$

then the average-population difference is given by

$$n_0 = N_0(1 + B_0 + A_0 n_2/n_0 + C_0 n_{-2}/n_0)^{-1}, \tag{27}$$

where

$$\frac{n_2}{n_0} = -\cfrac{C_2}{1 + B_2 - \cfrac{A_2C_4}{1 + B_4 - \cfrac{A_4C_6}{1 + B_6 - \cdots}}} \tag{28}$$

and

$$n_{-2}/n_0 = (n_2/n_0)^* . \tag{29}$$

The evaluation of the population density at the stationary regime allows us to determine the polarization of the medium. The latter quantity is given by

$$P(z,t) = \mathrm{Tr}(\hat{\mathcal{P}}\rho) = \mathcal{P}_x[\rho_{ab} + \rho_{ba}]. \tag{30}$$

Making use of (13c) and (12c), we get

$$P(z,t) = p_x[-i\Sigma_k L_k(\tilde{\beta}_1 n_{k+1} + \tilde{\beta} n_{k-1})e^{i(k\phi^- - \phi^+)} + \text{c.c}]. \tag{31}$$

Actually we need only to evaluate the polarization oscillating at the frequencies of the two modes: i.e., we require that

$$k\phi^- - \phi^+ = -\tfrac{1}{2}[(\omega_1 t + \phi_1)(1-k) + (\omega_2 t + \phi_2)(1+k)] = -(\omega_1 t + \phi_1) \tag{32a}$$

or

$$k\phi^- - \phi^+ = -(\omega_2 t + \phi_2). \tag{32b}$$

The former relation if fulfilled by $k = -1$, the latter by $k = 1$. Therefore only two terms in the summation (31) are relevant. The only population difference components which are necessary in the evaluation of the relevant part of $P(z,t)$ are n_0 and $n_{\pm 2}$, which in turn can be derived from Eqs. (27)–(29). We have, therefore,

$$P_1(z,t) = -i\mathcal{P}_x[\gamma_{ab} - i(\omega_1 + \dot{\phi}_1 - \omega)]^{-1}(\tilde{\beta}_1 n_0 + \tilde{\beta}_2 n_{-2}) \times e^{-i(\omega_1 t + \phi_1)} + \text{c.c.}, \tag{33a}$$

$$P_2(z,t) = -i\mathcal{P}_x[\gamma_{ab} - i(\omega_2 + \dot{\phi}_2 - \omega)]^{-1}(\tilde{\beta}_1 n_2 + \tilde{\beta}_2 n_0) \times e^{-i(\omega_2 t + \phi_2)} + \text{c.c.} \tag{33b}$$

The in-phase and the out-of-phase components of the polarization determine the dispersion and the absorption (or gain) of the mode under investigation. For example, the average power per unit length absorbed by the medium from the em field at frequency ω_1 is given by

$$G_1(z) = \langle \dot{P}(z,t) E_1 \cos(\omega_1 t + \phi_1) \sin(k_1 z) \rangle$$
$$= -2\hbar\omega \left(\mathrm{Re} \frac{\tilde{\beta}_1^2 n_0}{\gamma_{ab} - i(\omega_1 + \dot{\phi}_1 - \omega)} + \mathrm{Re} \frac{\tilde{\beta}_1 \tilde{\beta}_2 n_{-2}}{\gamma_{ab} - i(\omega_1 + \dot{\phi}_1 - \omega)} \right) \quad (34)$$
$$= G_{11} + G_{12} .$$

In the weak-field limit, $n_0 \sim N_0$ and $|n_{-2}| \ll n_0$, we get the familiar absorption law

$$G_1(z) \simeq -2\hbar\omega N_0 \gamma_{ab} \tilde{\beta}_1^2 / [\gamma_{ab}^2 + (\omega_1 + \dot{\phi}_1 - \omega)^2]. \quad (35)$$

A net absorption is achieved if $G_1 > 0$, i.e., when $N_0 < 0$ (the lower level is more populated than the upper one). Otherwise, $G_1 < 0$ and the em field gains power from the medium.

We note that the absorption (or gain) per unit length depends on z through $\tilde{\beta}_1$ and $\tilde{\beta}_2$, as a consequence of the stationary spatial pattern of the em field. In a running-wave configuration, the field peaks sweep all the cavity, and the z dependence disappears. The total absorption or gain of the mode can be evaluated by calculating the integral of $G_1(z)$ over the active medium length.

In general, gain (or absorption) of each mode has two contributions, as indicated in Eq. (34). In the limit that only one mode saturates the gain function we can give these two terms a simple interpretation. G_{11} represents that part of the gain which the mode oscillating at frequency ω_1 experiences directly through interaction with the inverted medium. However the medium itself is saturated by all the modes oscillating in the cavity and by this approach the whole effect of saturation is taken into account in the steady value of the population difference n_0 as shown by Eq. (27).

The second term G_{12} has a different origin: the nonlinear interaction between the two modes gives rise to an oscillation of the population difference at frequencies $n|\omega_1 - \omega_2|$ (being $n = \pm 1, \pm 2, \ldots$). This oscillation in turn is coupled with the polarization at frequency ω_2, giving rise to a component of the polarization at ω_1, which acts as a source for the mode oscillating at the same frequency. When both modes saturate the gain function, these two effects become intertwined, and saturation couples strongly the two oscillating modes.

In the following we consider two different cases. In the first one the intensity of the two modes is strong enough that the effect of the saturation of the inverted medium is important, but the pulsation difference can still be neglected as a coupling mechanism. This range of operation is referred to as weak-coupling regime (WCR).

In the second case the intensity in the cavity is such that the term G_{12} becomes relevant and the coupling of the modes through the pulsation of the population difference plays an important role. This operation range is referred to as strong-coupling regime (SCR).

III. WEAK-COUPLING REGIME

In this section we describe the two mode operation in the weak-coupling regime. Here, pulsations of population difference are not effective to induce a direct coupling between the modes. As a matter of fact, this regime is certainly achieved when the pulsations are vanishingly small, where the REA applies.

Although the continued fraction approach allows for an exact evaluation of n_0, the main features of this regime, i.e., spatial inhomogeneity effects, or saturating properties of the modes, come out already in the REA, where pulsations are neglected completely.

The condition of applicability of the REA has already been given in (23). At this point we stress a remarkable difference between the present approach and the approach of perturbative expansion in ascending powers of the field amplitudes. In the latter cases it is not possible to make a sharp distinction between terms coming from saturation and terms coming from pulsation of the population difference. Indeed, at third order in the field amplitudes, the coefficient of $\beta_1 \beta_2^2$ include a contribution coming from G_{11} and a contribution coming from G_{12}. Such a distinction is feasible by the continued fraction method. For instance, let us consider a two-mode operation in a ruby laser, with $\gamma_{ab} = 1.8 \times 10^{11}$ sec^{-1}, $\gamma_a = 0.3 \times 10^3$ sec^{-1}, and a frequency spacing $\delta = 0.16\gamma_{ab}$. As is shown below, spatial inhomogeneity allows for two mode operation, and the two modes can have comparable intensities. As a matter of fact, even with an intercavity energy flux of 1 Kw/cm^{-2}, for which the perturbation expansion breaks down, effects of saturation are relevant, while the effects of pulsations are still negligible.

As is well known 4(b) in the REA, the gain of the mode frequency ω_1 is given by

$$G_1(z) = 2\hbar\omega \tilde{\beta}_1^2(z) n_0 \,\mathrm{Re}[\gamma_{ab} - i(\omega_1 - \omega)]^{-1}, \quad (36)$$

and the stationary inversion density is

$$n_0(z) = N_0 \left[1 + 4 \frac{\tilde{\gamma}_{ab}}{\gamma_a \gamma_b} \left(\tilde{\beta}_1^2(z) \frac{\gamma_{ab}}{\gamma_{ab}^2 + (\omega_1 - \omega)^2} + \tilde{\beta}_2^2(z) \frac{\gamma_{ab}}{\gamma_{ab}^2 + (\omega_2 - \omega)^2} \right) \right]^{-1}. \quad (37)$$

The mode at frequency ω_2 experiences a gain whose analytical expression is the same as (36), with the obvious substitutions $\tilde{\beta}_1 \rightleftharpoons \tilde{\beta}_2$, $\omega_1 \rightleftharpoons \omega_2$. We have considered a stationary condition for the phases, i.e., $\dot{\phi}_1 = \dot{\phi}_2 = 0$.

In order to shown the relevance of spatial inhomogeneity, we compare the results obtained both by neglecting and considering the different spatial patterns of the standing waves. The first case is simply achieved by setting $k_1 = k_2$. The total gain is obtained by integration of Eq. (36) over the active medium length.

This integration yields[23]:

$$G_1 = \int_{L_1}^{L_2} G_1(z)\,dz = \frac{\hbar\omega}{2}\left(\frac{\mathcal{P}_x E_1}{\hbar}\right)^2 \frac{\gamma_{ab}}{\gamma_{ab}^2 + (\omega_1 - \omega)^2} \times N_0 \frac{L_2 - L_1}{\alpha}\left(1 - \frac{1}{\sqrt{1+\alpha}}\right), \quad (38)$$

where $L_2 - L_1$ is the length of the active medium, and α is given by

$$\alpha = \frac{\mathcal{P}_x^2}{\hbar^2}\frac{\tilde{\gamma}_{ab}\gamma_{ab}}{\gamma_a\gamma_b}\left(\frac{E_1^2}{\gamma_{ab}^2 + (\omega_1 - \omega)^2} + \frac{E_2^2}{\gamma_{ab}^2 + (\omega_2 - \omega)^2}\right). \quad (39)$$

When the saturating fields are sufficiently high, the total gain $G_1 + G_2$ becomes independent of the field amplitude as well as of the frequency. The sum of the two gain factors gives the maximum amount of stored power which can be "extracted" from the medium. It depends only on the medium length and on the inversion population density. A simple argument shows that, in this situation of complete homogeneity of the gain line, a regime can be achieved where only one mode oscillates. Let us denote by Λ the losses of the two modes. The equations of motion for the field intensities are simply

$$\dot{I}_1 = KG_1 - \Lambda I_1, \quad (40a)$$

$$\dot{I}_2 = KG_2 - \Lambda I_2, \quad (40b)$$

where K is a constant and $I = (\beta/\gamma_{ab})^2$. A stationary regime cannot be achieved as the condition

$$\dot{I}_1 = \dot{I}_2 = 0 \quad (41)$$

would imply

$$G_1/I_1 = G_2/I_2. \quad (42)$$

Equation (38) shows that (42) cannot be satisfied if $\omega_1 \neq \omega_2$. Therefore, if the spatial dependence of the two standing modes is neglected, two or more modes cannot oscillate simultaneously because the mode closer to the resonance condition $\omega_1 = \omega$ saturate the line of gain. Let us consider now the real situation, where $k_1 \neq k_2$, so that the spatial pattern of the standing modes is taken into account. Here the integration of G_1 and G_2 is performed by computer.

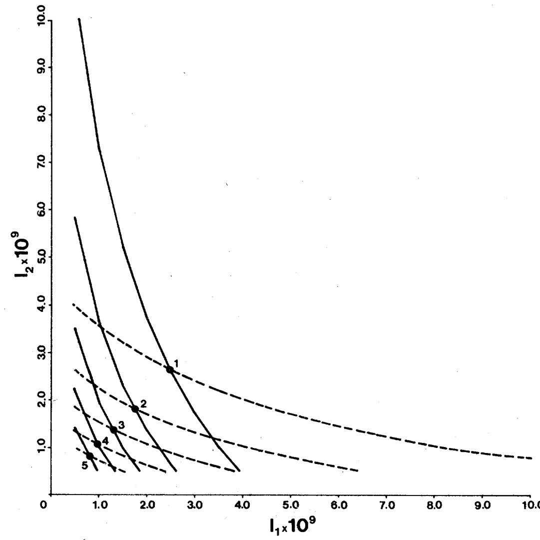

FIG. 1. Maps of $G_1(I_1, I_2)/I_1 = \Lambda_i/K$ (solid lines) and $G_2(I_1, I_2)/I_2 = \Lambda_i/K$ (dashed lines) are reported for difference values of Λ_i/K in the plane (I_1, I_2) in the WCR. 0.31, 0.41, 0.51, 0.62, 0.72 are the values of Λ_i/K corresponding to the curves intersecting in the points from 1–5, respectively. The ruby parameters at room temperature were chosen. Namely $\gamma_{ab} = 1.8 \times 10^{11}$ sec^{-1}, $\gamma_b = 0.3 \times 10^3$ sec^{-1}, $\gamma_b = 0.4 \times 10^3$ sec^{-1}. The frequency of mode 2 is equal to the resonant frequency ω of ruby material; $\omega = 2.71 \times 10^{15}$ rad sec^{-1}. The frequency of mode 1 is $\omega_1 = \omega + 0.16\gamma_{ab}$.

The maps of $G_1/I_1 = \text{const}$ and $G_2/I_2 = \text{const}$ are reported in the plane I_1, I_2 of Figs. 1 and 2, for two values of ω_1. The other frequency ω_2 is, in both cases, equal to the resonant frequency. A two-mode stationary regime is achieved in the

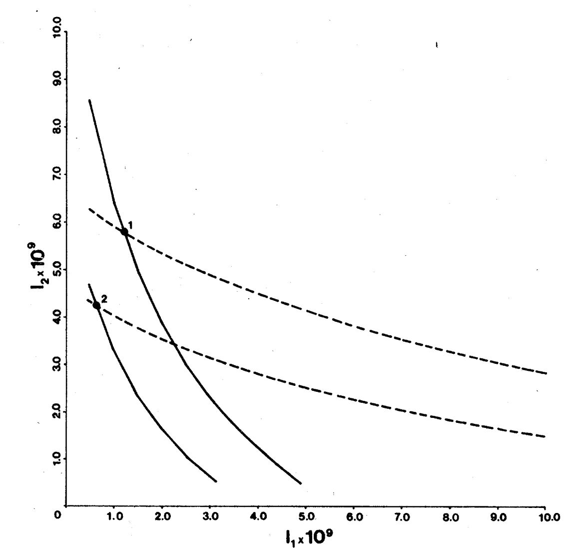

FIG. 2. Same maps of Fig. 1 are reported for different values of Λ_i/K (0.23 and 0.30 for 1 and 2, respectively) and for $\omega_1 = \omega + 0.84\gamma_{ab}$.

intersection point of these curves, where $G_1/I_1 = G_2/I_2$ [Eq. (42)]. The two curves intersect for both values of the frequency ω_1, showing that the spatial inhomogeneity effects are of relevance over the whole curve of gain. The curves reported in Figs. 1 and 2 are obtained considering the active medium at the center of the cavity, so that the two standing mode patterns do not overlap appreciably in the medium, and the two modes can gain from different regions.

The position of the medium enhances the effects of spatial inhomogeneity when the frequencies of the modes are very close to each other. In fact, the difference between k_1 and k_2 is multiple of π/L

$$k_2 - k_1 = n(\pi/L)\,, \tag{43}$$

therefore the two-mode patterns differ by the amount

$$|\sin^2(k_1 z) - \sin^2(k_2 z)| = |\sin(k_1+k_2)z| \times |\sin n\pi/Lz|\,. \tag{44}$$

Apart from the rapidly varying factor $\sin(k_1+k_2)z$ the difference reaches its maximum when

$$n(\pi/L)z = \tfrac{1}{2}\pi,\ \tfrac{3}{2}\pi, \ldots \tag{45}$$

and is very small in the regions of the active medium where

$$n(\pi/L)z = 0,\ \pi,\ 2\pi, \ldots\ . \tag{46}$$

At the center of the cavity $z \simeq \frac{1}{2}L$. Therefore the largest difference is achieved when n is an odd number.

Figure 3(a) shows the behavior of the curves for $n=1$, and Fig. 3(b) for $n=2$. According to (44) the spatial inhomogeneity is much more effective in the former case than in the latter one. If we had drawn the curves for the homogeneous case, Eq. (38), then we would have obtained two parallel straight lines with no crossing point.

From the analysis of the graphs one can draw the conclusion that the oscillation at the crossing point is stable.

Let us consider a single diagram of oscillation, reported in Fig. 4, corresponding to a value Λ' for the losses. If the two modes have intensities different from the stationary ones (point A in Fig. 4), then $\dot{I}_1$ and $\dot{I}_2$ are not equal to zero and can be evaluated by Eqs. (40a) and (40b). As is apparent from the graphs of Figs. 1 and 2, in the point A the value of KG_1/I_1 is greater than Λ', while the values of KG_2/I_2 is smaller. Therefore the increment of I_1 is positive, while the increment of I_2 is negative, so that the oscillation point gets closer to C (point A′). Briefly, any small displacement from the oscillation regime C causes a restoring force which pushes back the oscillating system to

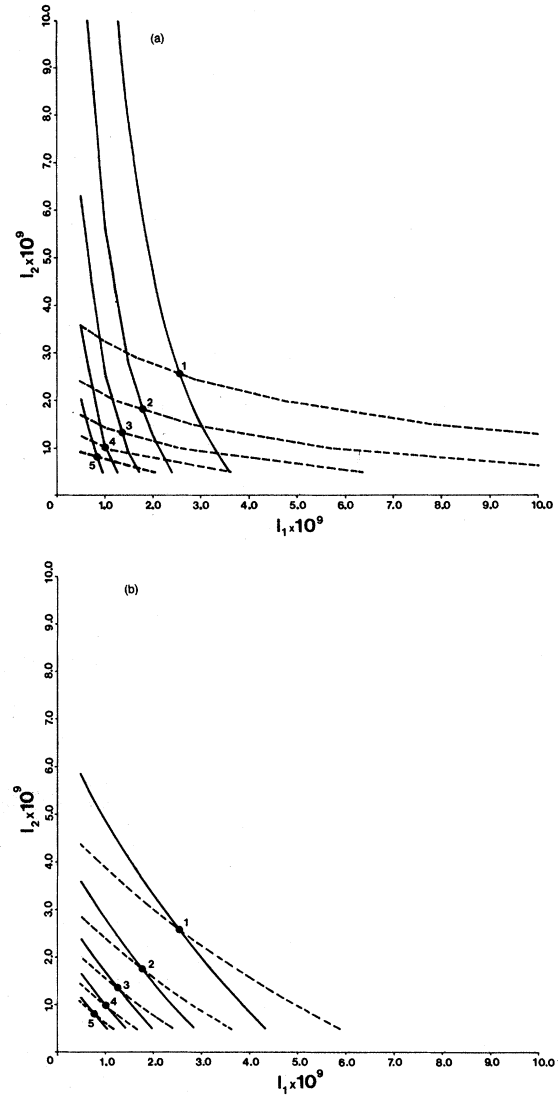

FIG. 3. Different crossing of G_1/I_1 and G_2/I_2 is stressed out when the difference of the wave vectors of the two modes is (a) an odd multiple of π/L ($n=1$), or (b) an even multiple of π/L ($n=2$).

the crossing point C.

We have restricted ourselves to a situation where the active atoms are stationary, and the stored energy does not diffuse. Diffusion effects may be of importance at high-intensity regimes. A description of these effects as well as their measurements in some lasers can be found in Ref. 25.

The mode interaction considered can be suitably described in terms of scattering of both traveling waves, which make up one standing mode on the Bragg diffraction pattern generated in the medium by the other standing mode.[26] However, this description becomes somewhat complicated when the

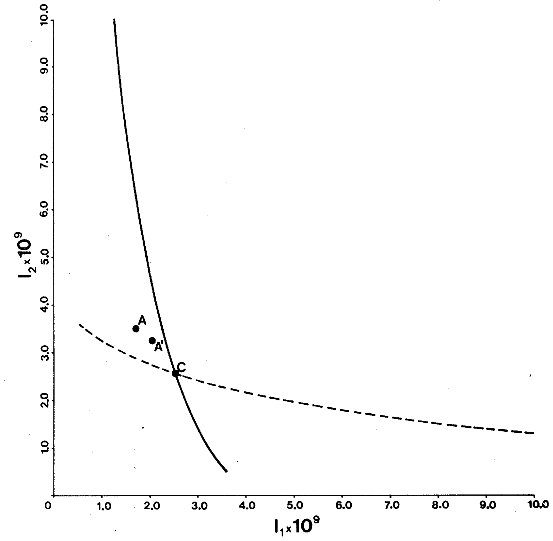

FIG. 4. Stability diagram for two-mode oscillation. The two curves refer to $G_1/I_1 = G_2/I_2 =$ const. The value chosen for the constant is arbitrary. The gradient of both the functions $G_i(I_1, I_2)/I_i$ points towards the left. Starting from A the oscillating point moves towards C. The same argument applies for any point of the phase plane, so that the oscillation at C turns out to be stable.

intensities are high enough that a perturbative expansion of Eq. (37) is inadequate to evaluate the inversion population density; we have therefore chosen to discuss the interaction phenomena in the I_1, I_2 plane by means of the phase curves. We end this section by drawing some conclusions. In the WCR the two modes interact through the saturation of the medium. However, a small variation in the intensity of the second mode has little influence on the gain of the first mode. This can be shown by using Eq. (38), although in this formula the spatial inhomogeneity is not taken into account. In the limit $\alpha \gg 1$, this formula reads

$$G_1 = r_1 I_1/(1 + \alpha_1 I_1 + \alpha_2 I_2), \tag{47}$$

where r_1, α_1, and α_2 can be found by inspection. Then

$$\left|\frac{\partial G_1}{\partial I_2}\right| = G_1 \frac{\alpha_2}{1 + \alpha_1 I_1 + \alpha_2 I_2}, \tag{48}$$

i.e., the variation of G_1 is a small fraction of the gain itself if the mode 1 is strong enough. Therefore, if a regime is reached in which the saturation of the inverted population dominates then the multimode operation (allowed by any kind of inhomogeneity) is stable and any variation in time of each mode has little influence on the others. Moreover, this interaction is phase independent, because the saturation of the medium occurs through the intensity of each mode. Therefore no constraint can exist which is capable of locking the phases of the modes. In Sec. IV we deal with another kind of interaction, which arises from the pulsation of the medium population difference.

IV. STRONG-COUPLING REGIME

In strong-field operation, one cannot ignore pulsations in the population difference, and the full expression (34) for the gain must be taken into account. The strong-coupling regime (SCR) is achieved when the Fourier component $|n_2|$ of the population difference becomes comparable with the static component n_0, i.e.,

$$|n_2/n_0| \simeq 1. \tag{49}$$

For most solid-state lasers, two oscillating modes are separated by a frequency δ much larger than the damping rate γ_{ab} of the population inversion. Therefore, when the two modes are close to the atomic transition frequency ω, (i.e., when $|\omega_1 - \omega|$, $|\omega_2 - \omega| \ll \gamma_{ab}$), the pulsations in the population appear for em field amplitudes sufficiently strong to compensate the frequency detuning. When these conditions are fulfilled Eq. (49) yields

$$4\tilde{\beta}_1\tilde{\beta}_2/\gamma_{ab}^2 \gtrsim |\omega_1 - \omega_2|/\gamma_{ab}, \tag{50}$$

i.e., the product of the Rabi frequencies of the two modes must be of the same order as the detuning of the two modes.

The relationship (50) involves the field amplitudes and comes out directly from the continued fraction approach.

The above reported limit and consequently the SCR cannot be described by a theory based on a series expansion in terms of powers of the em field amplitudes, since the latter diverges for high-intensity fields.

A new kind of interaction arises when the two modes are so close, or their amplitudes are so strong that the frequency shifts associated with the Rabi "flopping" are of the same order as the detuning of the two modes. As pointed out by Haroche and Hartmann,[9] the Rabi frequency shift, while absent in the stationary single mode operation, manifests itself in the nonlinear interaction between the two modes. An interesting interpretation of this phenomenon in terms of nonlinear amplification of a single amplitude-modulated em field has been given in the same paper.[9] In our case, two modes with the same amplitude may be considered as a single mode, with frequency $\frac{1}{2}(\omega_1 + \omega_2)$, whose amplitude is 100% modulated with a modulation frequency $\frac{1}{2}|\omega_1 - \omega_2|$, and the discussion of Haroche and Hartmann applies. We only note two differences in our case: first, the two modes are approximately of the same strength, and therefore both their Rabi frequencies appear in Eq. (50). Second, the local amplitudes [modified by the spatial pattern term $\sin(k_{\bar{1},2}z)$] are involved in Eq. (50), i.e., this effect is not uniform in the whole interaction volume.

The phenomenon has a resonance when $\omega_1 \sim \omega_2$ within the level's radiative width $\tilde{\gamma}_{ab}$. This resonance is a second-order effect, and its shape is very complicated, but in the limit $\tilde{\gamma}_{ab} \ll \gamma_{ab}$ it is dispersionlike. In fact, from the second term in the gain function (34) we have, at second order,

$$\mathrm{Re}\left(\frac{1}{\gamma_{ab}+i(\omega_1-\omega)}\,\frac{n_2}{n_0}\right) = -\mathrm{Re}\left[2\tilde{\beta}_1\tilde{\beta}_2\,\frac{1}{\gamma_{ab}+i(\omega_1-\omega)}\,\frac{1}{\tilde{\gamma}_{ab}+i(\omega_1-\omega_2)} \times\left(\frac{1}{\gamma_{ab}+i(\omega_1-\omega)}+\frac{1}{\gamma_{ab}+i(\omega_2-\omega)}\right)\right], \qquad (51)$$

Then in the range $\tilde{\gamma}_{ab} < |\omega_1 - \omega_2| \ll \gamma_{ab}$ the gain changes its sign passing from $\omega_2 < \omega_1$ to $\omega_1 < \omega_2$. Furthermore, the gain transfer is lower in the range $\omega_1 \sim \omega_2 \sim \omega$, while increases when ω_1 and ω_2 are detuned from the transition frequency ω. A photon is absorbed from the mode 2 and emitted in mode 1, or vice versa, and the net energy difference $\hbar(\omega_1 - \omega_2)$ is transferred from the medium to the em fields (or vice versa). The appearance of a dispersive part in the gain is shown in Fig. (5). Here the gain of mode 2, averaged over the whole length of the active medium, versus its frequency, which is swept across the frequency of mode 1, is plotted. In this graph the amplitudes of the two modes are kept constant at a value at which these phenomena are remarkable, although a value sufficiently weak so that the expression (51) for the second order gain applies.

We want to point out that a dispersive shape of the line of gain comes out also from apparently different physical situations. As is known a small signal can be amplified when passing through an absorbing medium in presence of a strong em field due to the nonlinear interaction arising with the temperature fluctuations. A theoretical analysis [27] of the process predicts a maximum amplification of the signal when its frequency is shifted by the amount $\nu = \Gamma_R/4$ from the frequency of the strong em field. Here, Γ_R is the lifetime of the temperature fluctuations. A symmetrical maximum of attenuation is also obtained at a frequency shift $\nu = -\Gamma_R/4\pi$, and the whole loss-gain curve displays the characteristic dispersionlike-shape around $\nu = 0$, in the steady-state regime. Experiments were performed to measure the gain function in the Rayleigh region of the spectrum,[28] but, due to the short duration of the laser pulses in comparison with the temperature-fluctuations lifetime, a stationary regime could not be reached, so that only a change of sign of amplification has been detected passing through $\nu = 0$. Another feature of interest in the SCR is that the stimulated scattering term, which gives rise to the direct coupling between the two modes, may be much greater than the emissive term when the two modes are far enough from the center of the atomic transition. It follows that a second-order laser emission occurs when a strong radiation field interacts with a sample of nonresonant atoms. This is the physical situation in the free electron laser (FEL), where the stimulated scattering provides

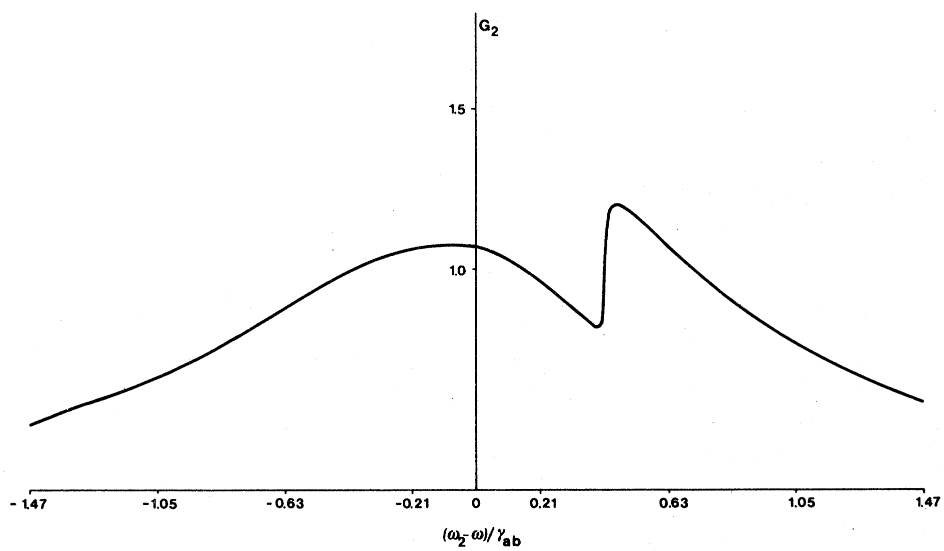

FIG. 5. Gain function G_2 (in arbitrary units) is reported versus the frequency of mode 2. The frequency of mode 1 is kept constant ($\omega_1 = \omega + 0.42\gamma_{ab}$). The intensities of the two modes are high enough to ensure the operation in the SCR. The values of $\omega_2 - \omega$ is reported on the x axis in units of γ_{ab}.

the source of gain (no absorption or emission of radiation occurs from a free electron). The basic process of the FEL turns out to be a Thomson (or Compton) scattering rather than a Rayleigh scattering, just because no bound state of the scatterer exists.[29]

If the gain for the two modes is provided mostly by the same part of the active medium, then, as already noted, the crossing term in the gain is greatly influenced by the overlapping of the spatial patterns of the two modes.

As an example, we show in Fig. 6 how the crossing term in the gain modifies the phase curves for the two-modes operation. We have noted earlier that in the WCR two modes cannot operate in a completely homogeneous line of gain, i.e., when $k_1 = k_2$. But, even in this case, the crossing term can induce two modes to oscillate simultaneously [Fig. 6(a)]. However, as can easily be shown using the same considerations as in Sec. III the oscillating regime, which occurs at the crossing point C, turns out to be unstable.

When the difference in the spatial patterns of the two modes is taken into account, the oscillation regime moves to stability. In Fig. 6(b) the phase curves of the two modes are reported, for the same oscillation frequencies ω_1 and ω_2, but with different k. It is worth comparing the graphs in this figure with those of Fig. 1, which have been obtained with the same parameters, but where the crossing term in the gain was ineffective. Figures 7(a) and 7(b) correspond to Figs. 3(a) and 3(b). In the SCR case shown in Fig. 7 the cross term of the gain introduces strong deviations in the phase curves. In particular the eight-shaped graph of Fig. 7(b), obtained by choosing $n = 2$, is of significance. Three oscillation regimes are obtained, the central being unstable, [corresponding to the stable but critical oscillation of Fig. 3(b)], while the others are stable. These two stable oscillations occur in a nearly symmetric way (owing to the large curve of gain, the two modes' frequencies are in a region of nearly equal unsaturated gain). In each oscillation point, one mode has a strong intensity and the other one a much lower intensity. Passing from WCR to SCR, the phase curves are distorted: the direct transfer of energy from one mode to the other gives the characteristic eight shape to the phase curves.

Although the phases of the modes do not appear in (34), a simple argument shows that this interaction mechanism is phase dependent when more than two modes are simultaneously oscillating. We have chosen the Fourier expansion (12) in order to eliminate any phase dependence in the Fourier coefficients n_k. This comes into evidence in the recurrence relation (17) for n_k, in which the

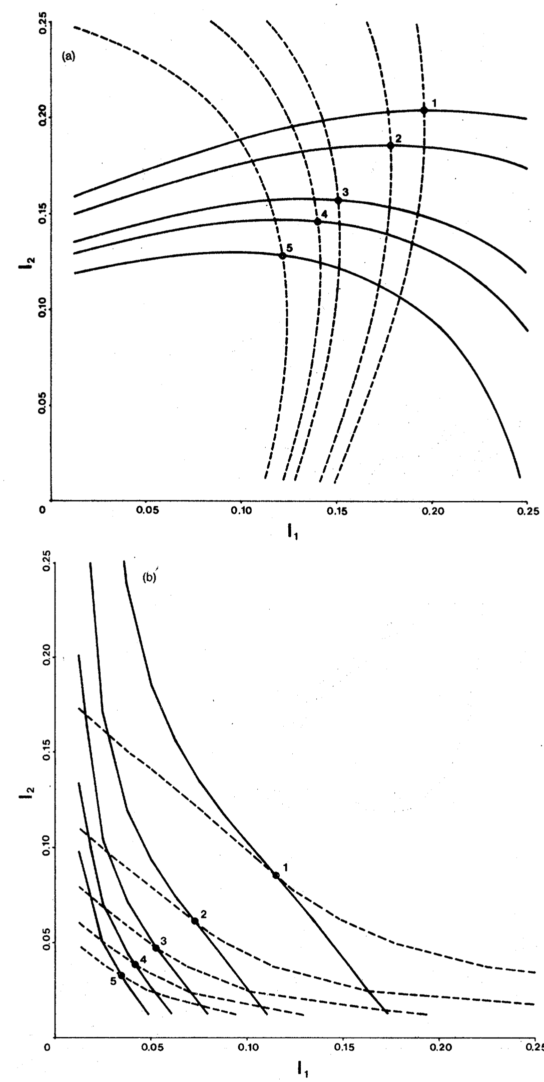

FIG. 6. Effect of spatial inhomogeneity is stressed out in the SCR. (a) The inhomogeneity is dropped out by evaluating the maps of G_1/I_1 (solid lines) and G_2/I_2 (dashed lines) setting $k_1 = k_2$. The oscillation at the crossing points is unstable. (b) The introduction of spatial inhomogeneity changes the intersection so that the oscillations result stable. A part from the value of Λ/K all the parameters are the same used to get the maps of Fig. 1.

phases of the two oscillating modes appear explicitly. Fourier expansions other than (12) could also have been chosen, and in that case the Fourier coefficients n_k would depend on the phases of the modes. However, the coupling term in the gain relation (34), namely

$$G_{12} = -2\hbar\omega \,\mathrm{Re}\left(\frac{\vec{\beta}_1 \vec{\beta}_2 n_{-2}}{\gamma_{ab} - i(\omega_1 - \omega)}\right) \tag{52}$$

is invariant for any choice of the Fourier expan-

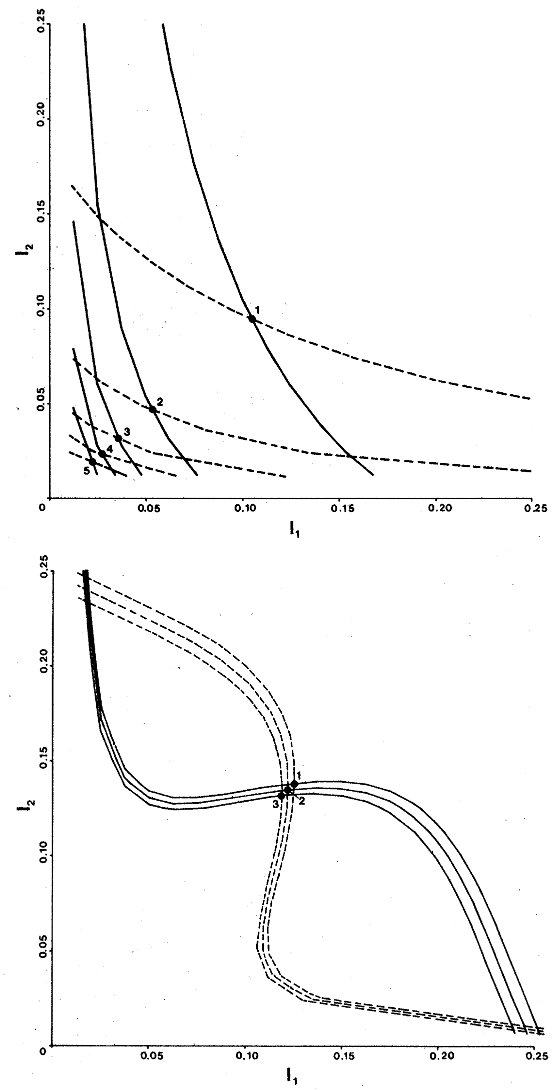

FIG. 7. The graphs in the SCR corresponding to Figs. 3(a) and 3(b) are reported. The effect of spatial competition is stressed out by the appearance of a "eight-shaped" form where $n=2$. The marked intersection points correspond to unstable oscillations. Stable oscillations are also possible, but with very different values of the intensities of the two modes.

sions; therefore if a phase factor appears in n_{-2}

$$n_{-2} \rightarrow e^{i\phi} n_{-2}, \tag{53}$$

an extra factor $e^{-i\phi}$ must balance it in (52). But these considerations are valid when only two oscillating modes are present. We conclude that the coupling terms G_{jl} are in general phase dependent. Therefore, at high-intensity operation, a coupling mechanism arises which scatters energy from one mode into another one, and this mechanism in phase dependent. We want to stress out the difference between the intensity dependent terms [as given by (47)] and the coupling terms arising in the SCR. The former ones are independent of the relative phases of the oscillating modes while the latter ones are quite sensitive to these phase relations and are able to provide a high rate of energy transfer among the modes.

V. COMPARISON WITH PREVIOUS THEORIES

As we have already noted in the introduction, our two-frequency (standing waves) problem for an array of stationary atoms is quite analogous to the two-frequency (running-waves) problem for moving atoms. Therefore, expression (27) is similar to that obtained by Menegozzi and Lamb[18] (see Eq. (6.19) in their paper), although some modifications must be introduced in order to allow for different physical situations. Moreover, one can show that the first term of the continued fraction expansion (34) is equivalent with the gain formula as given by Haroche and Hartmann[9] [see Eq. (45) of their paper], although it is obtained by a different method. The problem itself is of relevance in saturation spectroscopy,[30] but here we are not concerned with that subject.

The influence of the modal spatial patterns on the oscillations in a multimode operation has been stressed out by Statz and Tang.[24] This effect vanishes when diffusion of the excited states turns out to be significant.[25] However, the Statz and Tang third-order theory does not apply when the modes' intensities are strong so that the series expansion is invalid, a situation which frequently occurs in solid-state laser operation. Our formulas are valid in any range of intensities, but they have the disadvantage that the integration of the gain function over the finite length of the active medium cannot be performed analytically, owing to the complicate z dependence of the continued fraction expansion. A numerical integration of the gain function must be performed.

An interesting interpretation of the interaction among the modes in a solid-state laser has been given by Sargent.[26] As has been pointed out by Kogelnik and Shank,[31] a periodic index of refraction or gain scatters a running wave into another running in the opposite direction. In the scattering process, therefore, energy can be transferred from one standing mode (which is made up of two running waves) to another mode. In the stationary regime described in the present paper, the two running waves which constitute each standing mode have equal amplitudes and the scattered energy from each of them to the counter running wave of the other mode are equal. The scattering process as a whole is therefore contained in the self-gain,

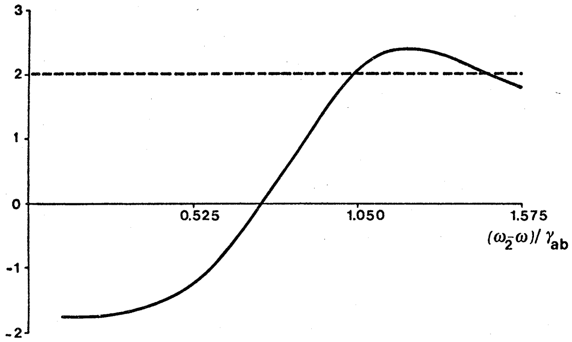

FIG. 8. The separate contributions to the gain function of one small amplitude mode in presence of a second strong mode versus the frequency shift with respect to the center of the line of gain are reported. The intensities of the two modes are $I_1 = 1$ and $I_2 = 10^{-10}$. The frequency of the strong mode is $\omega_1 = -\omega_2$. The dashed and solid lines represent the self- and cross-gain, respectively. We have dropped out the spatial in homogeneity effects by setting $k_1 = k_2$. In such a way the analogy with the problem treated by Haroche and Hartmann is closer.

formula (36) of this paper.

The cross term in the gain, Eq. (52), arises as a parametric coupling between the two modes and the inversion population density, oscillating at a frequency $\omega_1 - \omega_2$. Therefore the above mentioned mechanism can be interpreted as a parametric amplification of one mode in presence of a second one. It is noteworthy that this "second-order" gain may exceed by several order of magnitude the self gain if the two modes are close to each other, but far enough from the center of the line of gain. In this connection, the transfer of energy from one mode to the other can be interpreted as a Rayleigh scattering. At the lowest order of its appearance, its width is the radiative width of the levels involved in the transition, i.e., $\gamma_{ab} = \frac{1}{2}(\gamma_a + \gamma_b)$. The appearance of a nonlinear Rayleigh process has been pointed out also by Haroche and Hartmann.[9]

Another feature of interest is the gain behavior of a small amplitude mode in presence of a strong mode when the frequencies of the modes are symmetrically tuned with respect to the center of the line. This is the analogous of the problem discussed by Haroche and Hartmann, who considered two running, waves with the same resonant frequency ω_0, but oppositely directed wave vectors, interacting with a set of atoms moving with an axial velocity v. In the moving reference frame the two running waves have frequencies $\omega_0 + kv$ and $\omega_0 - kv$, respectively. We have reported on the graph of Fig. 8 the self-gain and the crossing term of the small amplitude mode, as a function of its detuning from the line center. The strong mode's frequency is symmetrically swept on the other side of the line gain. It appears that the self-gain is nearly constant in this frequency range $|\omega - \omega_0| \ll \gamma_{ab}$, while the shape of the crossing term displays the dynamical Stark effect as a peak in the gain, at a frequency linearly dependent on the amplitude of the strong mode's electric field. This effect has been extensively discussed in the paper by Haroche and Hartmann.

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