Vacuum fluctuation and retardation effects on long-range potentials

Larry Spruch and Edward J. Kelsey
Department of Physics, New York University, New York, New York 10003
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The Casimir-Polder R^{-7} retarded potential between two neutral (polarizable) atoms at a separation R, which replaces the nonrelativistic R^{-6} van der Waals potential for R sufficiently large, has been derived both by standard techniques *and* by assuming the atoms to be within a large volume with conducting walls and considering vacuum-fluctuation effects. The latter technique is here used to provide simple derivations of the R^{-5} potential between a charged point particle and a polarizable system—an addition to the usual R^{-4} polarization potential—and the R^{-3} two-photon contribution to the potential between two charged point particles. It also suggests a new, R^{-2} potential—an addition to the usual R^{-1} image potential—between a charged point particle and a conducting wall.

I. INTRODUCTION

The considerable interest in the possibility of detecting effects of the finiteness of the speed of light in experiments involving two atoms at large separation R stems from a classic paper by Casimir and Polder.[1] They showed that this "retardation effect" changes the interaction potential $U(R)$ between two hydrogen atoms, each in its ground state, from the Van der Waals–London R^{-6} dependence, which had been thought to be valid at all distances somewhat greater than a few Bohr radii, to an R^{-7} dependence at very large R. We suggest that retardation effects be looked for in a new area, in energy measurements on the high Rydberg states of a single atom. The motivation is the recent rapid advance in the precision of these measurements. We find that the primary effect is an R^{-5} *addition to* rather than replacement of the R^{-4} polarization potential of the outer electron and the polarizable inner ion. Using a dispersion-relation analysis, Bernabéu and Tarrach[2] showed very recently that the leading retardation addition to the R^{-4} potential of a particle with charge q_2 and mass m_2 and a *neutral* polarizable system is, asymptotically,

$$U(R) \sim \frac{\hbar q_2^2}{(4\pi m_2 c R^5)}[11\alpha_{d1}(0) + 5\beta_{d1}(0)] . \qquad (1)$$

$\alpha_{d1}(0)$ and $\beta_{d1}(0)$ are the static electric and magnetic dipole polarizabilities, respectively, of the neutral system. Though of great interest, it does not follow from their calculation that the result is applicable to the problem at hand, since the core is *charged*. In a recent paper,[3] Kelsey and Spruch used standard time-ordered perturbation theory to study retardation corrections for a high Rydberg state which has as well a very large angular momentum quantum number. They showed that the leading retardation correction, for, roughly, $10^2 \ll R/(\hbar^2/me^2) \ll 10^4$, is the first term in Eq. (1); $\alpha_{d1}(0)$ is here the static electric dipole polarizability of the (charged) inner ion. [The magnetic term was not obtained since it is of order $(e^2/\hbar c)^2$ smaller than the electric term for the problem we examined, and terms of that order were dropped.] The calculation is a very tedious one.

We here provide a short derivation of the R^{-5} retardation potential and some physical insight into its origin. The argument is based on effects of the vacuum fluctuation of the electromagnetic field. The few problems analyzed previously using such arguments gave results in exact agreement with those obtained using standard methods of calculation. Why vacuum-fluctuation arguments worked in the past in the problems to which they were applied is, to our knowledge, not completely understood, but the simplicity of the approach gives it considerable appeal, as a means of providing physical insight into known results and as a means of suggesting new results.

II. PRELIMINARY CALCULATION

A. Two finite systems

Although calculations using the vacuum-fluctuation approach can be much more than an order of magnitude simpler than, for example, calculations using time-ordered perturbation theory, the precise vacuum-fluctuation formulation can still require considerable effort. The physics behind the vacuum-fluctuation viewpoint is, however, rather simple. We will therefore use Sec. II to obtain results with a rudimentary version of the vacuum-fluctuation approach in which we simply ignore all details that might obscure the underlying physics. These include signs, numerical coefficients of order unity,

and angle-dependent factors whose integrals can be expected to be of order unity; we also ignore magnetic fields. The various results are rederived in Sec. III, but exact numerical coefficients are then obtained. No approximations whatsoever are made in Sec. III over and above those present in the standard vacuum-fluctuation approach.

In the past, the vacuum-fluctuation formulation has been limited to systems with internal degrees of freedom, and one has used the electric and magnetic polarizabilities associated with those degrees of freedom. The R^{-7} results for atom-atom interactions in that formulation are the same as the results obtained using other techniques. The new feature of the present paper is simply the realization that the vacuum-fluctuation formulation can be immediately extended to cases for which one or both of the systems is a charged point particle; one need merely use the polarizability appropriate to the new situation. (The polarizability of a free charged particle behaves as $1/\omega^2$, while at low frequencies systems with internal degrees of freedom have frequency-independent polarizabilities; this will be seen to be the root cause of the different R dependencies of different interacting systems.) The extended formulation is also applicable to charged systems with internal degrees of freedom.

The extended vacuum-fluctuation formulation can be used to derive the interaction between two systems where one or both of them is a charged point particle; the derivations of the interaction between a charged point particle and a charged or neutral system with internal degrees of freedom, and between two charged point particles, are very much simpler than derivations that use standard techniques. In addition, one new result has been obtained, a $1/R^2$ retardation correction to the $1/R$ image potential of a charged particle and a perfectly conducting wall.

We are interested initially in obtaining, as simply as possible, the dependence of $U(R)$ on R for a pair of systems; either "system" can be (i) neutral, with internal degrees of freedom, or (ii) charged, with internal degrees of freedom, or (iii) a charged point particle. We therefore begin by presenting a bare outline of the argument, omitting all factors of order unity, and signs. Further, in this section we neglect magnetic-field effects. We assume our two arbitrary systems are in the presence of a background standing-mode electric field

$$\vec{E}_b(\omega,\vec{r},t)=\vec{E}_b(\omega,\vec{r})\cos(\omega t).$$

All time-dependent fields and dipole moments will henceforth be understood to have a $\cos(\omega t)$ time dependence. $\vec{r}$ is measured with respect to an arbitrary origin and the wavelength of $\vec{E}_b(\omega,\vec{r})$ is assumed to be very large compared to the dimension of either system—the dipole approximation with respect to either system—but *not* necessarily compared to R. The $\vec{p}_i(\omega)$ factor of the electric dipole moment $\vec{p}_i(\omega,t)$ induced in system i by some field $\vec{E}(\omega,\vec{r},t)$ is $\vec{p}_i(\omega)=\alpha_i(\omega)\vec{E}(\omega,\vec{r})$; this equation defines the dynamic electric polarizability $\alpha_i(\omega)$ of system i. The energy $\mathcal{E}_i(\omega)$ of interaction of system i and $\vec{E}(\omega,\vec{r},t)$ is $\alpha_i(\omega)\vec{E}^2(\omega,\vec{r}_i)$; here and in subsequent expressions energies represent time-averaged energies. System 2 interacts with both $\vec{E}_b(\omega,\vec{r},t)$ and the field $\vec{E}_{1\to2}(\omega,\vec{r},t)$ set up by the oscillating dipole of system 1. Very schematically, the field at system 2 caused by the induced dipole at system 1 is

$$\vec{E}_{1\to2}(\omega,\vec{r}_2)\sim\left[\alpha_1(\omega)\vec{E}_b(\omega,\vec{r}_1)/R^3\right]g(\omega R/c). \tag{2}$$

The exact form of $\vec{E}_{1\to2}$ can be found in almost any text on electrodynamics. In reality, $\vec{E}_{1\to2}(\omega,\vec{r}_2)$ is not in the same direction as $\vec{E}_b(\omega,\vec{r}_1)$ but has a complicated angular dependence; we will assume that the angular integrations give results of order unity. The only properties of $g(x)$ of interest to us here are that $g(x)/x$ remains finite as x vanishes and that $g(x)$ is of order unity over most of the relevant range of x.

The energy of the polarizable system 2 in the presence of both $\vec{E}_b(\omega,\vec{r})$ and $\vec{E}_{1\to2}(\omega,\vec{r})$ is

$$\mathcal{E}_2(\omega)\sim\alpha_2(\omega)\left[\vec{E}_b(\omega,\vec{r}_2)+\vec{E}_{1\to2}(\omega,\vec{r}_2)\right]^2. \tag{3}$$

We have neglected fields at system 2 which arise from radiation by a dipole induced in system 2 by $\vec{E}_b(\omega,\vec{r}_2)$ which propagates to system 1 and is then backscattered to system 2. Such a field is proportional to $\alpha_1(\omega)\alpha_2(\omega)$, and its neglect generates an error in $\mathcal{E}_2(\omega)$ which is cubic in the polarizabilities. It would not then be consistent to retain the $\alpha_2(\omega)[\vec{E}_{1\to2}(\omega,\vec{r}_2)]^2$ term in Eq. (3), since it, too, gives a contribution which is cubic in the polarizabilities. Further, we need not even consider the term $\alpha_2(\omega)\vec{E}_b^2(\omega,\vec{r}_2)$ since it is independent of R. The interaction potential $U(\omega,\vec{R})$ of the two systems in the presence of $\vec{E}_b(\omega,\vec{r})$ is then, at sufficiently large R,

$$U(\omega,\vec{R})\sim\alpha_2(\omega)\vec{E}_b(\omega,\vec{r}_2)\cdot\vec{E}_{1\to2}(\omega,\vec{r}_2). \tag{4}$$

We next assume that the two systems are in a box with conducting walls of volume V, that R is very much smaller than the distance of either system from any wall, and that the background field has a distribution $N(\omega)\,d\omega$ of modes, given, as usual, by $V\omega^2\,d\omega/c^3$. We then have, from Eqs. (2) and (4),

$$U(\vec{R}) \sim \int U(\omega, \vec{R}) N(\omega)\, d\omega$$

$$\sim R^{-3} \int \alpha_1(\omega)\alpha_2(\omega)\, [\vec{E}_b(\omega, \vec{r}_1) \cdot \vec{E}_b(\omega, \vec{r}_2)]$$

$$\times g(\omega R/c)\; N(\omega)\, d\omega\,. \tag{5}$$

We now approximate the [ ] in Eq. (5) by $\vec{E}_b^2(\omega, \vec{r}_2)$ for $\omega < c/R$ and by zero for $\omega > c/R$, since there will be considerable cancellation for $\lambda = c/\omega < R$. The value of $\vec{r}_2$ is arbitrary, and we replace $\vec{E}_b^2(\omega, \vec{r}_2)$ by its average over $\vec{r}_2$, denoted by $\langle\ \rangle$ with $\vec{r}_2$ replaced by $\vec{r}$. We then have

$$U(R) \sim R^{-3} \int_0^{c/R} g\left(\frac{\omega R}{c}\right) \alpha_1(\omega)\alpha_2(\omega)$$

$$\times \langle \vec{E}_b^2(\omega, \vec{r}) \rangle N(\omega)\, d\omega\,.$$

At this stage we introduce quantum mechanics for the first time by replacing $\vec{E}_b(\omega, \vec{r})$ by $\vec{E}_{fl}(\omega, \vec{r})$, the vacuum fluctuation field whose spatial average is $\langle \vec{E}_{fl}^2(\omega, \vec{r}) \rangle V \approx \hbar\omega$. We arrive at

$$U(R) \sim \frac{\hbar}{(cR)^3} \int_0^{c/R} g\left(\frac{\omega R}{c}\right) \alpha_1(\omega)\alpha_2(\omega)\omega^3\, d\omega\,. \tag{6}$$

An identical contribution arises from the interaction of system 1 with $\vec{E}_{fl}$ and the dipole field generated by system 2.

If the ith system is neutral but has internal degrees of freedom, $\alpha_i(\omega)$ will be $\alpha_{di}(\omega)$, the dynamic electric dipole polarizability of system i, which has a well defined nonvanishing limit, $\alpha_{di}(0)$, as $\omega \sim 0$. Since the maximum frequency under consideration is the very low frequency c/R, we may replace $\alpha_{di}(\omega)$ by $\alpha_{di}(0)$. If the ith system is a point particle with charge q_i and mass m_i, we must use for $\alpha_i(\omega)$ the free polarizability $\alpha_{fi}(\omega) = -(q_i^2/m_i\omega^2)$; this follows directly from the definition of $\alpha_i(\omega)$. In general we have $\alpha_i(\omega) \approx \alpha_{di}(0) + \alpha_{fi}(\omega)$, and there are three kinds of contributions to $U(R)$. The retarded interaction between two neutral atoms, for example, is given by $U_{dd}(R)$. The interaction of the outer electron in a high Rydberg state with the ionic core includes a Coulomb R^{-1} term, a polarization R^{-4} term, and a retarded interaction $U_{df}(R)$. Finally, the retarded interaction between two charged point particles is given by $U_{ff}(R)$. Systems can have retarded interactions which are sums of the above. Thus, the retarded interaction between two He^+ ions is a combination of all three. We have

$$U_{dd}(R) \sim \frac{\hbar}{(cR)^3} \alpha_{d1}(0)\alpha_{d2}(0) \int_0^{c/R} g\left(\frac{\omega R}{c}\right) \omega^3\, d\omega$$

$$\sim \hbar c\, \alpha_{d1}(0)\alpha_{d2}(0)/R^7\,, \tag{7a}$$

where we set $x = \omega R/c$ and assume that

$$\int_0^1 g(x)\, x^3\, dx \sim 1\,.$$

Similarly, we have

$$U_{df}(R) \sim \hbar q_2^2\, \alpha_{d1}(0)/(m_2 c R^5)\,, \tag{7b}$$

where we assume that

$$\int_0^1 g(x)\, x\, dx \sim 1\,;$$

there is of course a similar form for subscripts 1 and 2 interchanged. Finally, we have

$$U_{ff}(R) \sim [\hbar/(cR)^3]\, (q_1^2/m_1)(q_2^2/m_2)\,, \tag{7c}$$

where we assume that

$$\int_0^1 [g(x)/x]\, dx \sim 1\,;$$

we here use a property of the quantity in square brackets assumed above.

B. Perfectly conducting wall and finite system

We now consider retardation interactions for system 1 not finite but a perfectly conducting wall. System 2 remains finite, and its center is at a distance R from the wall. The situation is in one sense profoundly different from that considered previously, for the interaction of radiation with a wall cannot be treated in the dipole approximation. We will proceed in two quite different ways.

First, we adapt the analysis above of the case of two finite systems to the case for which system 1 is a conducting wall. Very crudely, we approximate the wall by a sphere of radius CR, with C of order $\frac{1}{5}$, perhaps, with the center of system 2 at a distance R from the nearest point on the sphere, so that the separation of the centers of the two systems is $(1+C)R$. CR is then sufficiently small compared to $(1+C)R$ for the dipole approximation to be meaningful if not accurate, and CR is sufficiently large for the sphere—with polarizability $(CR)^3$—to roughly approximate a wall. We can immediately obtain the retarded interaction of a perfectly conducting wall with a neutral system with internal degrees of freedom from Eq. (7a), and of a perfectly conducting wall with a charged point particle from Eq. (7b); we need merely replace $\alpha_{d1}(0)$ by $(CR)^3$. [We must also replace R by $(1+C)R$, but we are assuming that C is small compared to 1.] With the subscript W denoting a perfectly conducting wall, we thereby obtain

$$U_{Wd}(R) \sim \left[\alpha_{d2}(0)\hbar c/R^4\right] C^3 \tag{7d}$$

and

$$U_{Wf}(R) \sim \left[\hbar q_2^2/(m_2 c R^2)\right] C^3 . \tag{7e}$$

The $1/R^2$ interaction does not seem to have been obtained before.

Although the forms of Eqs. (7d) and (7e) should be believed, it is difficult to make a meaningful estimate of C, and we will consider a second approach. The interaction of system 2 with a fluctuating field $\vec{E}_{fl}(\omega, \vec{r}_2)$ in a huge conducting cube of volume V, for system 2 far from any wall, is

$$U_{\text{isolated}} \sim \int \alpha_2(\omega) \vec{E}_{fl}^2(\omega, \vec{r}_2) N(\omega)\, d\omega ,$$

where $\vec{r}_2$ is measured from an arbitrary origin. This is essentially the self-energy of the system, but the divergence at high frequencies need not concern us, for the retarded interaction of interest is the difference between U_{isolated} and the interaction of system 2 with $\vec{E}_{fl}$ when system 2 has been moved so that its center is at a distance R from one of the walls; we will now measure distances from the point on the wall closest to system 2. The interaction of system 2 with $\vec{E}_{fl}$ will of course be modified by the presence of the wall. A rough indication of the modification can be obtained by assuming that modes with $\lambda < R$ are unaffected while modes with $\lambda > R$ no longer interact with system 2. The retarded interaction is then given by the difference, namely,

$$U_W(R) \sim \frac{\hbar}{c^3} \int_0^{c/R} \alpha_2(\omega) \omega^3\, d\omega ,$$

where we have again used $\langle \vec{E}_{fl}^2(\omega, \vec{r}) \rangle V \approx \hbar\omega$. Inserting $\alpha_2(\omega) = \alpha_{d2}(0)$ and $q_2^2/m_2\omega^2$ gives Eqs. (7d) and (7e), respectively, apart from the C^3 factor.

C. Range of validity of retarded interaction, and additive correction *versus* replacement

Equations (7a)–(7e) represent five different retarded interactions. The interaction of two neutral atoms is an example of (7a), while the interaction of the outer electron in a high Rydberg state with the ionic core includes the interaction in (7b). The interaction of two free electrons is an example of (7c). The interaction of a perfectly conducting wall with a neutral atom and with an electron provide examples of (7d) and (7e), respectively.

It follows from the nature of the derivation that the retarded interactions are valid for R asymptotically large. It is natural to ask the further question of the possibility of making at least a rough estimate of the value R_0 beyond which the retarded interaction approaches its asymptotic form. It will also be important to know when the retarded interaction is an additive correction to the interaction in the nonrelativistic approximation, and when it is a replacement of the interaction in the nonrelativistic approximation. Finally, we will make a brief comment on the possibility of estimating the interaction for R in the neighborhood of R_0 from a knowledge of the nonrelativistic interaction for R larger than a few Bohr radii—the domain beyond which interactions in the nonrelativistic approximation normally approach their asymptotic forms—and of the retarded interaction for $R \gg R_0$. The questions are well posed and can be answered by a complete analysis for any given pair of systems. We are here trying to obtain at least partial answers, using only the results obtained above.

As has been emphasized in a number of papers, the effect of retardation can be expected to be important if the time it takes light to travel to and fro, $2R/c$, is comparable to or larger than a significant characteristic period of either system. For an atom or an ion, a characteristic period will be of order $2\pi a_0/v_0 = 2\pi a_0/(e^2/\hbar)$, where $v_0 = e^2/\hbar$ is a characteristic velocity. Dropping a factor of π, we thereby arrive at $R_0 \approx a_0(\hbar c/e^2)$. We thus have an estimate for three of the cases discussed above, but not for the electron-electron and electron-wall cases.

We note that the nonrelativistic approximation is a self-consistent one; we can never have $R > R_0$ in that approximation since $R_0 \to \infty$ as $c \to \infty$.

We turn now to the question of additive corrections versus replacements. The interaction in the nonrelativistic approximation of two neutral atoms (which have no permanent dipole moments) can be thought of, crudely, as the interaction of two correlated virtual dipole moments. When retardation is taken into account, the correlation can be expected to be gradually reduced as R increases, and the interaction can be expected to fall off more rapidly than $1/R^6$; the correlation between the virtual dipoles can be maintained only if the atoms maintain communication by the transfer back and forth of photons, and, as the time of transfer increases with R, the communication link is weakened. (The argument is clearly intended to be at best suggestive.) For a neutral atom or positive ion with one of its electrons in a high Rydberg state, the electron-core interaction in the nonrelativistic approximation will include a $1/R$ Coulomb potential, a static electric dipole $1/R^4$ polarization potential, a static electric quadrupole $1/R^6$ polarization potential, and a nonadiabatic $1/R^6$ potential. The first three interactions remain when retardation is accounted for; the ionic charge is of course

specified, and the induced electric dipole and electric quadrupole have perfectly well-defined magnitudes and directions for the electron fixed, no matter how large a value R has. The retarded R^{-5} interaction given by (7b) will be an additive correction. However, the nonadiabatic nonrelativistic R^{-6} interaction depends upon the velocity and position of the electron; the correlation between motion within the ion and the position and velocity of the electron will be reduced as R increases, and the nonadiabatic interaction can be expected to fall off faster than R^{-6}; the present authors have checked this point formally in work that has not yet been published. For two electrons, the nonrelativistic Coulomb term persists when retardation is accounted for, and the R^{-3} term of Eq. (7c) is an additive correction.

For a perfectly conducting wall interacting with a neutral atom (with no permanent dipole moment), the nonrelativistic interaction is that between the virtual dipole moment of the atom and its perfectly correlated virtual dipole image; the interaction is therefore that between two dipoles, which behaves as $1/R^3$. On accounting for retardation, the correlation between the dipoles is gradually lost as R increases, and the interaction will fall off more rapidly than $1/R^3$. For a perfectly conducting wall and an electron, the nonrelativistic $1/R$ image potential will persist when retardation is accounted for, for the location of the image will continue to be $-\vec{R}$ no matter how large the value of R is; the R^{-2} retarded interaction will be an additive correction.

We now consider the two cases for which the retardation interaction is a replacement rather than an additive correction, namely, two interacting neutral atoms and a perfectly conducting wall interacting with a neutral atom. The nonrelativistic interactions are $\alpha_{d1}(0)\alpha_{d2}(0)\Delta E/R^6$ and $e^2a_0^2/R^3$, respectively, where $\Delta E \approx e^2/a_0$ is a characteristic atomic-excitation energy. Comparing with the appropriate retarded interactions, it is found for both cases that the nonrelativistic and retarded interactions are comparable at $R \approx R_0$. It follows for these two cases that one can expect the effective interaction to be reasonably well approximated by any interaction we can choose which represents a smooth transition from the nonrelativistic to the retarded interaction. This point has been made previously. It is not clear how one could make an estimate of the interaction at "intermediate distances" for the other three cases along the above lines.

Analysis from the viewpoint of vacuum fluctuations of the asymptotic interaction of two neutral atoms, and of a neutral atom and a perfectly conducting wall, have been given previously. Whatever merit the above derivations may have for these two cases is heuristic; particularly in the case of two neutral atoms, the introduction of a complete set of modes for the electromagnetic field in a very large container with perfectly conducting walls, and the exact treatment of those modes, generates a lengthy treatment, and one can easily lose the physical picture in the mathematical details. The three cases involving an electron—the electron in a high Rydberg state, the two electrons, and the electron and the perfectly conducting wall—have not previously been treated from the viewpoint of vacuum fluctuations. The essential step was the realization of the simple fact that the electron could itself be treated as a polarizable system [with polarizability $-e^2/m\omega^2$]. The asymptotic interactions for the high-Rydberg-state case and for the case of two electrons had previously been obtained using other approaches, while the retarded R^{-2} interaction of an electron and a perfectly conducting wall does not seem to have been obtained previously by any approach.

III. CALCULATION

For two finite systems, one can obtain the precise form for $U(R)$, rather than the approximate form given in Eq. (6), if one uses the identical physical approach but retains signs, all numerical coefficients, all terms, and angular factors, and includes magnetic effects, but it is not necessary to go to the trouble for almost all of the work has been done, and we simply note down an extension of a version given by Boyer.[4] The new form is

$$U(R) = -\frac{\hbar c}{\pi} \lim_{\lambda \to 0} \int_0^\infty dk\, k^6 e^{(-\lambda k)} \times [A(\omega) I(kR) + B(\omega) J(kR)] , \qquad (8)$$

where $A(\omega) \equiv \alpha_1\alpha_2 + \beta_1\beta_2$, $B(\omega) \equiv \alpha_1\beta_2 + \alpha_2\beta_1$, with the α's and β's ω dependent, and where $I(x) \equiv \sin(2x)\,(x^{-2} - 5x^{-4} + 3x^{-6}) + \cos(2x)\,(2x^{-3} - 6x^{-5})$, $J(x) \equiv \sin(2x)\,(-x^{-2} + x^{-4}) - 2\cos(2x)(x^{-3})$. Boyer was interested only in polarizabilities associated with internal degrees of freedom. We have taken his $\alpha_{d1}(0)$, $\alpha_{d2}(0)$, $\beta_{d1}(0)$, and $\beta_{d2}(0)$ inside the integral and replaced them by $\alpha_1(\omega)$, $\alpha_2(\omega)$, $\beta_1(\omega)$, and $\beta_2(\omega)$, respectively, assuming thereby that Eq. (8) is also applicable to cases where one or both of the systems is a charged point particle. [The applicability of Eq. (8) to electrons, say, might seem even more difficult to justify than for (heavy) polarizable systems, since the latter are far more easily localized, but for sufficiently large R this should not pose any serious new difficulty.] For the appropriate choices of $\alpha_i(\omega)$ and $\beta_i(\omega)$, we find that $U_{dd}(R)$ is

exactly the result obtained in a dispersion relation analysis by Feinberg and Sucher[5] as the generalization (which includes magnetic effects) of the Casimir-Polder result, a point noted previously.[4] (By *exactly*, we mean coefficient and all.) What is more interesting at present is that $U_{df}(R)$ is found to be *exactly* the result of Eq. (1).[6] Furthermore, U_{ff} is *exactly* the sum of one and two transverse-photon graphs in positronium as calculated previously[7] using standard perturbation theory. Though not properly founded, the present derivation is not only *very* much simpler but suggests the last two results are valid over a wider range of conditions than those encompassed in the original derivations.

Equation (8) was obtained by extending an expression for the interaction of two finite systems whose polarizabilities are associated with internal degrees of freedom to the case where polarizabilities associated with a free charged point particle are allowed. To make the extension, we had merely to replace the static-dipole polarizabilities, constants, by dynamic (frequency-dependent) dipole polarizabilities, and place them inside the integral over the frequency. An identical adaptation can be made for the case for which system 1 is a perfectly conducting wall. An expression has been obtained[8] for the retarded interaction with a finite second system whose polarizability is associated with internal degrees of freedom. To extend the validity of the expression to include the case where the polarizability of system 2 can be associated with that of a charged point particle, we need merely take $\alpha_{d2}(0)$ inside the integral as $\alpha_2(\omega)$. We thereby obtain

$$U_W(R) = -\frac{\hbar c}{16\pi R^4} \lim_{\lambda\to 0} \int_0^\infty \alpha_2(\omega) e^{-\lambda z} z^3 \frac{d^2}{dz^2} \frac{\sin z}{z}\, dz\,, \tag{9}$$

where $z = 2\omega R/c$. We immediately arrive at

$$U_{Wd}(R) \sim -(3/8\pi)\alpha_{d2}(0)\hbar c/R^4 \tag{10a}$$

for the polarizable system–wall interaction[8] and the apparently new result[9]

$$U_{Wf}(R) \sim \hbar q_2^2/(4\pi m_2 c R^2) \tag{10b}$$

for the free charge–wall interaction.

IV. DISCUSSION

The verification of the interaction $U_{Wf}(R)$ of Eq. (10b) may possibly represent in the future an attractive way of detecting a retardation effect. Its validity should first be checked by redoing the calculation using a more standard and reliable approach,[10] which would, at the same time, give the $U(R)$ for small R. The asymptotic result is almost surely valid, however, and for the purposes of this discussion we will assume it to be correct. One could not hope to detect the effect for an electron bound in a low-lying state by its image charge to a conducting wall, for the separation would be of the order of a_0, a Bohr radius, and at that distance it makes little sense to think of the conductor as a perfect plane. One might conceivably detect the effect for an electron bound in a low-lying state to a dielectric surface with a dielectric constant close to unity, or bound in a highly excited state to a conductor.

The extension to a dielectric medium would require techniques similar to those Lifshitz[11] used on the problem of the attractive force between two slabs of dielectric material separated by a large distance which was an extension of work by Casimir[12] on the attractive force between two metal plates. A possible way of performing such a test would be to perform precision experiments on electrons bound at great distances from a surface of liquid helium—the low temperature simplifies the analysis—along the lines of Grimes and Brown.[13] There are clearly a number of serious problems, since the effect is very small. Thus, since the bound states are at distances of order $100a_0$, the R^{-2} correction is of order $(\hbar/mc)/R \approx 10^{-4}$ relative to the binding energy in the Grimes-Brown experiment. There may therefore be a number of possible masking effects. In particular, the model dependence of the Grimes-Brown result would have to be reduced, perhaps by some other experiments which gave information on the surface. Further image-charge distortion in the liquid-helium surface could pose a problem. Finally, realistic checks of the results discussed in this paper may require separations R such that $\alpha_d(\omega)$ *cannot* be approximated by $\alpha_d(0)$ for $0 < \omega < R/c$ in which case a more detailed knowledge of the frequency dependence of the polarizability will be required.

The semiclassical analysis of the first three examples presented above might seem far removed from the standard approach, but it may not be too difficult to make the connection. The starting point, equivalent to summing nonrelativistic perturbation diagrams but hopefully arrived at directly, would be an attempt to express $U(R)$ in the form

$$U(R) = \sum_{\vec{\lambda},\vec{\lambda}'} \int d^3k\, d^3k'\, T_1(\vec{k},\vec{\lambda},\vec{k}',\vec{\lambda}') \times T_2(\vec{k},\vec{\lambda},\vec{k}',\vec{\lambda}') f(\vec{k},\vec{\lambda},\vec{k}',\vec{\lambda}',\vec{R})\,, \tag{11}$$

where $\vec{k},\vec{\lambda}$ and $\vec{k}',\vec{\lambda}'$ are the momenta and polarization vectors of the two virtual photons and

T_i is the off-shell inelastic amplitude for the scattering of light by the ith system; f would be a sum of terms, each containing a factor $\exp(\pm i\vec{k}\cdot\vec{R}\pm i\vec{k}'\cdot\vec{R})$. Because of the presence of the exponentials, only very small values of k and k' would give significant contributions for large R, and T_1 and T_2 would each reduce to low-energy elastic-scattering amplitudes, Thomson ($T_{\rm Th}\sim\omega^0$) for a charged particle and Rayleigh [$T_{\rm Ra}\sim\alpha_d(\omega)\omega^2$] for a polarizable system. [From this point of view, the basic elements are not the α's but the T's. In particular, we need never introduce the free polarizability $\alpha_f(\omega)=-e^2/m\omega^2$ for an electron, and, with $\alpha_d(\omega)$ replaced by $\alpha_f(\omega)$, find $T_{{\rm Ra},f}$ for the Rayleigh scattering amplitude for a free electron to be $T_{{\rm Ra},f}\sim\alpha_f(\omega)\omega^2\sim\omega^0$; using $T_{\rm Th}$ gives ω^0 directly.] Apart from providing a proper justification of Eq. (8)—it has been seen to give three correct answers—Eq. (11) might serve as a convenient starting point for obtaining terms in $1/R$ beyond the leading term; one would expand the T's in powers of k and k'.[14] Note, incidentally, that the first term beyond the Coulomb term in $U(R)$ for an electron in a high Rydberg state is not the R^{-4} polarization term but the R^{-3} charged particle–electron interaction, arising from $T_{\rm Th}$ for the ion, but the coefficient is extremely small. (There is an additional R^{-3} term if the nucleus or ionic core has a permanent quadrupole moment and the Rydberg electron is not in an s state.)

In order to verify the existence at very large R of the R^{-5} retardation potential between an electron and a polarizable ionic core in future bound-state experiments, it will be necessary to have reliable theoretical values of radiative and other relativistic corrections. That these shifts may be as large or larger than the shift produced by the R^{-5} potential will not be a source of difficulty once they have been calculated. A more serious problem is that the radiative *widths* of the states are larger than the R^{-5} shifts. For a helium atom with the outer electron with effective quantum numbers $n=15$ and $l=14$, the radiative width is approximately 100 times that of the shift caused by the R^{-5} term. As a consequence such an experiment will have to be performed with very good statistics.

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Note added in proof. (i) The proof that the nonadiabatic interaction between a charged particle and a polarizable system falls off faster than R^{-6} when retardation effects are included appears in Ref. 3. (ii) That retardation is significant if and only if $2R/c$ is comparable to any relevant period $2\pi/\omega_0$ of either system led us to argue that the atom-atom, atom-electron, and wall-atom retarded interactions are significant beyond $R/c\geq 1/\omega_0$. This same conclusion follows directly from our derivation in Sec. II. Thus $\alpha_d(\omega)$ can be approximated by $\alpha_d(0)$ only if ω is small compared to the lowest natural frequency ω_0 of the system. Since the effective range of ω is from 0 to order c/R, we can take $\alpha_d(\omega)$ outside the integral as $\alpha_d(0)$ only if $c/R\lesssim\omega_0$, the same estimate is arrived at using transit time arguments. [For the electron-electron and electron-wall retarded interaction, the integral does not contain $\alpha_d(\omega)$ and we cannot readily estimate the range beyond which retardation plays a role, nor were we able to make such an estimate using transit time arguments.] (iii) We note that the nonrelativistic R^{-6} atom-atom interaction involved the uncertainty principle for particles, while the retarded R^{-7} term is a consequence of the uncertainty principle for electromagnetic fields.

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