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Radiation-induced modification of the atomic momentum distribution in a traveling-wave resonant light field

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A quantum-mechanical transport equation of the Boltzmann type is proposed for the momentum-distribution function of atoms in the field of a quasiresonant traveling light wave. The physical meaning of the solution is discussed in terms of the linear momentum transferred from the photons to the atoms through a succession of photon-scattering processes, the number of which follows a Poisson law. A connection is established with the random-flight problem. The analytical expression of the radiation-modified atomic-momentum-distribution function is derived explicitly. The first and second moments of the distribution correspond to a drift (associated with an average force) and a smearing out (associated with an anisotropic diffusion tensor), respectively. The latter effect is shown to arise from the dispersion in both the number and the direction of the scattered photons.

I. INTRODUCTION

In most previous studies (see, e.g., Refs. 1–3), the description of the mechanical effect exerted on atoms (or molecules) by a traveling-wave, homogeneous, resonant light field has been limited to an effective force accounting for the average linear momentum transferred from the photons to the atoms. In this paper, we are interested in the spread of the atomic momenta around this average value. Up to now, this problem has been treated as a diffusion phenomenon leading to a Gaussian approximation (defined by two moments) of the atomic-momentum-distribution function.[4,5] In the present work, I derive an analytical expression which describes completely the shape (i.e., which yields all the moments) of the momentum-distribution function of the atoms in the presence of a radiation field. This is obtained as the solution of a transport equation valid both for low- and high-intensity light fields. The transport equation is established on the basis of a fully quantum-mechanical treatment, including the translational motion of the atom's center of mass, of the atom-radiation interaction.

II. TRANSPORT EQUATION IN MOMENTUM SPACE

A. Theoretical framework

We consider the interaction of a moving atom (or molecule) with quasimonochromatic traveling radiation in the vicinity of optical resonance. The Hamiltonian of the total system "atom plus field" is written as the sum

$$H = H_a + H_f + V, \tag{1}$$

where

$$H_a = H_e + \vec{\mathcal{P}}^2/2m \tag{1a}$$

is the free-atom Hamiltonian (which involves the Hamiltonian of the electron and the Hamiltonian of the center-of-mass motion),

$$H_f = \sum_k \hbar\omega_k a_k^\dagger a_k \tag{1b}$$

is the free-field Hamiltonian, and

$$V = -\vec{D}\cdot\vec{E}(\vec{r}) \tag{1c}$$

is the atom-field interaction Hamiltonian (in the electric-dipole approximation). The atom is described as a two-level system with an excited level a and a ground level b, both nondegenerate, separated by the energy difference $\hbar\omega_0$. Since we are especially interested in the recoil effects in the interaction with radiation, we have to quantify the external degrees of freedom of the atom. We denote by $\vec{r}$ and $\vec{p}$ the eigenvalues (in the Schrödinger and momentum representation, respectively) of the operators associated with the position and linear momentum of the atom's center of mass. The laser radiation is assumed to excite only the mode of momentum $\vec{k}_L$ and energy $\hbar\omega_L$ of the quantized electromagnetic field. For high laser intensities, the relative dispersion of the number of photons n_L in this mode is weak. The strength of the atom-laser field coupling is characterized by the Rabi frequency Ω_L. We assume that the atom-vacuum field interaction can create at most one photon in the other modes, labeled $\vec{k}$, of the electromagnetic field. Owing to this interaction, the atoms decay by spontaneous emission from level a to level b with a rate $\gamma = \tau^{-1}$, where τ is the radiative lifetime of the excited state.

We now introduce a representative ensemble of such atoms, described at the time $t=0$ at which the interaction takes place by the momentum-space distribution function $f_0(\vec{p})$. Within the above theoretical framework, I show in another paper[6] that the momentum-space distribution function $f(\vec{p}, t)$ of this ensemble at time t is a solution of the Boltzmann-type transport equation

$$\frac{\partial}{\partial t} f(\vec{p}, t) = -f(\vec{p}, t)\int W(\vec{p}\to\vec{p}\,')d^3p' + \int f(\vec{p}\,', t)W(\vec{p}\,'\to\vec{p})d^3p' , \tag{2}$$

with the initial condition $f(\vec{p}, t=0) = f_0(\vec{p})$. Here, the kernel $W(\vec{p}\to\vec{p}\,')$ represents the probability density per unit time to have a change of the atomic momentum from $\vec{p}$ to $\vec{p}\,'$ caused by interaction with radiation. The kernel may be put in the form

$$W(\vec{p}\to\vec{p}\,') = \gamma\chi(\vec{p})w(\vec{p}\,'-\vec{p}) , \tag{2a}$$

where the dimensionless resonance factor

$$\chi(\vec{p}) = \Omega_L^2/[(\omega_L-\omega_0-\vec{k}_L\cdot\vec{p}/m)^2+\tfrac{1}{4}\gamma^2+2\Omega_L^2] \tag{2b}$$

yields its average magnitude and the function

$$w(\vec{p}\,'-\vec{p}) = \int u(\hbar\vec{k})\delta(\vec{p}\,'-\vec{p}-\hbar\vec{k}_L+\hbar\vec{k})\hbar^3d^3k \tag{2c}$$

contains its angular properties. The latter is obtained by a translation $\hbar\vec{k}_L$ of the normalized isotropic density function defined by $u(\hbar\vec{k}) = 1/4\pi(\hbar k_L)^2$ over the sphere of radius $\hbar|\vec{k}| = \hbar k_L$, and $u(\hbar\vec{k}) = 0$ elsewhere. The integral over the final momenta

$$\int W(\vec{p}\to\vec{p}\,')d^3p' = \gamma\chi(\vec{p}) = \frac{1}{T(\vec{p})} \tag{3}$$

gives the probability per unit time that an atom of momentum $\vec{p}$ undergoes a momentum change.

B. Brief derivation of transport equation

Although Eq. (2) is quite intuitive, it is of interest to outline briefly how it can be established in order to gain some physical insight into the problem and to display the underlying assumptions. We first consider the low-field limit $\Omega_L \ll \gamma$. A complete basis of the physical space is generated by the tensorial product of the eigenvectors of the uncoupled Hamiltonians H_a and H_f, which we label $|\alpha\vec{p}n_L\vec{k}_Lm\vec{k}\rangle$, with $\alpha = a, b$ and $m = 0, 1$. By using general scattering theory (see, e.g., Ref. 7) and limiting ourselves to the lowest-order resonant photon scattering amplitude, we obtain the probability per unit time (for times such that $\gamma t \gg 1$) for the transition from the state $|b\,\vec{p}n_L\vec{k}_L0\rangle$ to the state $|b\,\vec{p}\,'(n_L-1)\vec{k}_L\vec{k}\rangle$:

$$S(\vec{p}\vec{k}_L\to\vec{p}\,'\vec{k}) = \frac{2\pi}{\hbar}\,\delta\!\left(\frac{\vec{p}\,'^2}{2m}+\hbar\omega_k-\frac{\vec{p}^2}{2m}-\hbar\omega_L\right)\left|\sum_{\vec{p}\,''}\frac{\langle b\vec{p}\,'(n_L-1)\vec{k}_L\vec{k}|V|a\vec{p}\,''(n_L-1)\vec{k}_L0\rangle\langle a\vec{p}\,''(n_L-1)\vec{k}_L0|V|b\vec{p}n_L\vec{k}_L0\rangle}{\vec{p}\,''^2/2m+\hbar\omega_0-\vec{p}^2/2m-\hbar\omega_L+i\frac{1}{2}\gamma}\right|^2 . \tag{4}$$

In order to derive the atom-reduced transition probability $W(\vec{p}\to\vec{p}\,')$, one has to sum this expression over the field variables n_L and $\vec{k}$. The two matrix elements contained in (4) involve the Dirac functions $\delta(\vec{p}\,'-\vec{p}\,''+\hbar\vec{k})$ and $\delta(\vec{p}\,''-\vec{p}-\hbar\vec{k}_L)$, respectively, which together lead to the momentum-conservation condition appearing in (2c). The form of the density function $u(\hbar\vec{k})$ of (2c) results from the energy-conservation condition $\delta(\vec{p}\,'^2/2m+\hbar ck-\vec{p}^2/2m-\hbar ck_L)$, which reduces to $k = k_L$ if we neglect the term $(\vec{k}_L-\vec{k})\cdot\vec{p}/mc$ of order v/c. When the summation is completed, the two matrix elements of (4) give rise to terms proportional to γ and Ω_L^2, respectively. If we neglect the recoil energy of the center of mass, $(\hbar\vec{k}_L)^2/2m$, in the energy denominator of (4), we recover the expression (2b) (the saturation term $2\Omega_L^2$ excepted).

Note that the momentum- and energy-conservation conditions apply to the entire photon scattering process. The same results can be obtained by extending the Wigner-Weisskopf theory of resonance fluorescence (see, e.g., Ref. 8) to the case where the atomic motion is quantized.[9] As emphasized by Heitler,[8] in the case of incoming monochromatic radiation one cannot separate the photon scattering process into absorption and spontaneous emission processes. During the scattering process, one therefore cannot determine whether the atom is in the ground state with momentum $\vec{p}$ or in the excited state with momentum $\vec{p}+\hbar\vec{k}_L$.

I show in the forthcoming paper[6] that the set of results (2) is also valid in the high-field limit $\Omega_L \gg \gamma$ [the saturation term of (2b) is now obtained, but not the natural-width term]. Here, it is convenient to use the "dressed-atom" approach[10] and extend it by treating the atomic motion quantum mechanically. Equation (2) follows from the fact that, under the secular approximation, the populations of the eigenlevels of the dressed atom (system "atom plus laser field") are coupled by rate equations describing the transfers due to interaction with the vacuum field. The eigenstates of the Hamiltonian of the dressed atom are linear superpositions of the eigenstates $|a(\vec{p}+\vec{k}_L)\times(n_L-1)\rangle$ and $|b\vec{p}n_L\rangle$ of the uncoupled Hamiltonians. Since the momentum operator $\vec{\mathcal{P}}$ does not commute with the atom-laser field interaction Hamiltonian, the momentum is not a "good quantum number" in this representation. Thus, we again have an indetermination of order $\hbar\vec{k}_L$ in the atomic momentum $\vec{p}$. However, owing to the uncertainty relation between $\vec{p}$ and $\vec{r}$, this is not very restrictive: in practice, already before the interaction with the field, one cannot specify completely the momentum of an individual atom.

III. ATOMIC-MOMENTUM-DISTRIBUTION FUNCTION

A. Solution of transport equation

I give in the following a simple solution, which requires only limited approximations, of the quantum-mechanical transport equation (2). At the start of the interaction, the wave function associated with the motion of the free atoms can be written in the Schrödinger representation as the superposition of plane waves:

$$\Psi(\vec{r},t=0)=(2\pi\hbar)^{-3/2}\int\tilde{\Psi}(\vec{p},t=0)\exp\left(i\frac{\vec{p}\cdot\vec{r}}{\hbar}\right)d^3p\,. \quad (5)$$

We suppose that the corresponding momentum-distribution function of the atoms $f_0(\vec{p})=|\tilde{\Psi}(\vec{p},t=0)|^2$ is a wave packet of small extension (of the order of $\hbar k_L$) centered on the momentum $\vec{p}_0$:

$$f_0(\vec{p})=g_0(\vec{p}-\vec{p}_0)\,. \quad (6)$$

For these atoms we then set

$$T(\vec{p})=T(\vec{p}_0)\,. \quad (7)$$

The approximation (7) amounts to neglecting the progressive Doppler detuning of the laser radiation induced by successive atomic recoils.[11] This is possible if the total momentum $\vec{P}$ acquired by an atom while interacting with the light satisfies $m^{-1}\vec{k}_L\cdot\vec{P}\ll\gamma$ or $m^{-1}\vec{k}_L\cdot\vec{P}\ll\Omega_L$. With this assumption, Eq. (2) transforms into the new one

$$\frac{\partial}{\partial t}f(\vec{p},t)=-\frac{1}{T(\vec{p}_0)}f(\vec{p},t)+\frac{1}{T(\vec{p}_0)}\int f(\vec{p}\,',t)w(\vec{p}-\vec{p}\,')d^3p'\,, \quad (8)$$

the Fourier transform of which is easily taken:

$$\frac{\partial}{\partial t}\tilde{f}(\vec{\alpha},t)=-\frac{1}{T(\vec{p}_0)}\tilde{f}(\vec{\alpha},t)+\frac{1}{T(\vec{p}_0)}\tilde{f}(\vec{\alpha},t)\tilde{w}(\vec{\alpha})\,. \quad (9)$$

Here, we have used notations of the type

$$\tilde{\varphi}(\vec{\alpha})=\int\varphi(\vec{p})e^{i\vec{\alpha}\cdot\vec{p}}d^3p\,. \quad (10)$$

The solution of Eq. (9) is readily obtained:

$$\tilde{f}(\vec{\alpha},t)=\tilde{f}(\vec{\alpha},0)\exp\left(-\frac{t}{T(\vec{p}_0)}[1-\tilde{w}(\vec{\alpha})]\right), \quad (11)$$

where

$$\tilde{f}(\vec{\alpha},0)=\int f(\vec{p},0)e^{i\vec{\alpha}\cdot\vec{p}}d^3p=\tilde{f}_0(\vec{\alpha}) \quad (12)$$

is the Fourier transform of the initial momentum-distribution function $f_0(\vec{p})$. Expression (11) may be rewritten in the form

$$\tilde{f}(\vec{\alpha},t)=\sum_{n=0}^{\infty}e^{-[t/T(\vec{p}_0)]}\frac{[t/T(\vec{p}_0)]^n}{n!}[\tilde{w}(\vec{\alpha})]^n\tilde{f}_0(\vec{\alpha})\,. \quad (13)$$

Finally, the inverse Fourier transform yields the radiation-modified wave packet at time t as a function of the initial wave packet:

$$f(\vec{p},t)=g(\vec{p}-\vec{p}_0,t)=\sum_{n=0}^{\infty}\Pi_n(\vec{p}_0,t)\times\int g_0(\vec{p}\,'-\vec{p}_0)w_n(\vec{p}-\vec{p}\,')d^3p'\,, \quad (14)$$

where

$$\Pi_n(\vec{p}_0,t)=e^{-[t/T(\vec{p}_0)]}\frac{[t/T(\vec{p}_0)]^n}{n!} \quad (14a)$$

is the Poisson distribution with parameter $t/T(\vec{p}_0)$, and $w_n(\vec{p}-\vec{p}\,')$ is the n-fold convolution with itself of the function

$$w_1(\vec{p}-\vec{p}\,')=w(\vec{p}-\vec{p}\,') \quad (14b)$$

for $n\geq 1$ [w is the function defined by (2c)], and reduces to

$$w_0(\vec{p}-\vec{p}\,')=\delta(\vec{p}-\vec{p}\,') \quad (14c)$$

for $n=0$ (δ is to be understood as the Dirac function).

In most practical cases, one shall have to allow for a distribution function $h(\vec{p}_0)$ of the centers of the wave packets associated with each atom at $t=0$ [for example, for a gas at thermal equilibrium, $h(\vec{p}_0)$ will be the Maxwell distribution function]. In such problems, the overall momentum-distribution function $F(\vec{p},t)$ at time t will be obtained by averaging expression (14) with respect to $\vec{p}_0$:

$$F(\vec{p},t)=\int g(\vec{p}-\vec{p}_0,t)h(\vec{p}_0)d^3p_0 \quad (15)$$

(the dependence of T on $\vec{p}_0$ should of course be accounted for in the integration).

If we consider now the limiting case where

$$g_0(\vec{p}-\vec{p}_0)=\delta(\vec{p}-\vec{p}_0)\,, \quad (16)$$

expression (14) reduces to

$$f(\vec{p},t)=g(\vec{p}-\vec{p}_0,t)=\sum_{n=0}^{\infty}\Pi_n(\vec{p}_0,t)w_n(\vec{p}-\vec{p}_0)\,. \qquad (17)$$

This describes the radiation-induced distribution of the momenta originating from a well-determined initial momentum $\vec{p}_0$, i.e., the atomic wave packet arising from the scattering of an atomic plane wave by the laser radiation.

B. Physical discussion

The physical meaning of the solution (17) is quite clear. On the one hand, if we denote by $N(t)$ the number of photon scattering processes which occur for an atom interacting with radiation from time 0 to time t, the Poisson distribution $\Pi_n(t)$ gives the probability for observing $N(t)=n$. This result was also obtained in other ways by Pusep[4] and by Cohen-Tannoudji and Reynaud.[10] For a phenomenon governed by a probability per unit time T^{-1}, the Poisson law with parameter t/T could be inferred *a priori* under the hypothesis of stochastic independence of nonoverlapping time intervals [in the present problem, this is true for time intervals large with respect to the correlation time ($\tau_c \sim\omega_0^{-1}\ll\Gamma$) of the vacuum fluctuations]. We recall that, for the Poisson distribution (14a), the mean $\bar{n}$ and variance σ^2 take the same value t/T.

On the other hand, the function $w_n(\vec{P})$ represents the probability density for the atom undergoing a momentum variation $\vec{P}=\vec{p}-\vec{p}_0$ after n photon scattering processes. The fact that $w_n(\vec{P})$ is the n-fold convolution of $w(\vec{p})$ [Eq. (14b)] results from the independence of the successive events. In the case $n=0$, which has the probability $e^{-\bar{n}}$, no momentum change can happen, in accordance with (14c).

In the language of mathematical probability theory (see, e.g., Ref. 12), the density function (17) is that of a random variable which is the sum of n mutually independent, continuous random variables having a common distribution, where n is in turn a discrete random variable having a Poisson distribution.

C. Explicit form of momentum distribution

The solution (17) is valid independently of the angular symmetry of the distribution $w(\vec{P})$, i.e., of the distribution $u(\vec{P})$. I now give its explicit form in the case of an isotropic distribution $u(\vec{P})$.[13] In the Appendix, I study the problem of the distribution of the length of the random vector $\vec{P}$ and relate it to the classical problem of random flights. However, the distribution of the vector $\vec{P}$ is completely determined by the one-dimensional distribution of its projection P_x on an arbitrary axis x:

$$\bar{f}(P_x,t)=\sum_{n=0}^{\infty}\Pi_n(\vec{p}_0,t)\bar{w}_n(P_x)\,, \qquad (18)$$

where $\bar{w}_n(P_x)$ is $\delta(P_x)$ for $n=0$, and is the n-fold convolution of the function $\bar{w}(P_x)=\bar{u}(P_x-\hbar k_{Lx})$ for $n\geq 1$. Since

$$\bar{w}_n(P_x)=\bar{u}_n(P_x-n\hbar k_{Lx})\,, \qquad (19)$$

we discuss first the properties of the distribution $\bar{u}_n(P_x)$.

The distribution of the projection P_x of a vector of fixed length $\hbar k_L$ and isotropically distributed random direction in three-dimensional space is the rectangular density such that $\bar{u}(P_x)=(2\hbar k_L)^{-1}$ over the interval $[-k_L,+k_L]$ and $\bar{u}(P_x)=0$ elsewhere. The distribution $u_2(P_x)$ is a triangular density. One can show by recurrence[12(a)] that the n-fold convolution of $u(P_x)$ may be written in the form

$$\bar{u}_n(P_x)=\frac{1}{(2\hbar k_L)^n(n-1)!}\times\sum_{\nu=0}^{n}(-1)^{\nu}C_{\nu}^{n}[P_x+(n-2\nu)\hbar k_L]_+^{n-1}\,. \qquad (20)$$

Here, C_ν^n is the binomial coefficient, and I have introduced the notation defined by $(X)_+=0$ for $X<0$ and $(X)_+=X$ for $X\geq 0$ [expression (20) remains valid for $n=1$, with the convention that $(X)_+^0=0$ for $X<0$ and $(X)_+^0=1$ for $X\geq 0$]. Note that the rectangular function $\bar{u}(P_x)$ may also be written as the well-known discontinuity integral of Dirichlet

$$\bar{u}(P_x)=\frac{1}{\pi}\int\frac{\sin\alpha\hbar k_L}{2\alpha\hbar k_L}\,e^{i\alpha P_x}\,d\alpha. \qquad (21)$$

An alternative form to (20) (which may be useful for some calculations) is therefore the Fourier transform

$$\bar{u}_n(P_x)=\frac{1}{\pi}\int\left(\frac{\sin\alpha\hbar k_L}{2\alpha\hbar k_L}\right)^n e^{i\alpha P_x}\,d\alpha\,. \qquad (22)$$

Some graphs of the distribution $\bar{u}_n(P_x)$ are shown in Fig. 1. The distribution is different from zero only in the interval $[-n\hbar k_L, n\hbar k_L]$. Like $\bar{u}(P_x)$, it is symmetrical with respect to its zero mean value. Its variance $\frac{1}{3}n(\hbar k_L)^2$ is n times the variance of $\bar{u}(P_x)$.

To get the expression of $\bar{w}_n(P_x)$, we merely have to replace P_x by $P_x-n\hbar k_{Lx}$ in expression (20) or (22). The translated distribution $\bar{w}_n(P_x)$ has the mean

$$\langle P_x\rangle_n=n\hbar k_{Lx}\,, \qquad (23)$$

and it keeps the variance

$$\langle P_x^2\rangle_n-\langle P_x\rangle_n^2=\tfrac{1}{3}n(\hbar k_L)^2\,. \qquad (24)$$

On the basis of the central-limit theorem,[12(b)] it is equivalent as $n\to\infty$ to the Gaussian (normal) distribution having the same mean and the same variance:

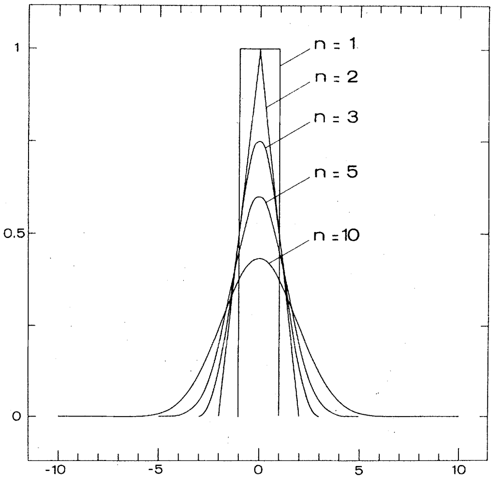

FIG. 1. Computer plots, for different n values, of the n-fold convolution of the centered uniform density over the interval $[-\hbar k_L, \hbar k_L]$. The horizontal and vertical axes are graduated in units of $\hbar k_L$ and $(2\hbar k_L)^{-1}$, respectively. If one denotes by $\vec{P}$ the momentum acquired by an atom through the scattering of n photons with momentum $\hbar\vec{k}_L$, the diagrams of the distribution $\bar{w}_n(P_x)$ of the projection of $\vec{P}$ are deduced from the above by a translation $n\hbar k_{Lx}$ parallel to the horizontal axis, while those of the distribution $\hat{w}_n(P)$ of the length of $\vec{P}$ are obtained by a translation $n\hbar k_L$.

$$\bar{w}_n(P_x) \sim \frac{1}{(\frac{2}{3}\pi n)^{1/2}\hbar k_L} \exp\left(-\frac{(P_x - n\hbar k_{Lx})^2}{\frac{2}{3}n(\hbar k_L)^2}\right). \quad (25)$$

Unlike $\bar{w}_n(P_x)$, the resultant distribution $\bar{f}(P_x, t)$ [Eq. (18)] is in general not symmetrical,[14] since it consists of a superposition, with weights governed by the Poisson law, of successive distributions $\bar{w}_n(P_x)$ having increasingly larger widths and centers displaced from each other (except for $k_{Lx} = 0$). Its moments are easily deduced from those of the former distributions by the relation

$$\langle P_x^m\rangle = \sum_{n=0}^{\infty} \Pi_n(\vec{p}_0, t)\langle P_x^m\rangle_n. \quad (26)$$

D. Average force and diffusion tensor

Let us investigate in particular the first two moments $m = 1, 2$ of the distribution $\bar{f}(P_x)$. The mean is simply

$$\langle P_x\rangle = \bar{n}\hbar k_{Lx}. \quad (27)$$

Through the relation

$$F_x = \langle P_x\rangle / t, \quad (28)$$

the effective resonant force originally introduced by Ashkin[1] is regained:

$$\vec{F} = \hbar\vec{k}_L / T(\vec{p}_0) = \gamma\hbar\vec{k}_L \frac{\Omega_L^2}{(\omega_L - \omega_0 - \vec{k}_L\cdot\vec{p}_0/m)^2 + \frac{1}{4}\gamma^2 + 2\Omega_L^2}. \quad (29)$$

The variance of the distribution is given by

$$\langle P_x^2\rangle - \langle P_x\rangle^2 = \tfrac{1}{3}\bar{n}(\hbar k_L)^2 + \sigma^2(\hbar k_{Lx})^2 = \bar{n}[\tfrac{1}{3}(\hbar k_L)^2 + (\hbar k_{Lx})^2]. \quad (30)$$

The first and second terms on the right-hand part of Eq. (30) represent the contributions to the variance of the random character of the direction of the momentum of the scattered photons, and of the uncertainty in the number of the scattered photons, respectively. While the first effect introduces an isotropic dispersion (with a spherical symmetry) of the atomic momenta, the second introduces an anisotropic dispersion (with a cylindrical symmetry around the direction of the propagation vector $\vec{k}_L$ of the laser radiation). One can associate with the variance (30) an effective diffusion coefficient (in fact the component of an anisotropic diffusion tensor having a principal axis along $\vec{k}_L$) defined by

$$2D_{xx}t = \langle P_x^2\rangle - \langle P_x\rangle^2. \quad (31)$$

This diffusion coefficient also has a resonant behavior:

$$D_{xx} = \frac{1}{2T(\vec{p}_0)}[\tfrac{1}{3}(\hbar k_L)^2 + (\hbar k_{Lx})^2] = \tfrac{1}{2}\gamma[\tfrac{1}{3}(\hbar k_L)^2 + (\hbar k_{Lx})^2] \times \frac{\Omega_L^2}{(\omega_L - \omega_0 - k_L\cdot\vec{p}_0/m)^2 + \frac{1}{4}\gamma^2 + 2\Omega_L^2}. \quad (32)$$

Rényi's generalization[15] of the central-limit theorem to sums of a random number of random variables can be used to prove that the distribution (18) is equivalent as $\bar{n} = t/T \to \infty$ to the Gaussian distribution:

$$\bar{f}(P_x, t) \sim (4\pi D_{xx}t)^{-1/2}\exp[-(P_x - F_x t)^2/4D_{xx}t]. \quad (33)$$

This is the approximation which has been made implicitly by Pusep.[4] It is also equivalent to the approximation of Baklanov and Dubetskii,[5] which amounts in fact to the use of the Fokker-Planck equation derived[6] from the above Boltzmann equation by expanding it to second order in $\hbar\vec{k}_L$ (however, these authors did not obtain the effect of the dispersion over n). Finally, I emphasize the fact that, for large values of $\bar{n}$, the asymmetrical distribution $\bar{f}(P_x, t)$ does not converge as rapidly to its Gaussian limit as $\bar{w}_n(P_x)$ does for large values of n. Of course, the Gaussian approximation does not hold at all for small values of $\bar{n}$ [for instance, it cannot yield the obvious singularity $\delta(P_x)$ of the distribution].

IV. CONCLUSION

The present study of the transport phenomena in momentum space accompanying the resonant interaction with radiation can be applied directly to the calculation of the photodeflection of an atomic beam,[1] for arbitrary relative directions of the laser beam and atomic beam,[14] or to the calculation of the photocooling of an atomic vapor.[16] It can be extended to the study of the associated effects in the conjugated position space (e.g., for the calculation of isotope separation of gases by radiation pressure[17]). Both spaces can be treated simultaneously by generalization of the transport equation (2) to the quantum phase-space distribution function $f(\vec{r},\vec{p},t)$.[6] This work can also serve as the basis for studying the problem of a standing-wave light field,[18] at least in the cases of low intensity or large detuning where the interaction may be treated as the sum of the interactions with two independent counter-propagating traveling waves.

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APPENDIX: CONNECTION WITH RANDOM-FLIGHT PROBLEM

If we were simply dealing with spontaneous emission processes, our problem would be exactly the well-known problem of random flights involving n successive steps of equal length $\hbar k_L$ in three-dimensional space. Following Feller,[12(c)] the distribution of the length P of the vector $\vec{P}$ after n steps ($n>2$) would then be

$$\hat{u}_n(P)=\frac{-P}{(2\hbar k_L)^n(n-2)!}\times\sum_{\nu=0}^{n}(-1)^\nu C_\nu^n[P+(n-2\nu)\hbar k_L]_+^{n-2} \qquad (34)$$

[(34) is minus the derivative of (20)]. This problem was originally investigated by Rayleigh. The standard reference given by physicists is to Chandrasekhar,[19] who, however, only derived the Fourier transform of the distribution (34).

However, in the present problem, we are dealing with photon scattering processes, and the distribution (34) has no physical reality. The momentum imparted to the atom at each process is the sum of a well-determined vector $\hbar\vec{k}_L$ and an isotropically distributed random vector of length $\hbar k_L$. In this case, one can show that the length of the resultant vector is uniformly distributed over the interval $[0, 2\hbar k_L]$. The distribution of the length P of the total momentum $\vec{P}$ acquired by the atom after n scattering processes is therefore

$$\hat{w}_n(P)=\frac{1}{(2\hbar k_L)^n(n-1)!}\sum_{\nu=0}^{n}(-1)^\nu C_\nu^n[P-2\nu\hbar k_L]_+^{n-1} \qquad (35)$$

[except for a translation $n\hbar k_L$, the graphs of $\hat{w}_n(P)$ are also given by Fig. 1].

In order to make the difference between the above two problems clear, the following geometrical representation may be of some use. In the first problem, the vector associated with one step joins the center of a sphere of radius $\hbar k_L$ to a random point on the surface of the sphere. In the second problem, the vector associated with one step joins a fixed point on the surface of a sphere of radius $\hbar k_L$ to a random point on the same surface.

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[13]Although the distribution $u(\vec{P})$ is necessarily even [i. e., such that $u(\vec{P})=u(-\vec{P})$], it is not necessarily isotropic. For example, in the case of circularly polarized light and an atomic transition between levels with angular momenta $J_b=0$ and $J_a=1$, the distribution $u(\vec{P})$ would have a cylindrical symmetry around the direction of the propagation vector $\vec{k}_L$ [although the radiation still connects only two (Zeeman) levels]. In such a case, one must in fact consider not only the conservation of linear momentum in the photon scatter-

ing process, but also the conservation of angular momentum.

[14]In a more specialized paper (unpublished), I shall plot the graphs of $\bar{f}(P_x, t)$ for some values of the parameters $\bar{n}$ and k_{Lx}. In fact, this paper will be devoted to the calculation of atomic-beam shapes modified by radiation pressure. Under typical experimental conditions, the distribution of the position x of the atoms at the detector is identical to the distribution of the projection P_x of their momentum at the end of the illuminated region. The two geometries that have already been used experimentally [J. L. Picqué and J. L. Vialle, Opt. Commun. 5, 402 (1972)] will be especially investigated. They correspond to the cases $k_{Lx} = k_L$, which allow the observation of the more efficient average deflection of the atomic beam, and $k_{Lx} = 0$, which allow the observation of the only broadening effect.

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