PHYSICAL REVIEW A VOLUME 19, NUMBER 3 MARCH 1979

Optimal configuration of a class of irreversible heat engines. I

Morton H. Rubin
University of Maryland Baltimore County, Catonsville, Maryland 21228
(Received 16 August 1978)

We determine the optimal configuration for a class of heat engines with finite cycling times. The engines studied are those for which irreversible process occur through the coupling of the engine to the environment, while the working fluid in the engine is assumed to undergo reversible transformations. For the case in which the only irreversible process is heat conduction, we derive the optimal configuration of these engines for two different operating goals: maximum average power output and maximum efficiency. We suggest figures of merit based on these optimal configurations which may be more useful than those based on reversible processes.

I. INTRODUCTION

One of the most useful results of classical thermodynamics is that the maximum work in a thermal process with fixed constraints is obtained when performing the process reversibly.[1-3] The power of this result lies in its universal nature, reversible processes lead to upper bounds on performance criteria for arbitrary processes. For example, the Carnot efficiency provides an upper bound on the efficiency of all cyclic heat engines operating between two fixed temperatures.

The universal nature of reversible processes has led to their use as standards for thermal processes.[3] However, it has been pointed out by several authors that the use of reversible processes as standards of performance is not really desirable because strictly reversible processes must be carried out infinitely slowly and, consequently, no power can be generated by such processes.[4-8] As a practical matter infinitely slowly means that the states of the working fluid must change slowly compared to relaxation times in the system and slowly enough so that frictional effects are negligible. While it is not too difficult to make adiabatic processes reversible to high accuracy, since many relaxation times may elapse in a time short compared to the duration of the process, for processes that involve heat transfer this is in general not the case. In order to make a heat-transfer process occur rapidly, it is necessary to have large temperature gradients and, consequently, irreversibility becomes important.

Since the production of power is important, it seems that we should look for new standards with which actual processes may be usefully compared. The general problem of finding a new standard for heat engines which have finite cycling times has been discussed elsewhere.[4-6,8] It is sufficient for our purposes to mention that there is no universal standard in the sense that reversible processes provide universal standards because irreversibility may occur in such a variety of ways. However, for a process determined by a set of operating constraints and goals, when the irreversible processes are specified, it becomes possible to define useful standards. These standards depend on the particular operational goal of the engine.

In this paper we wish to prove that for a certain type of heat engine, if linear heat conduction is the source of irreversibility, there are processes first studied by Curzon and Ahlborn,[9] which may be used as a standard when the operational goal is to maximize the power output per cycle or, in a different operating mode, to maximize the efficiency.

If the only irreversible process is heat conduction and if it may be assumed that the adiabatic branches of the cycle occur during a negligibly short fraction of the cycling time, our proof is quite general. If we wish to take into account the fact that the adiabatic branches do not occur in a negligibly short time, the problem becomes more difficult, and we must use the methods of optimal control theory. This problem is sufficiently more complicated that we have devoted a separate paper to it.

The work presented here differs from earlier work[5,9] in that the reservoir temperatures are treated as controllable parameters and it is shown that only the hottest and coldest reservoirs are needed. Our point of view is somewhat different than that of Refs. 5 and 9 and more in the spirit of Ref. 6.

The purpose of this paper is to find optimal operating procedures for a class of simple heat engines. The plan of the paper is as follows. In Sec. II we define our model heat engine and list our assumptions about its operation. In Sec. III we optimize the performance of the engine for two different operating goals, maximum average power output and maximum efficiency, and we suggest figures of merit for these cases in Sec. V. In Sec. V, we summarize our results and draw some conclusions.

II. MODEL HEAT ENGINE

Before specifying the subclass of heat engines we will study, it is convenient to first define a more general class of heat engines. We define an *endoreversible engine* to be an engine such that during its operation its working fluid undergoes reversible transformations. This class of engines includes all the reversible engines discussed in thermodynamics textbooks. More importantly for our work, the subclass of engines which are coupled to the external world via irreversible processes provide a useful starting point for the study of finite-time processes.[5,9]

We now define our model heat engine. The engine will be a standard ideal heat engine composed of a working fluid enclosed in a cylinder whose walls may be adjusted to be insulating adiabatic walls, or conducting diathermic walls. The cylinder contains a piston which is used to do work on the external world. We also assume that there is a set of heat reservoirs available.

We now list our assumptions about the operation of the engine.

(i) The engine is endoreversible.

(ii) The diathermic walls have constant thermal conductivity.

(iii) When the engine is in contact with a heat reservoir of absolute temperature T_R, the heat flux into the working fluid is given by a linear law

$$\dot{q} = \rho(T_R - T)\,, \tag{2.1}$$

where ρ is the thermal conductivity of the diathermic walls and T is the absolute temperature of the working fluid. Our first assumption assures us that, at each instant, the working fluid has a uniform temperature.

(iv) Each thermal reservoir has a constant temperature T_R where $T_L \leq T_R \leq T_H$.

(v) The work done by the engine in one cycle is given by

$$W = \int_0^\tau P\dot{V}\,dt\,, \tag{2.2}$$

where P and V are the pressure and volume of the working fluid, $\dot{V}$ means the time derivative of V, and τ is the cycling time of the engine.

A few comments upon these assumptions are necessary. The first assumption means that the fluid only undergoes reversible transformations, consequently, the entropy change of the fluid in one period is zero; i.e.,

$$\Delta S = \int_0^\tau \frac{\dot{q}}{T}\,dt = 0\,. \tag{2.3}$$

This constraint will be explicitly taken into account in our calculation. In the following paper when the dynamics of the working fluid is taken into account, this constraint is automatically satisfied by the equations of motion and the fact that the process is cyclic. Thus Eq. (2.3) is a substitute for the dynamics of the working fluid.

The second assumption is overly restrictive. It is a trivial matter to let the thermal conductivity of the walls be different when the engine is in contact with different reservoirs so $\rho \to \rho_R$. We shall not bother with this. It is also not difficult to show that if ρ_R is allowed to vary so that $0 \leq \rho_R \leq \rho_0$, the optimal solution requires $\rho_R = \rho_0$. This is in fact shown in the following paper.

The fourth assumption is necessary to avoid the unphysical cases of infinite or absolute zero temperatures.

The fifth assumption ensures us that there is no friction present. If there was friction in the coupling to the external world, e.g., sliding friction between the piston and the cylinder wall, then there would be lost work[10]; i.e.,

$$P\dot{V} = \dot{W} + L\dot{W}\,,$$

where $\dot{W}$ is the power generated by the engine and $L\dot{W}$ is the rate of power dissipated. Note that Eq. (2.2) is unaffected by the presence of a constant external pressure such as atmospheric pressure since such a contribution averages to zero for cyclic process.

Finally we conclude this section by noting that the heat input to the engine per cycle is

$$Q_1 = \int_0^\tau \dot{q}\,\theta(T_R - T)\,dt\,, \tag{2.4}$$

where $\theta(x)$ is the Heaviside step function, $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ if $x < 0$.

III. OPTIMAL PERFORMANCE OF A FRICTIONLESS, ENDOREVERSIBLE ENGINE

In this section we show that under assumptions (i)–(v), the Curzon-Ahlborn process[9] leads to the maximum average power output or, in a different configuration, the maximum efficiency.

A. Maximum average power

It is often desirable to choose as an operational goal for an engine its maximum average power output. In our case this means maximizing W since the cycling period τ is fixed.

It is convenient to first rewrite Eq. (2.2) by applying the first law of thermodynamics to the working fluid:

$$\dot{q} = \dot{U} + P\dot{V}\,, \tag{3.1}$$

where $\dot{U}$ is the rate of change of the thermal energy of the working fluid. Since the process is cyclic Eq. (2.2) may be rewritten using (3.1) as

$$W = \int_0^\tau \dot{q}\,dt\,. \tag{3.2}$$

Therefore we must maximize W subject to $\Delta S = 0$, i.e., we maximize

$$L = W - \lambda\Delta S = \int_0^\tau [\rho(T_R - T) - \lambda\rho(T_R - T)/T\,]dt \tag{3.3}$$

by varying T and T_R.

Note that only one thermodynamic variable describing the working fluid appears in Eq. (3.3). To understand why a second thermodynamic variable is absent, we must first recall that the temperature and any second independent variable, say the volume, are dynamical variables, i.e., they satisfy equations of motion. These equations of motion ensure the consistency of the equation of state of the working fluid with the first law at each instant of time. In our problem the equations of motion are replaced by the constraint given by Eq. (2.3) and, consequently, we assume that the second thermodynamic variable adjusts itself appropriately. This may require nonphysical variations such a discontinuous jumps in the state variables. The advantage of our nondynamical or quasistatic calculation is its simplicity and its generality in the sense that we do not have to specify the equation of state of the fluid. In the following paper we solve the dynamical problem when the working fluid is an ideal gas.

We now turn to the problem of finding the optimum solution to our problem. In order to take account of assumption (iv), we replace T_R by a new variational parameter ψ such that

$$T_R = \tfrac{1}{2}(T_H + T_L) + \tfrac{1}{2}(T_H - T_L)\tanh\psi\,, \tag{3.4}$$

where ψ is unconstrained. It is now a simple matter to maximize L

$$\delta L = \int_0^\tau dt\,\rho\left[\delta T_R\left(1 - \frac{\lambda}{T}\right) + \delta T\left(-1 + \lambda\frac{T_R}{T^2}\right)\right], \tag{3.5}$$

where

$$\delta T_R = \tfrac{1}{2}(T_H - T_L)\delta\psi/\cosh^2\psi\,. \tag{3.6}$$

Setting $\delta L = 0$ we find that when $\rho \neq 0$

$$T = \lambda \tag{3.7a}$$

or

$$\psi = \pm\infty \tag{3.7b}$$

and

$$T = (\lambda T_R)^{1/2}\,. \tag{3.8}$$

First we see that if for some part of the cycle Eq. (3.7a) holds so that $T = \lambda$, then from (3.8) $T = T_R$. During this part of the cycle there is no contribution to W; furthermore if the second variation of L is examined we find that this solution corresponds to a saddle point and so we discard it.

Therefore, Eq. (3.7b) must hold which means that $T_R = T_H$ or T_L, that is, we only need the hottest and coldest reservoirs. Thus we have two solutions

$$T_R = T_H\,,\quad T = T_h = (\lambda T_H)^{1/2} \tag{3.9}$$

and

$$T_R = T_L\,,\quad T = T_l = (\lambda T_L)^{1/2}\,. \tag{3.10}$$

In order to show that these are true maxima, we consider the variation of L up to second order.

$$\Delta L = \delta L + \int_0^\tau dt\,\rho\left[\delta T_R \delta T\,\frac{2\lambda}{T^2} - (\delta T)^2 \frac{2\lambda}{T^2}\frac{T_R}{T}\right].$$

Evaluating this at $T_r = (\lambda T_R)^{1/2}$ we find

$$\Delta L = \int_0^\tau dt\,\rho\left\{\delta T_R\left[1 - \left(\frac{\lambda}{T_R}\right)^{1/2}\right] + \frac{2}{T_R}\left[\delta T_R \delta T - (\delta T)^2\left(\frac{T_R}{\lambda}\right)^{1/2}\right]\right\}.$$

The first-order variation with respect to T_R does not vanish since the integrand attains it maximum with respect to T_R on the boundaries of the T_R interval.

For $T_R = T_H(T_L)$, the allowed variations of T_R are such those for which $\delta T_R \leq 0$ (≥ 0). Thus $\Delta L < 0$ for all allowed variation of T and T_R if $\sqrt{T_H} > \sqrt{\lambda} > \sqrt{T_L}$ which is indeed the case as we shall see.

It is clear that our solution as it stands consists of two disjoint points in the allowed region of the T_R-T plane. The discontinuity in T_R does not disturb us since it is standard practice in thermodynamics to imagine reservoirs are connected or disconnected from systems instantaneously. However, a discontinuity in the temperature of the working fluid is disturbing since we know that the fluid temperature satisfies an equation of motion—it is a dynamical variable. In order to overcome this difficulty, we shall imagine that the two isotherms are connected by adiabats. This means that we take $\rho = 0$ for two branches of the cycle. These two branches do not contribute to the integrals for W or ΔS.

It is interesting to note that if we had treated ρ as a variable where $0 \leq \rho \leq \rho_0$ then varying with respect to ρ implies that if Eqs. (3.9) or (3.10) hold and $\sqrt{T_H} > \sqrt{\lambda} > \sqrt{T_L}$ then $\rho = \rho_0$, otherwise $\rho = 0$.

We now complete the solution to our problem. We use Eqs. (3.9) and (3.10) in Eq. (2.3) to solve for λ,

$$\sqrt{\lambda} = x\sqrt{T_H} + (1-x)\sqrt{T_L}\,, \tag{3.11}$$

where

$$x = t_H/(t_H + t_L) \tag{3.12}$$

with t_H (t_L) the length of time that the cylinder is in contact with the high- (low-) temperature reservoir during each cycle. Finally,

$$W_{\max} = \rho\tau'(\sqrt{T_H} - \sqrt{T_L})^2\,, \tag{3.13}$$

$$Q_1 = \rho\tau'(\sqrt{T_H} - \sqrt{T_L})\sqrt{T_H}\,, \tag{3.14}$$

and

$$\eta = \frac{W_{\max}}{Q_1} = 1 - (T_L/T_H)^{1/2}\,, \tag{3.15}$$

where

$$\tau' = t_H t_L/(t_H + t_L)\,. \tag{3.16}$$

Thus the cycle consists of two isothermal transformations connected by two adiabatic transformations ($\rho \equiv 0$) which in this case are also isentropic. If t_A is the time required for the adiabatic transformations, then $t_A + t_H + t_L = \tau$.

The cycle derived above is the Curzon-Ahlborn cycle discussed in Ref. 9, and in Ref. 5. In Fig. 1 we have sketched such an engine. The internal engine operating between T_h and T_l is a Carnot engine so the efficiency is given by $1 - T_l/T_h$ which is equal to Eq. (3.15) since $T_h = (\lambda T_H)^{1/2}$ and $T_l = (\lambda T_L)^{1/2}$.

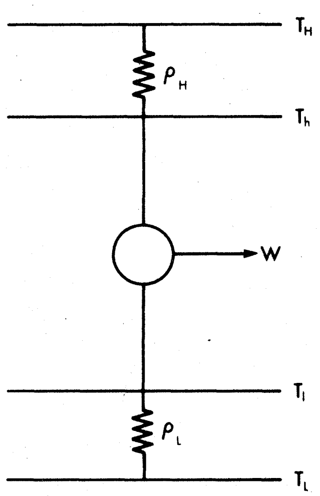

FIG. 1. Sketch of the endoreversible engine for the Curzon-Ahlborn cycle.

It is worth noting that the fact that we need use only the hottest and coldest reservoirs and that we do not need any other reservoirs is not a trivial result. For contrast we note that there are processes which require a continuum of reservoirs. For example, to charge a capacitor through a fixed resistance in finite time with a minimum of Joule heating requires a continuous source of voltage.

If we make some assumption about t_A we may maximize Eq. (3.13) with respect to t_H and t_L. Curzon and Ahlborn assume $t_A = (1-\gamma)(t_H + t_L)$ so $\tau = \gamma(t_H + t_L)$ and maximize $W_{\max}$ with respect to t_H, τ being held fixed. They show that $t_H = \tau/2\gamma$, consequently,

$$W_{\max} = (\rho\tau/4\gamma)(\sqrt{T_H} - \sqrt{T_L})^2\,. \tag{3.17}$$

As we shall show in the following paper the assumption of Curzon and Ahlborn does not lead to an optimal solution of the dynamical equations unless $\gamma \simeq 1$, i.e., $t_A \ll \tau$. This corresponds to the magnitude of the rate of change of the volume of the cylinder, $|\dot{V}|$, becoming very large during the adiabatic branches.

B. Maximum efficiency

It is often useful to run an engine at maximum efficiency rather than use some other operating goal such as we did in Sec. III A. The efficiency is, as usual, defined by

$$\eta = W/Q_1\,, \tag{3.18}$$

where W is defined Eq. (3.2) and Q_1 by (2.4). We will maximize η for a fixed input energy subject to Eq. (2.3) and assumptions (i)–(v). If we keep Q_1 fixed this problem is the same as maximizing

$$L = W - \lambda\Delta S - \mu Q_1\,. \tag{3.19}$$

It is simple to show that we again get a Curzon-Ahlborn cycle with

$$\eta_{\max} = 1 - \frac{T_L}{T_H}\frac{1}{1 - Q_1/Q_1^0}\,, \tag{3.20}$$

where

$$Q_1^0 \equiv \rho\tau' T_H\,, \tag{3.21}$$

with τ' given by (3.16) and $W = \eta_{\max} Q_1$.

Figure 2 is a sketch of a graph of the average power output W/τ plotted against the average power input, i.e., $W/\tau = \eta_{\max} Q_1/\tau$. It is evident that the efficiency decreases with increasing input energy, $d\eta_{\max}/dQ_1 < 0$, so that the greatest efficiency is attained at $Q_1 \to 0$ where the slope of

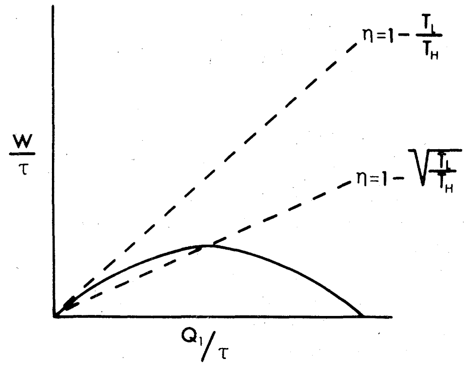

FIG. 2. Sketch of the graph of the average power output as a function of the average power input for the maximum efficiency.

the curve is the Carnot efficiency. The average power output increases with increasing input energy until it reaches the maximum average power output when the engine's operating conditions are those derived in Sec. III A. The efficiency and the output both decrease as Q_1 continues to increase until $W \to 0$ when T_h and T_l, the internal temperatures, become equal. The changes in Q_1 and W are generated by adjusting T_h and T_l. T_h decreases and T_l increases with increasing Q_1 until eventually $T_h = T_l$.

IV. FIGURES OF MERIT

Figure 2 illustrates two of the options available for operating an engine, in this case, high efficiency versus high power output. There are many different optimum solutions depending on particular operating goals, for example, economic considerations expand the range of operating goals enormously. We now wish to suggest some figures of merit for engines which satisfy the assumptions made in Sec. II.

First, suppose a cyclic heat engine is to be operated so that it produces the maximum average power output. Then a possible figure of merit would be

$$\epsilon_\rho = W/W_{\max}, \tag{4.1}$$

where W is the work per cycle done by the engine being considered and $W_{\max}$ is the work done by an idealized engine of the type discussed in Sec. III A as determined by Eq. (3.17). We see that $W_{\max}$ depends on ρ, γ, and τ as well as the temperature of the reservoirs. This dependence of $W_{\max}$ on the details of the engine and its operating characteristics is the result of including an irreversible process in our analysis. Of course, if we made our model engine more realistic by allowing for other irreversible processes the result would become more complicated.

It is remarkable that the efficiency of the process discussed in Sec. III A only depends on the reservoir temperature. It may be thought, because of this, a better figure of merit would be the ratio of the true efficiency to the efficiency given by Eq. (3.15). However, since this figure of merit may be greater than or less than 1, it does not seem as commendable as that of Eq. (4.1).

Next suppose a cyclic heat engine is to be operated at maximum efficiency. Then a figure of merit may be defined by

$$\epsilon_\eta = \eta_{\text{actual}}/\eta_{\max}$$

where η_{actual} is the measured efficiency of the engine and η is given by Eq. (3.20) evaluated for the actual energy input. Since $\eta_{\text{actual}} \le \eta_{\max}$, $\epsilon_\eta \le 1$. A definition similar to the definition of ϵ_η where $\eta_{\max}$ is replaced by η_{Carnot} is called the second-law efficiency[3] or the coefficient of utility.[2] $\epsilon_\eta \le \epsilon_{\text{2nd law}} \equiv \eta_{\text{actual}}/\eta_{\text{Carnot}}$.

V. CONCLUSIONS

We have obtained the optimal operating conditions for an idealized class of heat engines. What distinguishes these engines from those usually studied in classical thermodynamics is that these engines generate power. Although the processes we derive are highly idealized, we can use them to derive standards of performance which may be more useful than those based on reversible processes.[9]

[1] J. H. Keenan, *Thermodynamics* (Massachusetts Institute of Technology, Cambridge, Mass., 1970).
[2] Landau and Lifshitz, *Statistical Physics* (Addison-Wesley, Reading, Mass., 1958).
[3] M. H. Ross *et al.*, *Efficient Use of Energy* (AIP, New York, 1975), Part I, Chap. 2.
[4] H. T. Odum and R. C. Pinkerton, Am. Sci. 43, 331 (1955).
[5] B. Andresen, P. Salamon, and R. S. Berry, J. Chem. Phys. 66, 1571 (1977).
[6] B. Andresen, R. S. Berry, A. Nitzan, and P. Salamon, Phys. Rev. A 15, 2086 (1977).
[7] P. Salamon, B. Andresen, and R. S. Berry, Phys. Rev. A 15, 2094 (1977).
[8] M. H. Rubin, Am. J. Phys. 46, 647 (1978).
[9] F. L. Curzon and B. Ahlborn, Am. J. Phys. 43, 22 (1975).
[10] R. E. Sonntag and G. J. Van Wylen, *Introduction to Thermodynamics: Classical and Statistical* (Wiley, New York, 1971).