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Theory of electromagnetic beams

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A relatively simple method for calculating the properties of a paraxial beam of electromagnetic radiation propagating in vacuum is presented. The central idea of the paper is that the vector potential field is assumed to be plane-polarized. The nonvanishing component of the vector potential obeys a scalar wave equation. A formal solution employing an expansion in powers of w_0/l is obtained, where w_0 is the beam waist and l the diffraction length. This gives the same result for the lowest-order components of the transverse and longitudinal electric field of a Gaussian beam that was derived by Lax, Louisell, and McKnight using a more complicated approach. We derive explicit expressions for the second-order transverse electric field and the third-order longitudinal field corrections.

I. INTRODUCTION

The theory of laser beams[1] has proven to be very successful in describing the character of radiation fields associated with stable spherical resonators. There are, however, two difficulties with the usual treatment. First, the solution is based on a paraxial approximation to the scalar wave equation, and no procedure is given for obtaining higher-order corrections. Secondly, the reader is typically told that the disturbance which approximately satisfies the scalar wave equation is the transverse component of the electric field,[2] but the other two components are not worked out. These difficulties have been overcome in a paper by Lax, Louisell and McKnight.[3] Their theory starts with the exact Maxwell equations and expands the electric field vector in powers of w_0/l, where w_0 and l are the scaling parameters for the beam waist and diffraction length, respectively. A contribution of the present paper to the theory of paraxial beams is that our procedure is considerably more simple than the one just mentioned. This relative simplicity stems from our proposal that the electromagnetic *vector potential* is linearly polarized, so that the nonvanishing component of $\vec{A}$ obeys a scalar wave equation. (Note that it is entirely consistent to assume the vector potential of a paraxial beam of radiation is confined to a single direction, whereas clearly this is not the case for the electric or magnetic fields.) By contrast, since Lax, Louisell, and McKnight deal with the electric field, it is necessary for them to solve a vector wave equation for $\vec{E}$.

It is shown that our approach easily reproduces the results of Lax, Louisell, and McKnight for the zeroth- and first-order fields of a Gaussian mode propagating in free space. Additionally, we are able to evaluate the second-order term in the expansion for the vector potential, leading to second- and third-order corrections for the electric field of a Gaussian beam in vacuum. This result, which to my knowledge is new, should be of use in applications where there is a need to obtain accurate solutions for the electromagnetic field of a strongly focused laser beam.

The profitability of the present procedure is probably restricted to the free field case. When a current exists which depends on the electric field, in general, the assumption that the vector potential is plane polarized fails. There is then no longer any obvious advantage to the method.

II. DEVELOPMENT OF FORMALISM

Consider an electromagnetic field which, using complex notation, varies as $e^{i\omega t}$. In the Lorentz gauge the vector potential obeys the inhomogeneous wave equation

$$\nabla^2\vec{A}+k^2\vec{A}=-(4\pi/c)\vec{J}\,, \tag{1}$$

where $k=2\pi/\lambda$. Since the scalar potential is given in terms of the vector potential via the Lorentz condition,

$$\Phi=(i/k)\nabla\cdot\vec{A} \tag{2}$$

the fields $\vec{B}$ and $\vec{E}$ may be expressed in terms of $\vec{A}$ alone. We have

$$\vec{B}=\nabla\times\vec{A}\,, \tag{3}$$

$$\begin{aligned}\vec{E}&=-\nabla\Phi-ik\vec{A}\\&=-(i/k)\nabla(\nabla\cdot\vec{A})-ik\vec{A}.\end{aligned} \tag{4}$$

To describe a paraxial beam, we assume $\vec{A}$ is polarized in the transverse direction. Use a Cartesian coordinate system where $\vec{A}$ is along the x axis and the beam propagates along the z axis. Then Eq. (1) reduces in empty space to the scalar relation

$$\nabla^2A+k^2A=0, \tag{5}$$

where we understand $A = A_x$(and $A_y = A_z = 0$). Anticipating that the waves are nearly plane, we take

$$A(\vec{r}) = \Psi(\vec{r})e^{-ikz}, \tag{6}$$

where Ψ is a slowly varying function. Inserting (6) into (5) gives

$$\nabla^2\Psi - 2ik\frac{\partial\Psi}{\partial z} = 0 . \tag{7}$$

A Gaussian beam has a width parameter w_0 at the waist. Introduce the dimensionless transverse variables $x = w_0\xi$ and $y = w_0\eta$. There is also a characteristic diffraction or spreading length $l = kw_0^2$, so that we let $z = l\zeta$. With these new variables Eq. (7) becomes

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi - 2i\frac{\partial\Psi}{\partial\zeta} + s^2\frac{\partial\Psi}{\partial\zeta^2}, \tag{8}$$

where we have defined

$$s = w_0/l = 1/kw_0 .$$

Note that so long as the beam waist parameter w_0 is large compared to λ, then s is small compared to unity. Thus it is natural to seek solutions of Eq. (8) of the form

$$\Psi = \Psi_0 + s^2\Psi_2 + s^4\Psi_4 + \cdots . \tag{9}$$

It is seen the lowest-order functions Ψ_0 and Ψ_2 obey

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\Psi_0 - 2i\frac{\partial\Psi_0}{\partial\zeta} = 0, \tag{10a}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} - 2i\frac{\partial}{\partial\zeta}\right)\Psi_2 = -\frac{\partial^2\Psi_0}{\partial\zeta^2} . \tag{10b}$$

Equation (10a) is the starting point of traditional Gaussian beam theory. The fundamental mode solution is well known to be

$$\Psi_0 = \exp[-i(P + Q\rho^2)], \tag{11}$$

where

$$Q = 1/i + 2\zeta, \quad iP = -\ln iQ,$$

and we have put $\rho^2 = \xi^2 + \eta^2$. For completeness, we mention that when Q is decomposed into real and imaginary parts one obtains

$$iQ = w_0^2/w^2 + il/2R,$$

where

$$w(z) = w_0(1 + 4\zeta^2)^{1/2}$$

is the spot-size parameter, and

$$R(z) = z\left(1 + \frac{1}{4\zeta^2}\right)$$

is the radius of curvature of the wavefront that intersects the axis at z.

We now consider the electric field of the beam. Since $\vec{A} = A\hat{e}_1$, one obtains

$$\vec{E} = -\frac{i}{k}\nabla\left(\frac{\partial A}{\partial x}\right) - ikA\hat{e}_1$$
$$= \left(-\frac{i}{k}\frac{\partial^2 A}{\partial x^2} - ikA\right)\hat{e}_1 - \frac{i}{k}\frac{\partial^2 A}{\partial y\partial x}\hat{e}_2 - \frac{i}{k}\frac{\partial^2 A}{\partial z\partial x}\hat{e}_3 .$$

In terms of the dimensionless variables (ξ, η, ζ) this becomes

$$\vec{E} = -ik\left[\left(s^2\frac{\partial^2 A}{\partial\xi^2} + A\right)\hat{e}_1 + \left(s^2\frac{\partial^2 A}{\partial\xi\partial\eta}\right)\hat{e}_2 + \left(s^3\frac{\partial^2 A}{\partial\xi\partial\zeta}\right)\hat{e}_3\right].$$

Consequently, the transverse component of $\vec{E}$ involves only even powers of s, while E_z involves only odd powers of s. Since $A = \Psi e^{-ikz}$, we get

$$\vec{E} = -ike^{-ikz}\left[\left(s^2\frac{\partial^2\Psi}{\partial\xi^2} + \Psi\right)\hat{e}_1 + \left(s^2\frac{\partial^2\Psi}{\partial\xi\partial\eta}\right)\hat{e}_2 + \left(-is\frac{\partial\Psi}{\partial\xi} + s^3\frac{\partial^2\Psi}{\partial\xi\partial\zeta}\right)\hat{e}_3\right].$$

Using Eq. (9), one can write in powers of s:

$$\vec{E} = -ike^{-ikz}\left\{\left[\Psi_0 + s^2\left(\Psi_2 + \frac{\partial^2\Psi_0}{\partial\xi^2}\right) + \cdots\right]\hat{e}_1 + \left(s^2\frac{\partial^2\Psi_0}{\partial\xi\partial\eta} + s^4\frac{\partial^2\Psi_2}{\partial\xi\partial\eta} + \cdots\right)\hat{e}_2 + \left[-is\frac{\partial\Psi_0}{\partial\xi} - is^3\left(\frac{\partial\Psi_2}{\partial\xi} + i\frac{\partial^2\Psi_0}{\partial\xi\partial\zeta}\right) + \cdots\right]\hat{e}_3\right\}. \tag{12}$$

III. SOME APPLICATIONS OF THE FORMALISM

To lowest order in s, the components of $\vec{E}$ in the x-y plane are given by

$$E_x = -ik\Psi_0 e^{-ikz}$$

$$E_z = -ks\frac{\partial\Psi_0}{\partial\xi}e^{-ikz} = -\frac{2Qx}{l}E_x .$$

These expressions are in agreement with results given by Lax, Louisell and McKnight.[3] It is of interest that in the region of the beam waist, that is for $z^2 \ll l^2$, we have $Q \to -i$ and therefore the longitudinal component of the field is nearly in phase quadrature with the transverse component. At large distances from the beam waist, that is for $z^2 \gg l^2$, we have $Q \to l/2z$ and hence $E_z/E_x \to -x/z$. In this case, as strongly expected, the direction of the field lines nearly corresponds to those of spherical waves originating at the beam waist.

Finally, we wish to obtain the second-order function Ψ_2. For the purpose of gaining physical insight, let us for the moment consider a diverging spherical wave propagating from the origin. Such a wave has an exponential factor

$$\exp[-ik(z^2+r^2)^{1/2}],$$

where $r^2=x^2+y^2$. Along the z axis, the binomial expansion leads to the paraxial approximation for a spherical wave

$$\exp[-ik(z+r^2/2z-r^4/8z^3+\cdots)]. \tag{13}$$

We now judiciously expand this in the form

$$[1+is^2(l^3/8z^3)\rho^4+\cdots]\exp[-ikz-i(l/2z)\rho^2]$$

in which l, s, and ρ have the same meaning as before. Remembering that for $z^2\gg l^2$ the condition $Q\rightarrow l/2z$ holds, this line of reasoning suggests that Ψ_2 may be written

$$\Psi_2=(C+iQ^3\rho^4)\Psi_0,$$

where C depends on Q alone. The coefficient C is determined by inserting this function in Eq. (10b). Using the fact that

$$\frac{\partial^2\Psi_0}{\partial\xi^2}=(8Q^2-16iQ^3\rho^2-4Q^4\rho^4)\Psi_0$$

in which the relations

$$\frac{\partial Q}{\partial\xi}=-2Q^2,\ \frac{\partial P}{\partial\xi}=-2iQ$$

have been used, it is seen that C obeys

$$\frac{\partial C}{\partial\xi}=-i4Q^2,$$

yielding

$$C=2iQ+c,$$

where c is an arbitrary constant of integration. Evidently the boundary conditions of the problem are satisfied if one sets $c=0$, giving

$$\Psi_2=(2iQ+iQ^3\rho^4)\Psi_0\,. \tag{14}$$

If we include corrections up to third order, the electric field components in the x-z plane then become

$$E_x=-ike^{-ikz}[\Psi_0+s^2(-4Q\xi^2+iQ^3\rho^4)\Psi_0+\cdots], \tag{15a}$$

$$E_z=-ike^{-ikz}[-s(2Q\xi)\Psi_0+s^3(8Q^3\rho^2\xi-2iQ^4\rho^4\xi+4iQ^2\xi)\Psi_0+\cdots]. \tag{15b}$$

A similar approach, using the paraxial approximation (13) as a guide, could presumably be used to obtain the higher-order functions Ψ_4, Ψ_6, etc., as desired.

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[1]G. D. Boyd and J. P. Gordon, Bell Syst. Tech. J. 40, 489 (1961); G. D. Boyd and H. Kogelnick, Bell Syst. Tech. J. 41, 1347 (1962).

[2]For example, see A. Yariv, *Introduction to Optical Electronics*, 2nd Ed. (Holt, Rinehart, and Winston, New York, 1976), M4, 3.1.

[3]M. Lax, W. H. Louisell, and W. B. McKnight, Phys. Rev. A 11, 1365 (1975).