Quantum aspects of classical and statistical fields

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(Received 27 September 1977; revised manuscript received 22 June 1978)

A generating functional for a classical system described by coordinates satisfying nonlinear equations of motion is constructed in terms of which any functional of the system's phase-space trajectory may be expressed. Statistical behavior of the system arising from either random initial conditions or random stirring forces may be handled simply within this theory. The analogy with Feynman's action-integral formalism of quantum theory provides an alternative approach to the operator methods recently developed by Martin, Siggia, and Rose. The connection with the earlier work of Onsager, Machlup, and Graham is also pointed out.

I. INTRODUCTION

A method invented by Onsager and Machlup[1] 25 years ago and later generalized by Graham[2] enables one to write, in the form of functional integrals, the correlation functions of the system, i.e., the averages of products of the system's coordinates, when the equations of motion contain a stochastic force over which the averaging is performed. This method does not apply, however, when the averaging is performed only over the initial value data, for reasons which will be described later. Functional-integral representations of correlation functions have also been derived by Hosokawa[3] and Rosen[4] which apply whether the average is taken over a stochastic force or over the initial data. It has recently become apparent that such functional integrals may also be used to describe the response of the system to perturbations.[5,6] This theory therefore provides a complete formal description of classical statistical dynamics and is analogous to Feynman's action-integral formalism of quantum theory.

Since quantum theory may alternatively be formulated in terms of state vectors and operators, one is led to suspect that this possibility should also exist for classical systems. In fact a formal Heisenberg operator theory has recently been constructed by Martin, Siggia, and Rose,[7] to be referred to as MSR theory, in which correlation and response functions are represented by vacuum expectation values of time-ordered products of Heisenberg operators. A peculiar feature of MSR theory is the appearance of new operators "conjugate" to the coordinates of the system and satisfying boson-type commutation relations with them. As these authors recognized, the formalism so developed lacked a clear foundation but seemed to provide a good operative procedure for reproducing the perturbative solution. However, a justification of MSR theory has since been given[8] and the operators and state vectors defined more precisely. The connection between the functional-integral representations and MSR theory has also been pointed out.[5,6]

The purpose of this paper is to present a fuller discussion of the analogy with quantum mechanics starting from the functional-integral theory. For the sake of completeness a brief derivation of this theory is presented differing slightly from those previously given. This is based on a generating functional for nonstatistical motion of the system. The Heisenberg and Schrödinger representations are derived by the usual arguments involving the propagation kernel. Finally it is shown how the result of Graham may be recovered for the case of white noise stirring forces.

II. DERIVATION OF THE FUNCTIONAL-INTEGRAL REPRESENTATION

For simplicity we consider throughout a system with just one degree of freedom $Q(t)$ which satisfies an equation of motion

$$\dot{Q}+A(Q)=f(t)\ , \tag{2.1}$$

where A and f are given functions of Q and t, respectively. This equation of motion may be derived from the Lagrangian

$$\mathcal{L}(\tau)=P(\tau)[\dot{Q}(\tau)+A(Q(\tau))-f(\tau)]+G(Q(\tau)) \tag{2.2}$$

by requiring that the action integral $\int_{t_2}^{t_1} d\tau\, \mathcal{L}(\tau)$ should be stationary for arbitrary variations of the conjugate variable $P(\tau)$. The function $G(Q)$ can be chosen arbitrarily. An equation of motion for P is obtained from the condition that the action is stationary for variations of Q satisfying the fixed endpoint value condition

$$\delta Q(t_1) = \delta Q(t_2) = 0 \ .$$

We obtain

$$\dot{P} = P\frac{dA}{dQ} + \frac{dG}{dQ} \tag{2.3}$$

which clearly depends on the particular choice of G.

If the solution of (2.1) satisfies the initial condition $Q(0) = b$ then $Q(t)$ satisfies the integral equation

$$Q(t) = b + \int_0^t d\tau [f(\tau) - A(Q(\tau))] \ . \tag{2.4}$$

We now pose the problem of determining an operation for picking out the value attained by a functional $F[q]$ of $q(\tau)$ when $q(\tau)$ is the solution of (2.1) with a given initial value. It will be assumed that F depends on the form of q only in a finite time interval $(0, T)$ in order to simplify the discussion. We give a purely formal analysis which will be commented on later. Firstly we may write

$$F[Q] = \int D[q] F[q] \delta[q - Q] \ , \tag{2.5}$$

where $\delta[\ ]$ denotes the δ functional for functions of τ on the interval $(0, T)$, and $\int D[\ ]$ denotes a functional integral. This expression is not in a useful form since it involves the solution Q which may not be known explicitly. However, we may write

$$\delta(q(0) - b)\delta[\dot{q} + A(q) - f] = J^{-1}\delta[q - Q] \ ,$$

where J is the functional Jacobian $\det(\delta f/\delta Q)$ which will be nonzero if, as we assume, there is a one to one relationship between f and Q. This identity is analogous to the one for the δ function

$$\delta(B(x) - y) = [1/|B'(x_0)|]\delta(x - x_0) \ ,$$

where x_0 is the unique solution of the equation $B(x) = y$. We therefore obtain

$$F[Q] = \int_{\{q(0)=b\}} D[q] F[q] J \delta[\dot{q} + A(q) - f] \tag{2.6}$$

where, as the notation implies, the integral is taken over all functions $q(\tau)$ satisfying the given initial condition $q(0) = b$.

We now make use of the Fourier representation of the δ functional

$$\delta[x] = \mathfrak{N}^{-1}(T) \int D[p] \exp\, i\left(\int_0^T d\tau\, p(\tau) x(\tau)\right) \ .$$

Here $\mathfrak{N}(T)$ is an infinite normalization constant depending only on T which has the property

$$\mathfrak{N}(t + s) = \mathfrak{N}(t)\mathfrak{N}(s) \ .$$

Using this we obtain

$$F[Q] = \mathfrak{N}^{-1}(T) \int_{\{q(0)=b\}} D[q] \int D[p] F[q] J \exp\left(i\int_0^T d\tau\, p(\tau)\{\dot{q}(\tau) + A(q(\tau)) - f(\tau)\}\right) \ . \tag{2.7}$$

The simplest way to assign a meaning to the formal procedures outlined above is by means of a limiting process. The time interval $(0, T)$ is divided into a number N of equal subintervals of length $l = T/N$, and the differential equation (2.1) is replaced by a finite difference equation. The Jacobian J is then obtained as the limit of an ordinary Jacobian, and similarly the functional integral is defined as the limit of a multiple integral of order N as $N \to \infty$. It is found that the value obtained for J depends on the particular difference equation chosen. For example, taking

$$(Q_{n+1} - Q_n)/l + A(Q_n) = f_n \ ,$$

where $f_n = f(nl)$ and $Q_0 = b$, we find that the Jacobian relating the two sets of variables $Q_1, \ldots, Q_N$ and $f_0, \ldots, f_{N-1}$ is given by

$$\det\left(\frac{\partial f_n}{\partial Q_m}\right) = \frac{1}{l^N} \ .$$

However, taking the difference equation

$$(Q_n - Q_{n-1})/l + \tfrac{1}{2}\{A(Q_n) + A(Q_{n-1})\} = \tilde{f}_n \ ,$$

where $\tilde{f}_n = f(t_n^*)$ with $t_n^* = \frac{1}{2}(t_{n-1} + t_n)$ we find,[2,6] as $l \to 0$,

$$J = \frac{1}{l^N} \exp\left(-\frac{1}{2}\int_0^T d\tau\, A'(Q(\tau))\right) \ ,$$

where A' denotes dA/dQ. Correspondingly, the functional integrals are obtained by different limiting procedures. These different representations are presumably equivalent although there has been much discussion of this point.[9,10] We shall here adopt the representation most commonly used in quantum mechanics and described in detail by Katz.[11] The right-hand side of (2.7) is to be regarded as the limit as $N \to \infty$ of the expression

$$\frac{1}{(2\pi l)^N}\int dq_1\cdots\int dq_N\int dp_1\cdots\int dp_N\mathfrak{F}(q_1,\ldots,q_N)\exp\left(-\frac{1}{2}l\sum_{n=1}^{N}A'(q_n)\right)$$
$$\times\exp\left[il\sum_{n=1}^{N}\tilde{p}_n\left(\frac{q_n-q_{n-1}}{l}+\tfrac{1}{2}A(q_n)+\tfrac{1}{2}A(q_{n-1})-\tilde{f}_n\right)\right]\ , \tag{2.8}$$

where $\tilde{p}_n=p(t_n^*)$, $q_n=q(t_n)$, $\tilde{f}_n=f(t_n^*)$, and $\mathfrak{F}(q_1,\ldots,q_N)$ denotes the discrete approximation to the functional $F[q]$.

It is seen that we may write

$$F[Q]=\mathfrak{N}^{-1}(T)\int_{\{q(0)=b\}}D[q]\int D[p]F[q]\times\exp\left(i\int_0^T d\tau\ \mathcal{L}(\tau)\right)\ , \tag{2.9}$$

where $\mathcal{L}$ is the Lagrangian (2.2) with the particular choice of G

$$G(q)=-\tfrac{1}{2}iA'(q)\ .$$

Taking $F\equiv 1$ we see that

$$\mathfrak{N}(T)=\int_{\{q(0)=b\}}D[q]\int D[p]\exp\left(\int_0^T d\tau\ \mathcal{L}(\tau)\right)\ .$$

A particular case of (2.9) arises in the representation of correlation functions

$$Q(t_1)\cdots Q(t_n)=\mathfrak{N}^{-1}(T)\int_{\{q(0)=b\}}D[q]\int D[p]\ q(t_1)\cdots q(t_n)\exp\left(i\int_0^T d\tau\ \mathcal{L}(\tau)\right)\ .$$

The statistical distribution is sharp and the correlation functions reduce to products in the present case. The analogy with the canonical form of the Feynman action-integral formulation of quantum mechanics is apparent.

It is convenient to consider more-general integrals in which F is a functional of both p and q

$$\mathfrak{N}^{-1}(T)\int_{\{q(0)=b\}}D[q]\int D[p]F[q,p]\times\exp\left(i\int_0^T d\tau\ \mathcal{L}(\tau)\right)\ .$$

The definition of this in terms of a limiting procedure involves an expression like (2.8) but F is replaced by a function of $q(t_n)$ and $p(t_n^*)$ for $n=1,\ldots,N$ as described by Katz.[11] One application of this more-general integral is in the representation of the system's response to small changes in the force. The linear change of the functional $F[Q]$ is described by the functional derivative

$$\delta F[Q]/\delta f(t)$$

and from (2.9) we see that this is given by

$$\mathfrak{N}^{-1}(T)(-i)\int_{\{q(0)=b\}}D[q]\int D[p]p(t)F[q]\times\exp\left(i\int_0^T d\tau\ \mathcal{L}(\tau)\right)\ .$$

Similarly higher-order functional derivatives are given by integrals with several p factors in the integrand. The response functions of the system are the functional derivatives of q products such as

$$G(t;t')=\frac{\delta Q(t)}{\delta f(t')}\ ,$$

$$G(t_1,t_2;t_1',t_2')=\frac{\delta^2[Q(t_1)Q(t_2)]}{\delta f(t_1')\delta f(t_2')}\ ,$$

and it is clear that these are represented by integrals of the form

$$\mathfrak{N}^{-1}(T)(-i)^m\int_{\{q(0)=b\}}D[q]\int D[p]q(t_1)\cdots q(t_n)\times p(t_1')\cdots p(t_m')\times\exp\left(i\int_0^T d\tau\ \mathcal{L}(\tau)\right)\ . \tag{2.10}$$

From Eq. (2.4) we see that $\delta Q(t)/\delta f(t')$ is zero for $t'>t$ (the causality property), while it approaches the value 1 as $t'\to t_-$. The quantity thus has a jump discontinuity at $t'=t$ and $\delta Q(t)/\delta f(t)$ is not well defined unless we adopt a convention, whereas the corresponding functional integral

$$\mathfrak{N}^{-1}(T)(-i)\int_{\{q(0)=b\}}D[q]\int D[p]p(t)q(t)\times\exp\ i\int_0^T d\tau\ \mathcal{L}(\tau)$$

is well defined provided we have adopted a suitable limiting procedure. The particular one used here assigns a value $\frac{1}{2}$ to the above integral and we must define the equal-time response functions accordingly if (2.10) is to be valid for all values of the time arguments.

The functional integrals over p and q introduced above must be carried out in the order indicated although the introduction of a suitable convergence factor into the integrand would enable this condition to be relaxed.

A generating functional may be introduced in the obvious way

$$Z[\xi,\eta]=\mathfrak{N}^{-1}(T)\int_{\{q(0)=b\}}D[q]\int D[p]\exp\left(i\int_0^T d\tau\,[\xi(\tau)q(\tau)+\eta(\tau)p(\tau)+\mathcal{L}(\tau)]\right). \tag{2.11}$$

The correlation and response functions are then obtained from Z by functional differentiation with respect to the test functions ξ and/or η. For example,

$$G(t;t')=i[\delta^2 Z/\delta\xi(t)\delta\eta(t')]_{\xi\equiv\eta\equiv 0}\;.$$

III. HEISENBERG AND SCHRÖDINGER REPRESENTATIONS

In order to demonstrate the connection with the MSR operator theory and the Fokker-Planck theory, which correspond, respectively, to the Heisenberg and Schrödinger pictures of quantum mechanics, we introduce the propagation kernel

$$K(q_1,t_1|q_2,t_2)=\mathfrak{N}^{-1}(t_1-t_2)\int_{\substack{q(t_1)=q_1\\ q(t_2)=q_2}}D[q]\int D[p]\exp\left(i\int_{t_2}^{t_1}d\tau\,\mathcal{L}(\tau)\right) \tag{3.1}$$

the integral being taken over all trajectories passing through the two points (q_1,t_1), (q_2,t_2). This quantity gives the conditional probability density that $Q(t_1)=q_1$ given that $Q(t_2)=q_2$. The integral can be rewritten

$$\mathfrak{N}^{-1}(t_1-t_2)\int D[q]\int D[p]\delta(q(t_1)-q_1)\times\delta(q(t_2)-q_2)\exp\left(i\int_{t_2}^{t_1}d\tau\,\mathcal{L}(\tau)\right).$$

Clearly, as $t_1-t_2\to 0$ we have

$$K(q_1,t_1|q_2,t_2)\to\delta(q_1-q_2)\;.$$

From (3.1) we see that K has the semigroup property

$$\int dq_2\,K(q_1,t_1|q_2,t_2)K(q_2,t_2|q_3,t_3)=K(q_1,t_1|q_3,t_3)\;. \tag{3.2}$$

The functional integrals representing correlation and response functions can be expressed in terms of K. Taking, for example,

$$\frac{\delta Q(t_1)}{\delta f(t_2)}=(-i)\mathfrak{N}^{-1}(T)\int_{\{q(0)=b\}}D[q]\int D[p]q(t_1)p(t_2)\times\exp\left(i\int_0^T d\tau\,\mathcal{L}(\tau)\right)$$

we obtain for $t_1>t_2$,

$$-\int dq_1\int dq_2\,q_1K(q_1,t_1|q_2,t_2)\frac{\partial}{\partial q_2}K(q_2,t_2|b,0)\;, \tag{3.3}$$

where we have used the identity

$$\mathfrak{N}^{-1}(t_1-t_2)\int_{\substack{q(t_1)=q_1\\ q(t_2)=q_2}}D[q]\int D[p]p(\tau')\times\exp\left(i\int_{t_2}^{t_1}d\tau\,\mathcal{L}(\tau)\right)\to -i\frac{\partial}{\partial q_1}\delta(q_1-q_2)$$

as $t_1\to\tau'_+$ and $t_2\to\tau'_-$.

The propagation kernel may be regarded as the coordinate representation of the evolution operator X introduced in Ref. 8 since it may be seen from the definitions that

$$K(q',t'|q,t)=X(t',t)\delta(q-q')\;,$$

where X acts only on functions of the variable q. Using this result it may easily be verified that the expression (3.3) for the response function can be rewritten in the form derived from the operator theory of MSR. The same argument may be applied to higher-order correlation and response functions and we find, as in quantum theory, that a functional integral involving a product of p's and q's with different time values may be expressed as a vacuum expectation value of a time-ordered product of operators

$$(-i)^m\mathfrak{N}^{-1}(T)\int_{\{q(0)=b\}}D[q]\int D[p]p(t_1)q(t_2)\cdots p(t_m)\exp\left(i\int_0^T d\tau\,\mathcal{L}(\tau)\right)=(\Phi_0,T\{\hat{q}(t_1)q(t_2)\cdots\hat{q}(t_m)\}\Phi_0)\;. \tag{3.4}$$

In the operator representation the "state vectors" are functions of q, $\Phi_0 \equiv 1$ being the "vacuum state." The operators $q(t)$ and $\hat{q}(t)$ are Heisenberg operators

$$q(t)=E(t)qE^{-1}(t)\ ,$$

$$\hat{q}(t)=\left(E(t)\frac{\partial}{\partial q}E^{-1}(t)\right)^{\dagger}\ ,$$

where $E(t)=X(t,0)$ and $\dagger$ denotes the adjoint of an operator. The scalar product is defined by

$$(\Psi,\Phi)=\int dq\,\rho(q)\Psi(q)\Phi(q)\ ,$$

where $\rho(q)$ is the probability density function for the initial data which, in the present case, is just $\delta(q-b)$.

The commutation relations for the operators follow directly from the definitions and we see that

$$[q(t),\hat{q}(t)]=1\ .$$

It also follows that $(\Phi_0,\hat{q}(t)\Phi)=0$ for any state Φ. The equations of motion may also be deduced from the definitions but it is probably simpler to make use of the identity

$$\int D[p]\frac{\delta}{\delta p(t)}\exp\left(i\int_0^T d\tau\{\xi(\tau)q(\tau)+\eta(\tau)p(\tau)+\mathcal{L}(\tau)\}\right)=0$$

and a similar one involving $\delta/\delta q(t)$ to derive the Schwinger equations

$$\frac{1}{i}\frac{\partial}{\partial t}\frac{\delta Z}{\delta\xi(t)}+\left[\eta(t)-f(t)+A\left(\frac{1}{i}\frac{\delta}{\delta\xi(t)}\right)\right]Z=0\ ,$$

$$\frac{1}{i}\frac{\partial}{\partial t}\frac{\delta Z}{\delta\eta(t)}+\left[-\xi(t)+\frac{\delta}{\delta\eta(t_+)}A'\left(\frac{1}{i}\frac{\delta}{\delta\xi(t)}\right)\right]Z=0\ .$$

Replacing each term by its corresponding operator expression we can derive the following equations

$$\dot{q}(t)+A(q(t))=f(t)\ ,$$

$$\dot{\hat{q}}(t)=\hat{q}(t)A'(q(t)).$$

To establish a Schrödinger type representation we use the semigroup property to write

$$K(q,t+\epsilon\,|b,0)=\int dq'\,K(q,t+\epsilon\,|q',t)\times K(q',t\,|b,0)\ . \tag{3.5}$$

The propagation kernel for the infinitesimal interval $(t,t+\epsilon)$ can be obtained by an argument identical to that used in quantum mechanics.[11,12] It is important to keep in mind the particular definition of the functional integrals since this determines the operator ordering of the Hamiltonian. The definition used here leads to a Hamiltonian which is obtained from the classical one

$$-p[A(q)-f(t)]+\tfrac{1}{2}iA'(q)$$

by replacing products of p's and q's by symmetrized products of the corresponding Schrödinger operators. Hence the equation is

$$\frac{\partial}{\partial t}K(q,t\,|\cdots)=\frac{\partial}{\partial q}[A(q)-f(t)]K(q,t\,|\cdots)\,. \tag{3.6}$$

Similarly we find

$$\frac{\partial}{\partial t}K(q,t\,|\cdots)=[A(q)-f(t)]\frac{\partial}{\partial q}K(\cdots|q,t)\,. \tag{3.7}$$

IV. STATISTICS

So far in this discussion the initial data and the force $f(t)$ have been assumed to be given precisely. However, the form of the functional integrals is such that an average over either, or both, of these quantities can be performed. Let us consider the case where $f(t)$ is a white-noise function so that, denoting averages over f by angular brackets, we have

$$\langle f(t)f(t')\rangle=c\delta(t-t')$$

and

$$\left\langle\exp\left(i\int d\tau\,p(\tau)f(\tau)\right)\right\rangle=\exp\left(-\frac{c}{2}\int d\tau\,p^2(\tau)\right)\ .$$

The formulas obtained above for response and correlation functions, etc., all involve f in the form of a factor

$$\exp\left(-i\int_0^T d\tau\,p(\tau)f(\tau)\right)$$

in the integrand so that, when the average over f is performed, this factor is simply replaced by

$$\exp\left(-\tfrac{1}{2}c\int_0^T d\tau\,p^2(\tau)\right)\ .$$

The correlation functions, etc., are thus given by the same formulas as before, such as (2.9), (2.10), and (2.11), but with the Lagrangian given by

$$\mathcal{L}(\tau)=p(\tau)[\dot{q}(\tau)+A(q(\tau))]+\tfrac{1}{2}icp^2(\tau)-\tfrac{1}{2}iA'(q(\tau))\ . \tag{4.1}$$

The normalization constant is unchanged in the averaging being independent of f.

Since $\mathcal{L}(\tau)$ is still local in the time variable τ the propagation kernel satisfies the semigroup property and a Schrödinger equation (the Fokker-Planck equation) can be derived

$$\frac{\partial}{\partial t}K(q,t\,|\cdots)=\left[\tfrac{1}{2}c\left(\frac{\partial}{\partial q}\right)^2+\frac{\partial}{\partial q}A(q)\right]K(q,t\,|\cdots)\,. \tag{4.2}$$

The Heisenberg representation can be developed as before, the operator equations of motion following most simply from the Schwinger equations. We now obtain

$$\dot{q}(t)+A(q(t))=c\hat{q}(t)\ ,$$
$$\dot{\hat{q}}(t)=\hat{q}(t)A'(q(t))\ ,$$

so that the averaging has led to a coupling of the equations.

Since the new Lagrangian involves p only in quadratic form the functional integral over p can be performed explicitly and we recover the result of Graham. For example, the propagation kernel is given by

$$K(q_1,t_1|q_2,t_2)=\tilde{\mathfrak{N}}^{-1}(t_1-t_2)\int_{\substack{q(t_1)=q_1\\ q(t_2)=q_2}} D[q]\exp\left[-\int_{t_2}^{t_1}d\tau\left(\frac{1}{2c}[\dot{q}(\tau)+A(q(\tau))]^2-\tfrac{1}{2}A'(q(\tau))\right)\right]\ .$$

Since the integrand contains products of $\dot{q}$ and functions of q evaluated at the same time the integral must be defined correctly. The appropriate definition is the same as that given by Feynman and Hibbs[13] for the quantum-mechanical problem of a particle in an electromagnetic field.

Averages over the initial data may also be easily carried out. If the initial-value condition is incorporated into the integrand by means of a factor $\delta(q(0)-b)$ the average over b then gives $\rho(q(0))$, where ρ is the probability density function of the initial value. In the case where no stochastic force acts the integration over p simply gives back the δ functional and so does not lead to a useful functional-integral representation.

If a statistically stationary distribution exists described by the density function $R(q)$ then the Fokker-Planck equation gives

$$\left[\frac{1}{2}c\left(\frac{\partial}{\partial q}\right)^2+\frac{\partial}{\partial q}A(q)\right]R(q)=0\ .$$

The stationary nature of R also leads to the equation

$$\int dq'\,K(q,t\,|q',t')R(q')=R(q)\ .$$

It follows that the response and correlation functions in the stationary state are given by

$$\left\langle\frac{\delta Q(t_1)}{\delta f(t_2)}\right\rangle_{\rm stat}=-\int dq_1\int dq_2\,q_1K(q_1,t_1|q_2,t_2)\frac{\partial R(q_2)}{\partial q_2}\ ,$$
$$\langle\!\langle Q(t_1)Q(t_2)\rangle\!\rangle_{\rm stat}=\int dq_1\int dq_2\,q_1q_2K(q_1,t_1|q_2,t_2)R(q_2)\ .$$

Using these results we can establish the fluctuation-dissipation theorem

$$(2/c)\langle\!\langle Q(t_1)\dot{Q}(t_2)\rangle\!\rangle_{\rm stat}=\left\langle\frac{\delta Q(t_1)}{\delta f(t_2)}\right\rangle_{\rm stat}\ .$$

This relationship is not valid for systems with more than one degree of freedom unless the stationary state satisfies additional conditions.

The generalization of the functional-integral representations to the case of a Gaussian stirring force which is not white noise is straightforward. If f is of zero mean and has a correlation function $\langle\!\langle f(t)f(t')\rangle\!\rangle=c(t-t')$ then, instead of a term $c\int_0^T d\tau\,p^2(\tau)$ in the Lagrangian we have

$$\int_0^T d\tau_1\int_0^T d\tau_2\,c(\tau_1-\tau_2)p(\tau_1)p(\tau_2)\ .$$

The Lagrangian is now nonlocal in time, the analogy with quantum mechanics breaks down, and a Schrödinger formalism does not exist. However, a Heisenberg operator theory can still be formulated by means of the Schwinger equations which can be derived as before. It is also possible to deal with situations in which the random force appears in the equation of motion multiplied by a function of the system coordinates.[6] Finally it should be mentioned that the Schwinger equations provide a simple way to obtain both "bare" and renormalized perturbation series.[7,14]

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