Superfluorescence: Quantum-mechanical derivation of Maxwell-Bloch description with fluctuating field source

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For a pencil-shaped extended medium with Fresnel number equal to 1, we have *quantum mechanically* derived a description of the initiation of superfluorescence in terms of Maxwell-Bloch equations with a fluctuating source due to the *zero-point fluctuations* of the vacuum field. By the introduction of classical behavior, these equations are extended to include nonlinear behavior due to decreasing atomic inversion. The principal assumption in the derivation is that the main features of superfluorescence are governed by the interaction of atoms with field modes inside two small solid angles around the pencil axis. The delay τ_D, defined as the time at which the mean-squared tipping angle of the collective Bloch vector attains the value 1, turns out to be given by $\tau_D = (\tau_R/4)[\ln(2\pi N)^{1/2}]^2$, where τ_R is the radiation time for collective decay and N is the number of atoms. The corresponding effective initial tipping angle roughly equals $2/(N)^{1/2}$. A Fokker-Planck equation is derived to describe the statistics of the initial development of the tipping angle. The variance $\Delta\tau_D$ of the delay of the superfluorescence pulse satisfies approximately $\Delta\tau_D = 2.3/\ln N$. A brief comparison with previous treatments of superfluorescence is given.

I. INTRODUCTION

Superfluorescence (SF) is the *cooperative* emission from a large number of initially inverted atoms, *without initial macroscopic dipole moment.*

SF was predicted by Dicke[1] in 1954. Its first observation was made in optically pumped HF gas by Skribanowitz *et al.*[2] in 1973. Since then several observations of SF on different atomic and molecular transitions have been reported.[3-9] All observations pertain to samples with linear dimensions that are large compared with the wavelength λ_0 of the emitted radiation.

Since Dicke's proposal,[1] numerous papers have been devoted to the theory of SF (Refs. 10–28). These theories differ in the sort of approximations used. The well-known theory of Bonifacio and Lugiato[26] gives a fully quantum-mechanical but mean-field treatment of SF; it neglects the spatial variation of the field envelope throughout the sample. To account fully for propagation effects, MacGillivray and Feld[24,27] introduced a semiclassical theory in which the initiation of SF is described by a fluctuating polarization source. Several recent papers link together aspects of quantum-mechanical and semiclassical theories.[29-32] Numerical calculations are usually based on Maxwell-Bloch equations with an effective initial tipping angle of the collective Bloch vector.[24,27,31-34] However, there is no agreement on the value to be assigned to that angle.[14,24,27,29-34]

We report a *quantum-mechanical* derivation of Maxwell-Bloch equations including a fluctuating field source which accounts for the stochastic initiation of SF. We consider a pencil-shaped active volume with cross section $S \gg \lambda_0^2$, length L, and Fresnel number $\mathfrak{F} \equiv S/\lambda_0 L = 1$. The atomic density ρ is so large that the characteristic time for collective decay along the pencil axis $\tau_R \simeq 4\pi\tau_N/(\rho\lambda_0^2 L)$ is much shorter than the natural lifetime τ_N of a single atom. In order to be able to ignore dipole-dipole interaction between the atoms we require $\rho(\lambda_0/2\pi)^3 \ll 1$. Atomic motion is disregarded ($T_2^* = \infty$) but the atoms occupy random positions. Only two-level atoms are considered.

Our principal assumption is that the initiation, propagation, and collective growth of SF is governed by the interaction of atoms with field modes inside two small solid angles ($\simeq \lambda_0^2/S$) around the pencil axis. This treatment in terms of two end-fire modes is made possible by our explicit assumption of Fresnel number $\mathfrak{F} = 1$, i.e., the diffraction angle $\lambda_0/\sqrt{S}$ equals the geometric angle $\sqrt{S}/L$. The reason for singling out the end-fire modes is, of course, the largest logarithm of gain along the pencil axis (see Refs. 11, 25, 28).

We further assume that the gain of field modes outside the specified solid angles can be ignored. The interaction with these modes manifests itself in lateral spontaneous emission and scattering of light via the atom's induced dipole moment. The incoherence of the lateral emission is guaranteed by the random positions of the atoms. In our treatment of SF it appears as damping of the upper level occupation and of the dipole moment of the atoms. The division into end-fire modes and other modes corresponds to phenomena on different time scales: the collective phenomena on a time scale τ_R and the incoherent phenomena on a time scale τ_N. Since $\tau_R \ll \tau_N$, the latter are of minor

importance.

In the Heisenberg picture, our analysis depicts SF as follows: The operator electric field acting on an atom consists of free-field running waves constituting the vacuum field $\vec{E}_{vac}$, the dipole radiation field $\vec{E}_{dip}$ due to all other atoms (dipoles), and the atom's own radiation reaction field $\vec{E}_{RR}$. End-fire modes couple to the collective operator Bloch vector of the atomic system. The end-fire modes of the vacuum field $\vec{E}_{vac}$ may be considered as a fluctuating *field* source that causes the collective Bloch vector to start jittering in a sort of Brownian motion about its upright position. The dipole field $\vec{E}_{dip}$ is responsible for the drift motion of the Bloch vector: the tipping angle grows exponentially with time. Propagation effects and, in the nonlinear regime, the decrease of inversion, eventually determine the shape of the SF emission. Its delay τ_D depends on the properties of the fluctuating field source, i.e., on *the zero-point fluctuations of the vacuum waves in the specified solid angles.* Finally, the radiation reaction field $\vec{E}_{RR}$ merely accounts for incoherent energy loss on the time scale τ_N.

The delay τ_D is usually expressed in terms of the effective initial tipping angle. We find this angle to be about equal to $2/\sqrt{N}$, where N is the number of atoms in the active volume. We also discuss the statistics of the time delays with the aid of a Fokker-Planck equation for the probability density of the tipping angle.

The organization of this paper is as follows: In Sec. II, we discuss equations of motion with subsections covering radiation reaction, collective atomic operators, and equations of motion for slowly varying envelope amplitudes (SVEA). In Sec. III we discuss Maxwell-Bloch equations, with subsections covering initial development of SF, properties of the fluctuating source, and classical and statistical behavior. Section IV covers initial motion of the tipping angle, its subsections dealing with the response to the Langevin force, mean-squared tipping angle, and statistics of the SF pulse. Finally, in Sec. V, we discuss connections with other work and conclusions.

II. EQUATIONS OF MOTION

We consider N two-level atoms in a pencil-shaped volume $V=SL$ (density $\rho=N/V$). The position of atom j is denoted by $\vec{r}_j (j=1,\ldots,N)$. The positions are random, but atomic motion is disregarded ($T_2^*=\infty$). At the initial time $t=0$ all atoms are inverted and the radiation field is empty apart from the zero-point energy. The two-level atoms are described by the (Schrödinger) operators

$$R_j^+ = |e\rangle_{jj}\langle g|, \quad R_j^- = |g\rangle_{jj}\langle e|,$$
$$R_j^3 = \tfrac{1}{2}[\,|e\rangle_{jj}\langle e| - |g\rangle_{jj}\langle g|\,], \tag{1}$$

where $|e\rangle_j$ and $|g\rangle_j$ denote the excited- and ground-state vectors of the jth atom, respectively. These operators satisfy the angular momentum commutation relations for spin $\frac{1}{2}$ value.

The interaction Hamiltonian in the electric dipole approximation is given by[35]

$$H=\hbar\omega_0\sum_{j=1}^{N} R_j^3+\sum_{\lambda}\hbar\omega_\lambda a_\lambda^\dagger a_\lambda-\sum_{j=1}^{N}\vec{\mu}_j\vec{E}(\vec{r}_j), \tag{2}$$

where ω_0 is the two-level transition frequency and $\vec{\mu}_j=\vec{\mu}R_j^+ + \vec{\mu}^* R_j^-$ is the electric dipole operator of the jth atom. The electric field $\vec{E}$ consists of the sum of $\vec{E}^{(+)}$ and $\vec{E}^{(-)}$, where the positive frequency part is

$$\vec{E}^{(+)}(\vec{r})=i\sum_{\lambda}\vec{e}_\lambda e^{i\vec{k}\cdot\vec{r}}a_\lambda, \tag{3}$$

the negative frequency part $\vec{E}^{(-)}=(\vec{E}^{(+)})^\dagger$ and

$$\vec{e}_\lambda=(2\pi\hbar ck/\mathcal{V})^{1/2}\vec{\epsilon}_\lambda. \tag{4}$$

Here $\mathcal{V}$ is the quantization volume, $\lambda=(\vec{k},\sigma)$ where $\vec{k}$ is the wave vector and σ denotes the state of polarization, $\omega_\lambda=ck$, $\vec{\epsilon}_{\vec{k}\sigma}$ is the polarization vector, a_λ the annihilation operator, and $a_\lambda^\dagger$ the creation operator of the field in the state λ. These operators satisfy $[a_{\lambda'}, a_\lambda^\dagger]=\delta_{\lambda',\lambda}$. In (2) we have disregarded the self-energy and contact interaction term $2\pi\int|\vec{P}|^2 dV$, where $\vec{P}=\sum_j\vec{\mu}_j\delta(\vec{r}-\vec{r}_j)$. Consistently, we do not consider Lamb shifts.

To obtain a close though formal analogy between the quantum-mechanical and the classical equations of motion we use the Heisenberg picture. Moreover, we make the rotating wave approximation (RWA). This deserves some comment. The RWA is not suited[36] to deal with the dipole-dipole interaction between the atoms. The dipole-dipole dephasing time T_Δ is of the order of $\tau_N/[\rho(\lambda_0/2\pi)^3]$. Since we have assumed $\rho(\lambda_0/2\pi)^3\ll 1$ the dipole-dipole interaction can be ignored in our treatment of SF. The RWA is then also a valid assumption.

In the RWA, the Heisenberg equations of motion read

$$\frac{dR_j^3}{dt}=\frac{i}{\hbar}[R_j^+(t)\vec{\mu}\cdot\vec{E}^{(+)}(\vec{r}_j,t) - R_j^-(t)\vec{\mu}^*\cdot\vec{E}^{(-)}(\vec{r}_j,t)], \tag{5}$$

$$\frac{dR_j^-}{dt}=-i\omega_0 R_j^-(t)-\frac{2i}{\hbar}R_j^3(t)\vec{\mu}\cdot\vec{E}^{(+)}(\vec{r}_j,t), \tag{6}$$

$$\frac{da_\lambda}{dt}=-i\omega_\lambda a_\lambda(t)+\frac{1}{\hbar}\sum_j e^{-i\vec{k}\cdot\vec{r}_j}g_\lambda^* R_j^-(t), \tag{7}$$

where the coupling constant $g_\lambda=\vec{\mu}\cdot\vec{e}_\lambda$. The equa-

tions of motion for R_j^+ and $a_\lambda^\dagger$ follow from (6) and (7) by Hermitian conjugation.

Averages of an operator $Q(t)$ in the Heisenberg picture with respect to the radiation field reservoir (Re), atomic system (A), and total system (T) are given by $\langle Q\rangle_{\mathrm{Re}} = \langle \mathrm{vac}|Q|\mathrm{vac}\rangle$, $\langle Q\rangle_A = \langle I|Q|I\rangle$, and $\langle Q\rangle_T = \langle \mathrm{vac}, I|Q|\mathrm{vac}, I\rangle$. We have assumed here that at $t=0$ the radiation field is in the vacuum state $|\mathrm{vac}\rangle$, i.e., the radiation field reservoir Re is at zero temperature. The state $|I\rangle$ stands for all atoms inverted.

A. Radiation reaction

At $t=0$, the only field acting on the atoms is the vacuum field $\vec{E}_{\mathrm{vac}}$. Its positive frequency part is given by

$$\vec{E}_{\mathrm{vac}}^{(+)}(\vec{r}_j, t) = i\sum_\lambda a_\lambda(0)\vec{e}_\lambda e^{i\vec{k}\cdot\vec{r}_j - i\omega_\lambda t} \tag{8}$$

and corresponds to the free-field solution of (7). At later times the field acting on atom j consists of three parts,

$$\vec{E}(\vec{r}_j, t) = \vec{E}_{\mathrm{vac}}(\vec{r}_j, t) + \vec{E}_{\mathrm{dip}}(\vec{r}_j, t) + \vec{E}_{\mathrm{RR}}(\vec{r}_j, t)\,. \tag{9}$$

Here $\vec{E}_{\mathrm{dip}}$ is the electric field due to all other atoms, also called the Lorentz field. Its positive frequency part is given by

$$\vec{E}_{\mathrm{dip}}^{(+)}(\vec{r}_j, t) = \frac{i}{\hbar}\sum_{I\neq j}\sum_\lambda{}' g_\lambda^*\vec{e}_\lambda e^{i\vec{k}\cdot(\vec{r}_j - \vec{r}_I)} \times \int_0^t dt'\, e^{-i\omega_\lambda(t-t')}R_I^-(t')\,. \tag{10}$$

The source field $\vec{E}_{\mathrm{RR}}$, also called operator radiation reaction field,[36] satisfies

$$\vec{E}_{\mathrm{RR}}^{(+)}(\vec{r}_j, t) = \frac{i}{\hbar}\sum_\lambda g_\lambda^*\vec{e}_\lambda\int_0^t dt'\, e^{-i\omega_\lambda(t-t')}R_j^-(t')\,. \tag{11}$$

Although the total electric field $\vec{E}$ commutes at all times with R_j^+, R_j^-, and R_j^3, the fields $\vec{E}_{\mathrm{vac}}$, $\vec{E}_{\mathrm{dip}}$, and $\vec{E}_{\mathrm{RR}}$ separately do not. To deal with these fields we must choose once and for all the ordering of atomic and field operators in (5) and (6). Whether in spontaneous emission of a two-level atom, the decay time T_1 of the upper level occupation and the decay time T_2 of the dipole moment should be attributed to vacuum field fluctuations or to quantum-electrodynamic radiation reaction depends on that ordering.[37,38] There is no ordering which attributes T_1 entirely to the vacuum fluctuations.[37] In normal ordering both T_1 and T_2 can be considered to be due entirely to radiation reaction.[37] In that case there is a close but formal analogy between the quantum-electrodynamic treatment of radiative damping and Lorentz's classical analysis.[38] We choose normal ordering in the following so that $a_\lambda^\dagger$ appears to the left and a_λ to the right of atomic operators.

The radiation reaction field $\vec{E}_{\mathrm{RR}}$ does not add to the creation of spatial coherence between the atoms. We calculate the effect of $\vec{E}_{\mathrm{RR}}$ before introducing collective atomic operators. Approximating $R_j^-(t')$ in (11) by the free-atom evolution

$$R_j^-(t') = R_j^-(t)e^{-i\omega_0(t'-t)}\,, \tag{12}$$

we find

$$-\frac{i}{\hbar}\vec{\mu}\cdot\vec{E}_{\mathrm{RR}}^{(+)}(\vec{r}_j, t) = R_j^-(t)\left(\frac{1}{\hbar^2}\sum_\lambda |g_\lambda|^2\int_0^t e^{i(\omega_0-\omega_\lambda)\tau}d\tau\right). \tag{13}$$

Clearly the imaginary part of the expression in large parenthesis diverges. This corresponds in Lorentz's classical analysis with the divergence of the electron's Coulomb self-energy for vanishing electron radius (see Ref. 39). Neglecting this contribution, since it yields the Lamb shift, we find for $t \gg 1/\omega_0$ (Ref. 38)

$$(-i/\hbar)\vec{\mu}\cdot\vec{E}_{\mathrm{RR}}^{(+)}(\vec{r}_j, t) = \tfrac{1}{2}\gamma R_j^-(t)\,, \tag{14}$$

where the decay rate

$$\gamma = \frac{2\pi}{\hbar^2}\sum_\lambda |g_\lambda|^2\delta(\omega_\lambda - \omega_0) \tag{15}$$

corresponds to the natural lifetime via $\tau_N^{-1} = \gamma$. Using

$$\sum_\sigma \epsilon_i^\sigma\epsilon_j^\sigma = \delta_{ij} - \hat{k}_i\hat{k}_j \quad (i,j = x,y,z)\,, \tag{16}$$

where $\hat{k}_i = k_i/k$, we find as usual

$$\gamma = \tfrac{4}{3}|\vec{\mu}|^2 k_0^3/\hbar\,. \tag{17}$$

From (5), (6), (14), and the equal-time commutation rules of the atomic operators, we finally obtain

$$\frac{dR_j^3}{dt} = -\gamma(R_j^3 + \tfrac{1}{2}) + \frac{i}{\hbar}[R_j^+\vec{\mu}\cdot\vec{E}^{(+)}(\vec{r}_j, t) - \vec{\mu}^*\cdot\vec{E}^{(-)}(\vec{r}_j, t)R_j^-]\,, \tag{18}$$

$$\frac{dR_j^-}{dt} = -(i\omega_0 + \tfrac{1}{2}\gamma)R_j^- - \frac{2i}{\hbar}R_j^3\vec{\mu}\cdot\vec{E}^{(+)}(\vec{r}_j, t)\,, \tag{19}$$

where now $\vec{E} = \vec{E}_{\mathrm{vac}} + \vec{E}_{\mathrm{dip}}$. Note that normal order is employed in (18) and (19).

Having thus incorporated the radiation reaction we could proceed by incorporating $\vec{E}_{\mathrm{dip}}$ in a similar way. In many papers on SF, essentially such a procedure is followed.[12,13,15,16,19] However, to keep the calculations tractable retardation effects

are then left out of account (see Ressayre and Tallet[29]). We proceed in a different manner and derive operator valued Maxwell-Bloch equations which fully describe the propagation effects.

B. Collective atomic operators

The gain of waves traveling through the active medium is largest for end-fire modes of propagation. We assume that, since $\mathfrak{F}=1$, a plane-wave treatment of the two end-fire modes is sufficient to describe the gross features of the SF pulse. We ignore other modes. The end-fire modes only couple to certain collective atomic operators. We divide the pencil into slices of thickness d, oriented perpendicular to the pencil axis. The center position of a slice is indicated by the discrete variable x. Since $\rho S(\lambda_0/2\pi) \simeq 2\tau_N/\tau_R$ and $\tau_N \gg \tau_R$ we may choose d so that each slice contains many atoms but is still thin compared with $\lambda_0/2\pi$. We introduce the collective atomic operators

$$\mathfrak{R}_x^{+,3}(t)=\frac{1}{N_s}\sum_{j\in\{j\}_x} R_j^{+,3}(t)\,, \tag{20}$$

where N_s is the mean number of atoms in a slice and $\{j\}_x$ denotes the collection of atoms with (fixed) positions $\vec{r}_j$, where $x-d/2<x_j<x+d/2$. The collective atomic operators $\mathfrak{R}_x^-(t)$ satisfy

$$\frac{d\mathfrak{R}_x^-}{dt}=-(i\omega_0+\gamma/2)\mathfrak{R}_x^- -\frac{2i}{\hbar N_s}\sum_{j\in\{j\}_x} R_j^3\vec{\mu}\cdot\vec{\mathbf{E}}^{(+)}(\vec{r}_j,t)\,. \tag{21}$$

So far the positions of the atoms have remained fixed. In the spirit of the plane-wave end-fire mode assumption we require the collective operators to be essentially independent of the particular random positions of the atoms in a slice. Accordingly, we introduce a slice averaged electric field in (21) by integrating over $\vec{r}_j$ in a slice and dividing by Sd. Since $\rho(\lambda_0/2\pi)^3 \ll 1$, we may, at the same time, extend, with negligible error, the summation in the dipole field (Eq. 10) to all atoms. Then we find

$$\frac{d\mathfrak{R}_x^-}{dt}=-(i\omega_0+\gamma/2)\mathfrak{R}_x^- -\frac{2i}{\hbar}\mathfrak{R}_x^3\vec{\mu}\cdot\vec{\mathbf{E}}^{(+)}(x,t)\,, \tag{22}$$

where now $\vec{\mathbf{E}}(x,t)=\vec{\mathbf{E}}_{\rm vac}(x,t)+\vec{\mathbf{E}}_{\rm mat}(x,t)$,

$$\vec{\mathbf{E}}_{\rm vac}^{(+)}(x,t)=i\sum_\lambda F(k_y,k_z)a_\lambda(0)\vec{e}_\lambda e^{ik_x x-i\omega_\lambda t} \tag{23}$$

and the matter field is given by

$$\vec{\mathbf{E}}_{\rm mat}^{(+)}(x,t)=\frac{iN_s}{\hbar}\sum_{x'}\sum_\lambda F(k_y,k_z)g_\lambda^*\vec{e}_\lambda \times\int_0^t dt'\,e^{ik_x(x-x')-i\omega_\lambda(t-t')}\mathfrak{R}_{x'}^-(t')\,. \tag{24}$$

The diffraction function is

$$F(k_y,k_z)=\frac{1}{S}\int_S dy\,dz\,e^{ik_y y+ik_z z} \tag{25}$$

and finds its origin in the averaging over the atomic positions in a slice. Approximately, it equals one for wave vectors $\vec{k}(k\simeq k_0)$ inside the solid angles $\Delta\Omega\simeq\lambda_0^2/S$ around the x axis and vanishes outside these angles. Also $|k_x|\simeq k$; in the following we neglect the difference between $|k_x|$ and k, thereby ignoring diffraction losses. The new electric field $\vec{\mathbf{E}}(x,t)$ corresponds to the Maxwell field in the plane-wave end-fire modes.

The field $\vec{\mathbf{E}}_{\rm mat}(x,t)$ equals the equiphase plane average of the microscopic matter field at an "aufpunkt" $\vec{r}=(x,y,z)$ ($\vec{r}\neq\vec{r}_l$ for all l), i.e.,

$$\vec{\mathbf{E}}_{\rm mat}(\vec{r},t)=\frac{i}{\hbar}\sum_{x'}\sum_\lambda g_\lambda^*\vec{e}_\lambda\int_0^t dt'\,e^{-i\omega_\lambda(t-t')}\sum_{l\in\{l\}_{x'}}\mathfrak{R}_l^-(t')\left\{\frac{1}{Sd}\int_{x-d/2}^{x+d/2}dx_l\int\int_S dy_l\,dz_l\,e^{i\vec{k}\cdot(\vec{r}-\vec{r}_l)}\right\}. \tag{26}$$

The necessity of such equiphase plane averaging for cases where the mean interatomic distance is much larger than $\lambda_0/2\pi$ was stressed by Kramers[40]; see also Reiche.[41]

By the particular procedure used to derive (22)–(24), the operator character of the atomic variables is strictly preserved. If, by ensemble averaging (in x),[42] continuous collective operators had been introduced, the operation of the resulting operators on the Hilbert space of 2^N atomic states would not be well defined.[32] A continuum description of SF requires quantization *after* the introduction of the continuum variables.

The equation of motion for $\mathfrak{R}_x^3(t)$ is found to be given by

$$\frac{d\mathfrak{R}_x^3}{dt}=-\gamma(\mathfrak{R}_x^3+\tfrac{1}{2})+\frac{i}{\hbar}[\mathfrak{R}_x^+\vec{\mu}\cdot\vec{\mathbf{E}}^{(+)}(x,t) -\vec{\mu}^*\cdot\vec{\mathbf{E}}^{(-)}(x,t)\mathfrak{R}_x^-]\,. \tag{27}$$

Equations (22)–(24) and (27) describe the coupling of the end-fire modes and the collective atomic operators.

The damping rate $\frac{1}{2}\gamma$ of $\mathcal{R}_x^-$ and $\mathcal{R}_x^+$ can now be understood as (Rayleigh) scattering, via the induced dipole moments, of end-fire mode light out of the specified solid angles. The damping rate γ of $\mathcal{R}_x^3$ is due to lateral spontaneous emission. The random positions of the atoms ensure the incoherence of the lateral scattering and emission.

C. Equations of motion for slowly varying envelope amplitudes

The end-fire modes of propagation of $\vec{E}_{vac}$, given by (23), start to drive the atomic system at $t=0$. The resulting dipole moment with components $\mathcal{R}_x^+(t)$ and $\mathcal{R}_x^-(t)$ induces the matter field $\vec{E}_{mat}$, as given by (24). Due to the frequency selection by the atomic transition, the Fourier spectrum of atomic and field variables vanishes outside a narrow range of frequencies around ω_0. Ignoring the coupling between left (L; towards negative x values) and right (R) traveling end-fire modes, we can thus introduce

$$\mathcal{R}_x^-(t)=\mathcal{R}_R^-(x,t)e^{ik_0x-i\omega_0t}+\mathcal{R}_L^-(x,t)e^{-ik_0x-i\omega_0t}, \quad (28)$$

where $\mathcal{R}_R^-$ and $\mathcal{R}_L^-$ are slowly varying envelope amplitudes (SVEA), both in t and x. Using (22) we find $\mathcal{R}_R^-(x,t)$ to satisfy

$$\left(\frac{\partial}{\partial t}+\tfrac{1}{2}\gamma\right)\mathcal{R}_R^-(x,t)=2\mathcal{R}_x^3A_R^{(+)}(x,t)+F_R^{(+)}(x,t), \quad (29)$$

where

$$A_R^{(+)}(x,t)=\frac{N_s}{\hbar^2}\sum_{x'}\sum_{\sigma}\sum_{\vec{k}\in\Delta\Omega_R}|g_{\vec{k}\sigma}|^2 \times\int_0^t dt'\exp\{i(k-k_0)[x-x'-c(t-t')]\}\times\mathcal{R}_R^-(x',t') \quad (30)$$

and

$$F_R^{(+)}(x,t)=\frac{2}{\hbar}\mathcal{R}_x^3\sum_{\sigma}\sum_{\vec{k}\in\Delta\Omega_R}g_{\vec{k}\sigma}a_{\vec{k}\sigma}(0)\exp[i(k-k_0)(x-ct)]. \quad (31)$$

The "Rabi frequency" $\vec{A}_R^{(+)}(x,t)=-i\vec{\mu}\cdot\vec{E}_{mat,R}^{(+)}(x,t)/\hbar$, where $\vec{E}_{mat,R}$ is the (R) SVEA of the matter field. The operator $F_R^{(+)}(x,t)$ represents the Langevin force due to the (R) end-fire modes in the zero-point fluctuations. We evaluate (30). Let the quantization volume $\mathcal{V}$ approach infinity. The main contribution to the resulting k integral arises from $k\simeq k_0$. The sum $\sum_\sigma|g_{\vec{k}\sigma}|^2$ is calculated using (16) and $\vec{k}\in\Delta\Omega_R$, where $\Delta\Omega$ can be calculated by integrating the diffraction function (25) with respect to k_y and k_z. The value of $\Delta\Omega$ depends somewhat on the specific shape of the cross section. We take $\Delta\Omega=(2\pi)^2k_0^{-2}S^{-1}$. Putting[43]

$$\int_0^\infty dk\,e^{i(k-k_0)\alpha}=2\pi\delta(\alpha), \quad (32)$$

we finally obtain

$$A_R^{(+)}(x,t)=\frac{1}{\tau_R}\cdot\frac{N}{N_s}\sum_{x'\le x}\mathcal{R}_R^-(x',t-(x-x')/c), \quad (33)$$

where $N/N_s=L/d$ is the number of slices and τ_R is the radiation damping time of the collective system,

$$\tau_R^{-1}=J\frac{\lambda_0^2}{4\pi S}N\tau_N^{-1}. \quad (34)$$

The ratio

$$J=(|\mu_y|^2+|\mu_z|^2)/(\tfrac{2}{3}|\vec{\mu}|^2) \quad (35)$$

measures the amount of anisotropy in the two-level atomic transition. For instance, for a dipole moment along a transverse direction $J=\frac{3}{2}$, while for circular polarization around a transverse direction $J=\frac{3}{4}$. The ratio $\lambda_0^2/4\pi S$ is the fraction of solid angle into which the SF emission goes.

When x and x' in (33) are considered as continuous variables, and $A_R^{(+)}$ and $\mathcal{R}_R^-$ are consistently considered as c numbers, the SVEA $A_R^{(+)}(x,t)$ satisfies the Maxwell equation

$$\left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)A_R^{(+)}(x,t)=\frac{1}{\tau_R(L/c)}\mathcal{R}_R^-(x,t) \quad (36)$$

with the no-backscattering boundary condition $A_R^{(+)}(0,t)=0$. We use (36) for operators too, but its solution is then to be understood in the form of Eq. (33).

The equations of motion for (L) SVEA of atomic and field variables are given by (29), (31), and (36) with $R\to L$ and $t-x/c\to t+x/c$. The (R) and (L) SVEA's are coupled by the equations of motion for $\mathcal{R}^3(x,t)\equiv\mathcal{R}_x^3(t)$. We find from (27),

$$\left(\frac{\partial}{\partial t}+\gamma\right)(\mathcal{R}^3(x,t)+\tfrac{1}{2})=-[\mathcal{R}_R^+(x,t)A_R^{(+)}(x,t)+\mathcal{R}_L^+(x,t)A_L^{(+)}(x,t)+\text{H.c.}]+G(x,t). \quad (37)$$

The Langevin force $G(x,t)$ depends linearly on both the (L) and (R) SVEA of $\mathcal{R}^+$ and $\mathcal{R}^-$. Its explicit expression will not be given since we shall not need it.

Nonvanishing reservoir averages of correlation functions of the Langevin forces determine the statistics of the initiation of SF. The origin of these forces does not lie in the material itself, but in the zero-point fluctuations. Material fluctuating sources will and can be left out of account in this treatment of SF from extended and rarefied $(\rho\lambda_0/2\pi)^3 \ll 1$ systems.

III. MAXWELL-BLOCH EQUATIONS

In the following we will only be interested in correlation functions of the type

$$g_R^{pq} = \langle\langle (\mathcal{R}_R^-)^p (\mathcal{R}_R^+)^q \rangle\rangle_T, \quad g_L^{pq} = \langle\langle (\mathcal{R}_L^-)^p (\mathcal{R}_L^+)^q \rangle\rangle_T \tag{38}$$

with the same arguments (x,t) in the atomic variables, and where p and q are non-negative integers. The antinormal order in (38) is chosen consistent with the fact that only atoms in the excited state detect ("feel") the zero-point fluctuations; at $t=0$, $g_R^{pq}=0$ and $g_L^{pq}=0$ for all p and q. Since the atomic system never develops a macroscopic dipole moment g_L^{10}, g_R^{10}, g_L^{01}, and g_R^{01} vanish. The second-order correlation functions $g_R^{11}(x,t)$ and $g_L^{11}(x,t)$ measure the development of the mean-squared tipping angle of the collective Bloch vector corresponding to (R) and (L) end-fire modes, respectively. The higher-order correlation functions determine the statistics of the SF pulses. The restriction to correlation functions of the type (38) is not a serious limitation (see *Note added in proof*). All correlation functions of $\mathcal{R}_{R,L}^+$ and $\mathcal{R}_{R,L}^-$ can be expressed in terms of $g_{R,L}^{pq}$ through the equal time commutation rules for atomic operators.

With the above restriction in mind we derive in Sec. III A Maxwell-Bloch equations with a fluctuating source to describe the initial stages of SF. The corresponding regime of times is determined by the range of validity of the assumption $\mathcal{R}^3(x,t) = \mathcal{R}^3(x,0)$. This regime presumably includes many collective radiation times but is certainly much smaller than the natural lifetime τ_N. Section III B discusses the properties of the fluctuating source. Together III A and III B contain all ingredients needed in Sec. IV for the evaluation of the initial motion of the tipping angle. In III C we discuss the transition to classical behavior in view of its use in the nonlinear regime.

A. Initial development of SF

We assume $\mathcal{R}^3(x,t) = \mathcal{R}^3(x,0)$. The (R) and (L) SVEA's of atomic and field variables are then decoupled. We consider only (R) components and omit the subscript R for the moment. The equation of motion for $\mathcal{R}^-(x,t)$ contains an intrinsic operator character due to the presence of the drift term $2\mathcal{R}^3(x,0)A^{(+)}(x,t)$. To obtain a (formal) classical description we consider the expectation values $\mathcal{R}_{\alpha\beta}^- = \langle\alpha|\mathcal{R}^-|\beta\rangle$, $A_{\alpha\beta}^{(+)}$ and $F_{\alpha\beta}^{(+)}$, where $|\alpha\rangle$ and $|\beta\rangle$ are representatives of a complete set of 2^N normalized atomic system eigenfunctions. For $|\alpha\rangle \neq |I\rangle$ but else arbitrary, the equations of motion for $\mathcal{R}_{I\alpha}^-$ and $A_{I\alpha}^{(+)}$ are sourceless [$\mathcal{R}^3(x,0)$ and $\alpha_{\vec{k}\sigma}^-(0)$ commute, $\mathcal{R}_{I\alpha}^3(x,0)=0$] and show no explicit operator character [$\mathcal{R}_{II}^3(x,0)=\frac{1}{2}$]. Since initially $\mathcal{R}_{I\alpha}^- = 0$, we have $\mathcal{R}_{I\alpha}^-(x,t)=0$ as long as the approximation $\mathcal{R}^3(x,t)=\mathcal{R}^3(x,0)$ is valid. Also $\mathcal{R}_{\alpha I}^+(x,t)=0$. The correlation function g^{pq} attains the particularly simple form

$$g^{pq} = \langle\langle (\mathcal{R}_{II}^-)^p (\mathcal{R}_{II}^+)^q \rangle\rangle_{\mathrm{Re}}. \tag{39}$$

For the II diagonal elements we obtain

$$\frac{\partial}{\partial t}\mathcal{R}_{II}^- = -\frac{\gamma}{2}\mathcal{R}_{II}^- + A_{II}^{(+)} + F^{(+)}, \tag{40}$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) A_{II}^{(+)} = \frac{1}{\tau_R(L/c)}\mathcal{R}_{II}^-, \tag{41}$$

where now the Langevin force is given by

$$F^{(+)}(x,t) = \frac{1}{\hbar}\sum_{\sigma}\sum_{\vec{k}\in\Delta\Omega_R} g_{\vec{k}\sigma}\, a_{\vec{k}\sigma}^-(0)\exp[i(k-k_0)(x-ct)]. \tag{42}$$

The variables $\mathcal{R}_{II}^-$ and $A_{II}^{(+)}$ are still operators on the radiation field reservoir Re, but Eqs. (40) and (41) show no explicit operator character anymore. However, that character does express itself via the Langevin force $F^{(+)}$. For instance

$$\langle F^{(+)}(x,t)F^{(-)}(x',t')\rangle_{\mathrm{Re}}$$

attains a finite value while

$$\langle F^{(-)}(x,t)F^{(+)}(x',t')\rangle_{\mathrm{Re}}$$

vanishes. This is consistent with the fact that the zero-point fluctuations only affect the excited state of an atom. To calculate the correlation function g^{pq} we only need antinormally ordered correlation functions of the Langevin force. On a formal basis we may thus treat the operators $\mathcal{R}_{II}^-$ and $A_{II}^{(+)}$ as if they are (complex valued) c numbers r and Ω, and consider $F^{(+)}$ as a classical fluctuating force f. Reservoir average $\langle\cdots\rangle_{\mathrm{Re}}$ and classical ensemble average $\langle\cdots\rangle$ are then to be identified.

B. Properties of the fluctuating field source

The Langevin force $f_R(x,t)$, corresponding to $F_R^{(+)}(x,t)$ defined by (42), is a function of $\tau = t - x/c$ only; the subscript R has been reintroduced now. The properties of $f_R(x,t)$ can be summarized as follows:

$$\langle f_R(\tau_1)\cdots f_R(\tau_p) f_R^*(\tau_1')\cdots f_R^*(\tau_q')\rangle = 0 \tag{43}$$

for $p \neq q$,

$$\langle f_R(\tau_1)\cdots f_R(\tau_p) f_R^*(\tau_1')\cdots f_R^*(\tau_p')\rangle = \sum_P \prod_{j=1}^{p} \langle f_R(\tau_j) f_R^*(\tau_{Pj}')\rangle, \quad (44)$$

where the sum is over all possible permutations of times, and

$$\langle f_R(\tau) f_R^*(\tau')\rangle = (1/N\tau_R)\delta(\tau - \tau'). \quad (45)$$

The first and second property, (43) and (44), can be verified immediately using the generalized Wick's theorem for Boson operators.[44] These properties state that the Langevin force f_R constitutes a two-dimensional Gaussian process. Using $\langle a_\lambda(0) a_\lambda^\dagger(0)\rangle_{\mathrm{Re}} = \delta_{\lambda\lambda}$ we find

$$\langle f_R(\tau) f_R^*(\tau')\rangle = \frac{1}{\hbar^2} \sum_{\mathbf{k}\in\Delta\Omega_R} \sum_\sigma |g_{\mathbf{k}\sigma}|^2 \exp[-ic(k-k_0)(\tau-\tau')]. \quad (46)$$

Since our only interest is in variations of $\langle f_R(\tau) f_R^*(\tau')\rangle$ over intervals of time $|\tau - \tau'| \gg \tau_c$, where $\tau_c \simeq 1/\omega_0$ is the correlation time of the reservoir, we may use (32) with $\alpha = c(\tau' - \tau)$; (see Ref. 44, p. 426.) Then the third property of f_R, described by (45), immediately follows.

The properties of f_L are obtained from (43)–(45) by replacing $R \rightarrow L$ and putting $\tau = t + x/c$. The forces f_L and f_R are fluctuating Gaussian field sources which cause the collective Bloch vectors r_L and r_R to start jittering, in a sort of Brownian motion, about their upright position. The correlation time of the forces is taken to be infinitely short as described by (45).

C. Classical and statistical behavior

For increasing tipping angles it is reasonable to assume that the operator character of both atomic and field variables gets less and less important. We therefore assume that after some time a classical regime emerges where all operators are c numbers and are identified as follows: $r_{L,R} = (\mathcal{R}_{L,R}^-)_{II}$, $r_{L,R}^* = (\mathcal{R}_{L,R}^+)_{II}$, $\Omega_{L,R} = (A_{L,R}^{(+)})_{II}$, $\Omega_{L,R}^* = (A_{L,R}^{(-)})_{II}$, and $n = 2\mathcal{R}_{II}^3$. The equations of motion for these variables read

$$\frac{\partial n}{\partial t} = -\gamma(n+1) - 4\,\mathrm{Re}(r_L\Omega_L^* + r_R\Omega_R^*), \quad (47)$$

$$\frac{\partial r_{L,R}}{\partial t} = -\tfrac{1}{2}\gamma r_{L,R} + n\Omega_{L,R} + f_{L,R}, \quad (48)$$

$$\left(\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}\right)\Omega_{L,R} = \frac{1}{\tau_R(L/c)} r_{L,R}. \quad (49)$$

The + and − signs in (49) correspond to (R) and (L) waves, respectively. Note that the Langevin force $G(x,t)$ is never important since initially $G_{II} = 0$. Equations (47)–(49) are Maxwell-Bloch equations including the classical Gaussian noise source of Sec. IIIB which accounts for the stochastic initiation of SF. These equations are identical in form to those of MacGillivray and Feld.[27] However, our noise source f is a *field* source, not a polarization source as in their work, and has different properties.

In our classical picture the result of each individual SF experiment corresponds to a single and unique path of development of the SF pulse determined by one representative out of all possible noise source functions $f(\tau)$. The ensemble averages to be calculated in Sec. IV then must be understood as averages over many repeated experiments. In the classical regime the amplitude of the matter field is already large compared with the vacuum field and mainly determines the further built up of SF. The behavior of SF becomes classical at a time of the order of a few τ_R and long before nonlinear behavior due to decreasing inversion sets in. The classical description in terms of the Maxwell-Bloch equations (47)–(49) finds its specific use in the nonlinear regime where the assumption $\mathcal{R}^3(x,t) = \mathcal{R}^3(x,0)$ as used in Sec. IIIA is no longer valid.

IV. INITIAL MOTION OF THE TIPPING ANGLE

In this section we investigate consequences of the theory presented in Secs. I–III for the initial motion of the tipping angle of the collective Bloch vector. We use the atomic variables $r_{L,R}$ and $r_{L,R}^*$, formally introduced in IIIA, with due regard to proper ordering of the corresponding operators. However, as far as the main purpose of this section is concerned, i.e., the study of the statistical behavior of the SF pulses, we may ignore the operator character and consider them to be classical variables as was done in IIIC. In particular, we discuss the response of the collective atomic variable $r_{L,R}$ to the fluctuating Langevin field source $f_{L,R}$ in Sec. IVA, the evolution of the mean-squared tipping angle of the collective Bloch vector in IVB, and the statistical behavior of the SF pulse with special emphasis on the statistics of the time delays in IVC.

The tipping angle θ is introduced as follows: Formally disregarding incoherent damping, the left traveling waves, and the fluctuating force, we find from (47) and (48), omitting the subscript R,

$$4rr^* + n^2 = 1. \quad (50)$$

Expressing r in its amplitude and phase via $r = |r| \exp i\phi$, Eq. (50) is satisfied by

$$2|r| = \sin\theta\,, \quad n = \cos\theta\,. \tag{51}$$

If $\Omega = |\Omega| \exp i\phi$ and ϕ is independent of t, we have according to (48), the relation

$$\theta(x,t) = \theta(x,0) + \int_0^t 2|\Omega(x,t')|\,dt'\,. \tag{52}$$

Equations (51) and (52) illustrate that θ may be interpreted as the tipping angle of the collective Bloch vector $2r$. In the following we discuss the initial development of the tipping angle. Then $n = 1$, $\theta = 2|r|$, and incoherent damping may be left out of account.

A. Response to the Langevin force

In dimensionless coordinates $T = (t - x/c)/\tau_R$ and $X = x/L$, the equations of motion read

$$\frac{\partial r}{\partial T} = \hat{\Omega} + \hat{f}(T)\,, \tag{53}$$

$$\frac{\partial \hat{\Omega}}{\partial X} = r\,, \tag{54}$$

where $\hat{\Omega} = \Omega\tau_R$ and $\hat{f} = f\tau_R$. The initial condition will be given on the light "cone" for right traveling waves

$$r(X, T=0) = 0\,. \tag{55}$$

Consistent with the derivation of (54) (see Sec. III) the backscattered field is neglected:

$$\hat{\Omega}(X=0, T) = 0\,. \tag{56}$$

Equations (53)–(56) are solved by constructing the Green's functions $r_G(X, T; T')$ and $\hat{\Omega}_G(X, T; T')$, satisfying

$$\frac{\partial r_G}{\partial T} = \hat{\Omega}_G + \delta(T - T')\,, \tag{57}$$

$$\frac{\partial \hat{\Omega}_G}{\partial X} = r_G\,, \tag{58}$$

and the boundary conditions

$$r_G(X, T < T'; T') = 0\,, \quad \hat{\Omega}_G(X=0, T; T') = 0\,. \tag{59}$$

Integration of (57) with respect to T from $T' - 0$ to $T' + 0$ yields

$$r_G(X, T = T' + 0; T') = 1\,, \tag{60}$$

since the "field" $\hat{\Omega}_G$ does not change in the passage of the δ-function pulse. No backscattering implies

$$r_G(X=0, T > T'; T') = 1\,. \tag{61}$$

This relation also follows from integration of (57) at $X = 0$ from $T' - 0$ to T, using (59). For $T > T'$, r_G satisfies

$$\frac{\partial^2 r_G}{\partial T \partial X} = r_G\,. \tag{62}$$

This equation together with the symmetric boundary conditions (60) and (61) is solved by change of variables to $2[X(T - T')]^{1/2}$ (cf. Burnham and Chiao[45]). By linear superposition we obtain

$$r_R(X, T) = \int_0^T dT'\, I_0[2\sqrt{X(T - T')}]\, \hat{f}_R(T')\,, \tag{63}$$

where I_0 is the modified Bessel function of order 0 and the subscript R has been reintroduced.

Initial condition (55) corresponds to (swept) excitation of the atomic system by a right traveling short electromagnetic pulse. For the SVEA of left traveling components the initial conditions are not given on the corresponding light "cone." Different initial conditions are also met for uniform (lateral) excitation. The resulting problems can be solved using the Green's function technique and Riemann's integration method.[46] However, if the escape time L/c is short compared with τ_R, the results are essentially described by equations of the form (63).

B. Mean-squared tipping angle

Starting from its upright position the collective Bloch vector r_R moves down on the Bloch sphere. According to (63), (45), and $\theta_R^2 = 4 r_R r_R^*$, its mean-squared tipping angle increases as

$$\langle \theta_R^2(X, T) \rangle = \frac{4}{N} \int_0^T dT'\, I_0^2(2\sqrt{XT'})\,. \tag{64}$$

Its phase ϕ_R remains completely random: $\langle r_R(X, T) \rangle = 0$, manifesting the absence of a preferential direction of motion down from the top of the Bloch sphere. We note once again that the ensemble average corresponds to an average of many repeated experiments. The initial development of $\langle \theta_R^2 \rangle$ is determined by diffusion,

$$\langle \theta_R^2 \rangle^{1/2} = \left(\frac{4}{N}\frac{\tau}{\tau_R}\right)^{1/2}, \quad (\tau \ll \tau_R)\,. \tag{65}$$

For $\tau \gg \tau_R$, the gain due to the atomic inversion makes the increase of $\langle \theta_R^2 \rangle$ at the end face $x = L$ purely exponential,

$$\langle \theta_R^2 \rangle^{1/2} \simeq (2\pi N)^{-1/2} \exp[2(\tau/\tau_R)^{1/2}]\,. \tag{66}$$

We define the delay of the SF pulse as the time needed for $\langle \theta_R^2 \rangle$ to become equal to one at the end face. From (66) we now arrive at a major quantitative result of our quantum-mechanical and fully retarded treatment of SF: the delay

$$\tau_D \simeq \frac{\tau_R}{4}(\ln\sqrt{2\pi N})^2\,. \tag{67}$$

In contrast, mean-field quantum-mechanical treat-

ments of SF yield a linear dependence of τ_D on $\ln N$ (cf. Refs. 19, 25).

If we introduce an effective initial tipping angle θ_0 to simulate the effect of the fluctuating force f_R on $\langle\theta_R^2\rangle$, we obtain

$$\theta_{\mathrm{eff},R}^2(X,T)=\theta_0^2 I_0^2(2\sqrt{XT})\,. \tag{68}$$

At the end face and for $\tau \gg \tau_R$ the effective tipping angle increases as

$$\theta_{\mathrm{eff},R}=\theta_0(4\pi)^{-1/2}(\tau/\tau_R)^{-1/4}\exp[2(\tau/\tau_R)^{1/2}]\,. \tag{69}$$

The increase is slower than exponential because the effective initial tipping angle can be considered to simulate a noise source acting for a finite time only. In order to get equal delays from (66) and (69), we thus need a value of θ_0, which is somewhat larger than $2/\sqrt{N}$, namely,

$$\theta_0=[(4/N)\ln(2\pi N)^{1/8}]^{1/2}. \tag{70}$$

With the present interpretation of the effective tipping angle, (70) represents the value of θ_0 to be used in numerical calculations on the basis of the Maxwell-Bloch equations.

The above obtained exponential increase of the tipping angle ceases to be valid for large tipping angles. In the effective initial tipping angle description, and ignoring chirp by taking ϕ_R to be constant, nonlinearity due to decreasing inversion can be easily included. We find the Sine-Gordon equation

$$\frac{\partial^2\theta_{\mathrm{eff},R}}{\partial X\partial T}=\sin\theta_{\mathrm{eff},R}\,, \tag{71}$$

with the initial and boundary conditions

$$\theta_{\mathrm{eff},R}(X,T=0)=\theta_{\mathrm{eff},R}(X=0,T)=\theta_0\,. \tag{72}$$

Such a set of equations has been studied by Burnham and Chiao.[45] The effective tipping angle $\theta_{\mathrm{eff},R}$ exhibits oscillatory behavior as a function of X and T, called "ringing". This is due to exchange of energy between the atomic system and the field.

C. Statistics of the SF pulse

We define the probability density function $P_R^{(a)}(\alpha,\alpha^*,x,t)$ of the random variable r_R by

$$P_R^{(a)}(\alpha,\alpha^*,x,t)=\langle\delta(\alpha-r_R)\delta(\alpha^*-r_R^*)\rangle\,. \tag{73}$$

The superscript a refers to the chosen antinormal order of the corresponding atomic operators $\mathcal{R}_R^-$ and $\mathcal{R}_R^+$. Ensemble averages of the form $\langle r_R^p r_R^{*q}\rangle$ can be expressed as moments of the distribution function,

$$\langle r_R^p r_R^{*q}\rangle=\int \alpha^p\alpha^{*q}P_R^{(a)}(\alpha,\alpha^*,x,t)\,d^2\alpha\,, \tag{74}$$

where the two-dimensional integral is over the complex α plane. A convenient representation of the probability density is obtained by expressing the δ functions in Fourier transforms. We obtain

$$P_R^{(a)}(\alpha,\alpha^*,x,t)=\frac{1}{\pi^2}\int d^2\xi\, e^{-i\xi\alpha-i\xi^*\alpha^*}C_R(\xi,\xi^*,x,t)\,, \tag{75}$$

with the characteristic function

$$C_R(\xi,\xi^*,x,t)=\langle e^{i\xi r_R}e^{i\xi^* r_R^*}\rangle\,. \tag{76}$$

By definition the characteristic function can be expressed in a linked average[47]

$$C_R(\xi,\xi^*,x,t)=\exp[\langle e^{i\xi r_R}e^{i\xi^* r_R^*}-1\rangle_{\mathrm{Li}}]\,. \tag{77}$$

The term in square brackets is expressed in terms of linked averages of the form

$$\langle f_R(\tau_1)\cdots f_R(\tau_p)f_R^*(\tau_1')\cdots f_R^*(\tau_q')\rangle_{\mathrm{Li}}$$

by expansion into a power series in ξ and ξ^* and use of (63). All of these averages except

$$\langle f_R(\tau)f_R^*(\tau')\rangle_{\mathrm{Li}}=\langle f_R(\tau)f_R^*(\tau')\rangle$$

vanish due to the Gaussian properties (43) and (44) of f_R. The characteristic function is thus given by

$$C_R(\xi,\xi^*,x,t)=\exp(-|\xi|^2\langle r_R r_R^*\rangle)\,. \tag{78}$$

Using (75) we obtain the two-dimensional Gaussian distribution

$$P_R^{(a)}(\alpha,\alpha^*,x,t)=\frac{\exp(-|\alpha|^2/\langle r_R r_R^*\rangle)}{\pi\langle r_R r_R^*\rangle}\,. \tag{79}$$

The Gaussian properties of the fluctuating Langevin force f_R ensure that the stochastic variation of the collective Bloch vector r_R is a Fokker-Planck process in two dimensions on the Bloch sphere. The corresponding Fokker-Planck equation with dimensionless coordinates X (fixed) and T reads

$$\frac{\partial P}{\partial T}=\frac{4}{N}I_0^2(2\sqrt{XT})\frac{\partial^2 P}{\partial\alpha\,\partial\alpha^*}\,, \tag{80}$$

where we have used $\langle r_R r_R^*\rangle=\frac{1}{4}\langle\theta_R^2\rangle$ and (64). Introducing, by analogy with $r_R=\frac{1}{2}\theta_R\exp(i\phi_R)$, amplitude and phase of α through $\alpha=\frac{1}{2}\eta\exp(i\chi)$, we find that the probability $p_R\,d\eta^2\,d\chi$ for θ_R^2 to be in between η^2 and $\eta^2+d\eta^2$ and ϕ_R to be in between χ and $\chi+d\chi$, is given by

$$p_R\,d\eta^2\,d\chi=\frac{d\chi}{2\pi}\frac{\exp(-\eta^2/\langle\theta_R^2\rangle)}{\langle\theta_R^2\rangle}\,d\eta^2\,. \tag{81}$$

This again illustrates that, starting from its upright position, there is no preferential direction of motion for the collective Bloch vector on the Bloch sphere. Moreover, it shows that the values η^2 that θ_R^2 can take are exponentially distributed with mean value $\langle\theta_R^2\rangle$. Using (81) one can calculate

all higher-order correlation functions $\langle\theta_R^p\rangle$. As expected for a two-dimensional Gaussian distribution $\langle\theta_R^4\rangle = 2\langle\theta_R^2\rangle^2$.

It should be emphasized that the Fokker-Planck description given here is restricted to small tipping angles. Once decreasing inversion makes the equations of motion nonlinear, a complete description of the stochastic behavior of the SF pulse requires the introduction of correlation functions of the type $\langle(\mathcal{R}_R^-)^p(\mathcal{R}^3)^n(\mathcal{R}_R^+)^q\rangle_T$ (cf. Ref. 20).

Finally, we consider briefly the statistics of the time delays of the SF pulse, i.e., the fluctuations of the delay observed over many repeated experiments. A discussion of the statistical properties of these pulses in terms of only time delays assumes that the pulses have exactly the same shape, but varying positions on the time axis. Such pulses are furnished by extending (66) to

$$\theta_R = (\theta_1/2\pi)\exp[2(\tau/\tau_R)^{1/2}]\,, \tag{82}$$

where in accordance with (81) the random variable θ_1 has a probability density

$$p(\theta_1^2) = \langle\theta_1^2\rangle^{-1}\exp[-\theta_1^2/\langle\theta_1^2\rangle] \tag{83}$$

with $\langle\theta_1^2\rangle = 2\pi/N$. The statistics of the time delays

$$\tau_D(\theta_1) = \tfrac{1}{4}\tau_R[\ln(\theta_1/2\pi)]^2 \tag{84}$$

is thus interpreted in terms of the stochastic behavior of the "initial" tipping angle $\theta_1/2\pi$ as described by (83). The mean delay $\langle\tau_D\rangle$ is given by

$$\langle\tau_D\rangle = \int_0^\infty d\theta_1^2\, p(\theta_1^2)\tau_D(\theta_1) = \frac{\tau_R}{16}\left(y^2+\frac{\pi^2}{6}\right), \tag{85}$$

where $y = \gamma_E + \ln(2\pi N)$ and $\gamma_E = 0.577\,21\cdots$ (Ref. 48) is Euler's constant. The mean delay differs only slightly from the delay (67) of the "mean" pulse. The mean-squared delay satisfies

$$\langle\tau_D^2\rangle = \langle\tau_D\rangle^2 + \left(\frac{\tau_R}{16}\right)^2\left[\tfrac{2}{3}\pi^2y^2 + 8\zeta(3)y + \tfrac{11}{90}\pi^4\right], \tag{86}$$

where the Riemann ζ function $\zeta(3) = 1.202\,05\cdots$.[48] The variance $\Delta\tau_\rho = [\langle\tau_D^2\rangle/\langle\tau_D\rangle^2 - 1]^{1/2}$ varies roughly (error $<10\%$ for $10^5 < N < 10^{40}$) as $2.3/\ln N$. Similar inverse logarithmic behavior has been derived by Degiorgio[18] from an "intuitive" model, but the proportionality constant was 1.3 instead of 2.3. The difference is due to the fact that Degiorgio used the linear dependence of τ_D on $\ln\theta_1$ as obtained from a mean-field treatment of SF.

V. CONNECTIONS WITH OTHER WORK AND CONCLUSIONS

Most fully quantum-mechanical treatments of SF have started from master equations in some sort of Markov approximation.[12,13,15,16,19] However, in general this amounts to leaving retardation out of account. Recently, this aspect has been discussed at length by Ressayre and Tallet.[29]

The semiclassical treatment of SF given by MacGillivray and Feld[27] is substantiated by our work. However, their fluctuating source differs from ours. The effective initial tipping angle derived by MacGillivray and Feld[27] equals $(\sqrt{2\pi N})^{-1/2}(\alpha L)^{-3/4}$, where $\alpha L = T_2'/\tau_R$ is the amplitude gain at line center. In our opinion T_2', being much larger than τ_R in order to have ideal conditions for SF, should not be involved in the initiation of SF. Bonifacio and Lugiato[26] have justified the $2/\sqrt{N}$ effective initial tipping angle from a fully quantum-mechanical treatment.

The time delay of the SF pulse derived by us satisfies $\tau_D = (\tau_R/4)[\ln\sqrt{2\pi N}\,]^2$. Eberly and Rehler[14] as well as Banfi and Bonifacio[25] obtain $\tau_D \simeq \tau_R \ln\sqrt{\mu N}$, where $\mu \simeq \lambda_0^2/S$. This would correspond to an effective initial tipping angle $\sim 1/\sqrt{\mu N}$. The origin of these incorrectly determined delays appears to lie in the implicit assumption that all spontaneous emission goes into the specified solid angles (see also Ref. 29).

Our principal assumption has been that, for $\mathfrak{F} = 1$, SF is described properly by the interaction of atoms with plane-wave end-fire modes. Consistently, field inhomogeneities in a cross section of the pencil are neglected. The problem of the so-called ray formation was first tackled by Ernst and Stehle[11] by using a Wigner-Weisskopf type of approach. Bonifacio and Lugiato[26] have dealt with an all-mode quantum-mechanical treatment of SF. The spirit of their work is that the resulting Maxwell-Bloch equations for the end-fire modes contain extra loss terms. The magnitude of these losses is subject to discussion.[27,31,32,49]

We have also neglected the coupling between the two opposite running end-fire modes. One mode acts as a diffraction grating for the other, which thus introduces an internal reflectivity into the problem. The effect of this coupling is unknown in detail. It may, however, affect the above mentioned losses.

Inhomogeneous broadening has been ignored in our treatment: $T_2^* = \infty$. Ressayre and Tallet[23] and Bonifacio and Lugiato[26] have included inhomogeneous broadening in their description of SF under the assumption that atomic frequencies and positions are uncorrelated.

In conclusion, for a medium with density ρ satisfying

$$\lambda_0/(2\pi^2 L) \ll \rho(\lambda_0/2\pi)^3 \ll 1\,,$$

a pencil-shaped active volume with a Fresnel number $\mathfrak{F} = 1$ and with the above assumptions, we have quantum mechanically derived a description of the initiation of SF in terms of Maxwell-Bloch equa-

tions with a fluctuating field source. By the *ad hoc* introduction of classical behavior these equations are extended to include the description of nonlinear behavior due to decreasing atomic inversion. The initial development of SF is discussed in terms of both the behavior of the mean-squared tipping angle and the statistics of the tipping angle. The quantum-classical correspondence suggested in this paper is related to the quantum-c-number correspondence introduced by Glauber[50] and Sudarshan[51] for the free-field case, and extended to the case of interacting fields and multitimes by Haken and Weidlich[52] and by Lax.[47,53]

The need still exists, and this may be the most difficult problem in the theory of SF, for an *all* mode and fully retarded quantum-mechanical description of the *initiation* of SF from an arbitrarily shaped but extended active volume. Also, the transition to "classical" behavior of the SF pulse and the limits of validity of the present approach for higher density, especially with regard to the role of the dipole-dipole interaction, deserves further discussion.

Note added in proof. In the meantime, Glauber and Haake have published a quantum-mechanical theory of SF using normally ordered correlation functions [R. Glauber and F. Haake, Phys. Lett. A 68, 29 (1978)]. They arrive at essentially the same results for the average behavior of the initial growth of SF. The interpretation that the quantum-mechanical average corresponds to an average over many repeated experiments, not to the result of one individual experiment (see our Secs. III and IV), is not given in their paper.

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