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Reduction of the Bethe-Salpeter equation to an equivalent Schrödinger equation, with applications

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We propose a new relativistic two-body formalism which reduces to a nonrelativistic Schrödinger theory for a single effective particle. The formalism is equal in rigor to that of Bethe and Salpeter, and considerably simpler to apply. We illustrate its use by computing $O(\alpha^6)$ terms in the ground-state splitting of muonium and positronium involving infinite Coulomb exchange.

I. INTRODUCTION

The high-precision measurements of the ground-state hyperfine splittings (hfs) in muonium[1] ($e^-\mu^+$) and positronium[2] (e^-e^+) allow a sensitive test of our understanding of two-body bound states in quantum field theories and particularly in quantum electrodynamics (QED). This is the second of two papers in which we examine alternatives to the Bethe-Salpeter[3] (BS) equation for organizing and computing bound-state energies in spinor field theories. In the first paper (I),[4] the BS equation was reduced to an equivalent Dirac equation by placing one particle effectively on mass shell. This approach is natural when the binding is nonrelativistic *or* when the ratio of constituent masses is large (e.g., in high $Z\alpha$ atoms or perhaps in D mesons). Analytic solutions were found for a Coulomb-like kernel, and a systematic perturbation theory developed. The bound-state equation reduced to the Dirac-Coulomb equation when one particle's mass became infinite.

Here we propose an alternative approach which may be more convenient when calculating high-order corrections for nonrelativistic systems (e.g., muonium or positronium). We reduce the exact BS equation to an equivalent Schrödinger equation with reduced mass. Among the advantages of such an approach are: (i) Approximating the kernel by a simple Coulomb interaction results in a zeroth-order problem of great simplicity. The wave functions are essentially just the usual Schrödinger wave functions for the hydrogen atom. (ii) The corrections to this zeroth-order problem can be elaborated in a systematic perturbation series. (iii) The unperturbed two-particle Green's function can be expressed in a number of simple analytic forms. This is important when computing contributions from second order perturbation theory, as we demonstrate below. (iv) As the exact unperturbed wave functions are finite at the origin, the expectation value of the one-photon annihilation kernel (in positronium) is finite. This is not the case in the BS approach, where this quantity can be made finite only after an infinite order (in α) renormalization of the annihilation vertices. In the formalism described below, all infinities related to renormalization can be removed order by order in precisely the way on-shell amplitudes are treated. This greatly simplifies the analysis and numerical evaluation of high-order terms (Appendix A). (v) The spinor structure of the wave functions is that of free-particle Dirac spinors, facilitating the use of computers for performing spinor algebra. This is quite important in view of the large number of diagrams remaining to be computed before theoretical and experimental determinations of hfs can be compared. (vi) The constituents are treated symmetrically, and hermiticity is explicitly maintained. (vii) In the limit of zero binding, the Green's function and wave functions reduce to the correct relativistic functions describing two free particles (at zero relative time).

Unlike I, none of the fine structure of levels with differing angular momenta is incorporated into the unperturbed QED solutions. The fine structure of atoms with constituents of equal mass is quite different in character from that of atoms with a large mass ratio. It is difficult to create a formalism which naturally acommodates both cases and still admits analytic solutions comparable in simplicity to those presented below.

The most recent measurements of hfs test theory to $O(\alpha^2\Delta E_0)$ in positronium and to $O(\alpha^2 m_e/m_\mu \Delta E_0$, $\alpha^3\Delta E_0)$ in muonium, where ΔE_0 is the leading contribution in each case. The relevant terms of $O(\alpha^3\Delta E_0)$ can be computed in the Dirac limit (m_μ

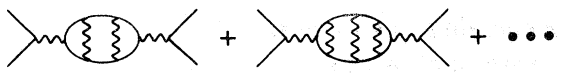

FIG. 1. Diagrams having three loops and more which contribute to $O(\alpha^6)$ hfs.

$\to\infty$) and will be discussed in a later paper (see also Ref. 5). Few $O(\alpha^2\Delta E_0,\ \alpha^2 m_e/m_\mu \Delta E_0)$ terms have been computed.[6] All zero-, one-, and two-loop (irreducible) kernels contribute to this order. In addition there exists an infinity of diagrams with three or more loops which contribute. These diagrams involve multiple Coulomb exchange. They arise when static interactions are treated in second-order perturbation theory (Fig. 1). To illustrate the use of our formalism, we compute all $O(\alpha^6)$ hfs terms of this sort.

In Sec. II we introduce a formalism describing bound states of two fermions with arbitrary mass. The analysis is similar to that in I and will only be outlined here. In Sec. III we rewrite the bound-state equation as a Schrödinger equation for a single effective particle and solve it for a Coulomb-like kernel. We outline the entire calculation of $O(\alpha^6)$ hfs in Sec. IV. We then compute the contributions requiring second-order perturbation theory. We also quote the analogous results obtained using the formalism of I. In Appendix A we comment upon certain aspects of renormalization theory for bound states, and finally, in Appendix B, we briefly discuss the relation between our formalism and the more conventional BS treatment.

II. BOUND-STATE FORMALISM (TWO FERMIONS)

Most bound-state formalisms follow from a Lippman-Schwinger equation for the truncated two-particle Green's function G_T having the general form (integrations over relative four momenta are implicit)[7]

$$G_T(P)=K(P)+K(P)\Lambda S(P)G_T(P)\,. \tag{2.1}$$

Here $K(P)$ is the interaction kernel at total momentum P, Λ is a spinor projection operation, and $S(P)$ is a two-particle propagator. The kernel is determined by the choice of Λ and $S(P)$:

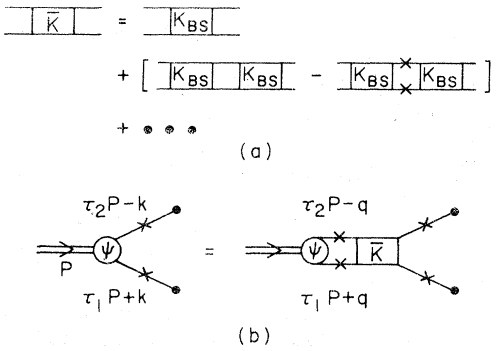

FIG. 2. (a) Definition of the effective kernel in terms of the BS kernel (b) The bound-state equation.

$$K(P)=G_T(P)[1+\Lambda S(P)G_T(P)]^{-1}$$
$$=G_T(P)-G_T(P)\Lambda S(P)G_T(P)+\cdots\,. \tag{2.2}$$

Given the expansion of G_T, this equation defines the expansion in α for K.

The BS equation is obtained by choosing[3]

$$S(k,P)=[i/(\tau_1\not{P}+\not{k}-m_1)][i/(\tau_2\not{P}-\not{k}-m_2)]\,,$$
$$\Lambda=1,\quad \tau_i=[m_i/(m_1+m_2)],\quad i=1,2\,.$$

In this case the kernel $K_{\rm BS}$ is the sum of all two-particle irreducible diagrams. This kernel is dominated by the static single-photon-exchange kernel in non-relativistic QED atoms. When the kernel is static, integrations over relative energy k^0 are easily done, resulting in a three-dimensional formalism with propagator (in the center of momentum frame)[8]

$$\Lambda S(kP)=2\pi i\delta(k^0)$$
$$\times\left(\frac{\Lambda_+^{(1)}(\vec{k})\Lambda_+^{(2)}(-\vec{k})}{P^0-E_1(k)-E_2(k)}-\frac{\Lambda_-^{(1)}(\vec{k})\Lambda_-^{(2)}(-\vec{k})}{P^0+E_1(k)+E_2(k)}\right),$$
$$E_i(k)=(\vec{k}^2+m_i^2)^{1/2},\quad i=1,2\,,$$
$$\Lambda_\pm^{(i)}(\vec{k})=[E_i(k)\gamma^0\mp(\vec{k}\cdot\vec{\gamma}-m_i)]/2E_i(k).$$

The $\Lambda_-\Lambda_-$ term contributes only to $O(\alpha^5)$, suggesting that a useful formalism could be constructed with $\Lambda=2\pi i\delta(k^0)\Lambda_+^{(1)}\Lambda_+^{(2)}$ and $S(\vec{k}P)^{-1}=P^0-E_1-E_2$.[9]

It is convenient at this point to introduce the two-particle Green's function evaluated at zero relative energy and having external fermion propagators

$$\bar{G}(\vec{k}\vec{q}P)=\frac{\Lambda_+^{(1)}(\vec{k})\Lambda_+^{(2)}(-\vec{k})}{P^0-E_1(k)-E_2(k)}\left((2\pi)^3\delta^3(\vec{k}-\vec{q})+\int\frac{d^3\vec{r}}{(2\pi)^3}i\bar{K}(\vec{k}\vec{r}P)\bar{G}(\vec{r}\vec{q}P)\right)$$
$$=\frac{\Lambda_+^{(1)}(\vec{k})\Lambda_+^{(2)}(-\vec{k})}{P^0-E_1(k)-E_2(k)}\left((2\pi)^3\delta^3(\vec{k}-\vec{q})+\bar{G}_T(\vec{k}\vec{q}P)\frac{\Lambda_+^{(1)}(\vec{q})\Lambda_+^{(2)}(-\vec{q})}{P^0-E_1(q)-E_2(q)}\right), \tag{2.3}$$

where $\bar{G}_T$ is related to the complete four-point function (2.1) by

$$\bar{G}_T(\vec{k}\vec{q}P) = \lim_{k^0,q^0\to 0} iG_T(kqP)\,. \tag{2.4}$$

Equation (2.3) is exact only if $\bar{K}$ is defined as in Eq. (2.2). In terms of the two-particle irreducible BS kernel $K_{\rm BS}$, we have [Fig. 2(a)]:

$$\bar{K}(\vec{k}\vec{q}P) = K_{\rm BS}(kqP)\Big|_{k^0=q^0=0} + \int \frac{d^4r}{(2\pi)^4} K_{\rm BS}(krP)\left(\frac{i}{\tau_1\not{P}+\not{r}-m_1}\frac{i}{\tau_2\not{P}-\not{r}-m_2} - 2\pi i\delta(r^0)\frac{\Lambda_+^{(1)}(\vec{r})\Lambda_+^{(2)}(-\vec{r})}{P^0-E_1(r)-E_2(r)}\right)$$
$$\times K_{\rm BS}(rqP)\Big|_{k^0=q^0=0} + \cdots\,. \tag{2.5}$$

As discussed above, $\bar{K}\simeq K_{\rm BS}\big|_{k^0=q^0=0}$ is a very good approximation when the binding is nonrelativistic, and the remaining terms in (2.5) may then be incorporated perturbatively.

Equation (2.3) is far simpler than the BS equation because we have chosen to consider $G_T(kqP)$ only at $k^0=q^0=0$.[10] The location of bound-state poles is unaffected by the relative energy of the constituents,[11] and so there is no need to retain this excess degree of freedom when computing energy levels or decay rates ($\Gamma=-2\,\mathrm{Im}\Delta E$). Furthermore when the BS kernel is static, G_T is independent of k^0 and q^0, and solving (2.3) is then equivalent to solving the BS equation (Appendix B).

Like $G_T(kqP)$, $\bar{G}(\vec{k}\vec{q}P)$ has poles at the m_1m_2 bound-state energies P_n^0:

$$\bar{G}(\vec{k}\vec{q}P)\to[\psi_n(\vec{k})\bar{\psi}_n(\vec{q})/(P^0-P_n^0)] \text{ as } P^0\to P_n^0\,. \tag{2.6}$$

Substituting (2.6) into (2.3) and evaluating at the pole, we obtain the bound-state equations [Fig. 2(b)]

$$[P^0-E_1(k)-E_2(k)]\psi(\vec{k}) = \Lambda_+^{(1)}(\vec{k})\Lambda_+^{(2)}(-\vec{k})\int\frac{d^3q}{(2\pi)^3}i\bar{K}(\vec{k}\vec{q}P)\psi(\vec{q})\,, \tag{2.7a}$$

$$\Lambda_-^{(1)}(-\vec{k})\psi(\vec{k}) = \Lambda_-^{(2)}(+\vec{k})\psi(\vec{k}) = 0\,. \tag{2.7b}$$

Note that the spinor structure of $\psi(\vec{k})$ follows immediately from (2.7)

$$\psi(\vec{k}) = \sum_{\lambda\lambda'}\frac{u^{(1)}(\vec{k}\lambda)u^{(2)}(-\vec{k}\lambda')}{[4E_1(k)E_2(k)]^{1/2}}\phi(\vec{k})_{\lambda\lambda'}\,, \tag{2.8}$$

where $u(\vec{k}\lambda)$ is the usual free-particle Dirac spinor ($\bar{u}u=2m$). Defining

$$\tilde{G}(\vec{k}\vec{q}P)_{\lambda'\mu',\lambda\mu} = \frac{u^{(1)}(\vec{k}\lambda')^\dagger u^{(2)}(-\vec{k}\mu')^\dagger}{[4E_1(k)E_2(k)]^{1/2}} \times\bar{G}(\vec{k}\vec{q}P)\gamma_0^{(1)}\gamma_0^{(2)}\frac{u^{(1)}(\vec{q}\lambda)u^{(2)}(-\vec{q}\mu)}{[4E_1(q)E_2(q)]^{1/2}}\,,$$
$$\tilde{K}(\vec{k}\vec{q}P)_{\lambda'\mu',\lambda\mu} = \frac{\bar{u}^{(1)}(\vec{k}\lambda')\,\bar{u}^{(2)}(-\vec{k}\mu')}{[4E_1(k)E_2(k)]^{1/2}} \times\bar{K}(\vec{k}\vec{q}P)\frac{u^{(1)}(\vec{q}\lambda)u^{(2)}(-\vec{q}\mu)}{[4E_1(q)E_2(q)]^{1/2}}\,, \tag{2.9}$$

we can rewrite Eqs. (2.3), (2.6), and (2.7):

$$\tilde{G}(\vec{k}\vec{q}P) = \frac{1}{P^0-E_1(k)-E_2(k)}\left((2\pi)^3\delta^3(\vec{k}-\vec{q}) + \int\frac{d^3r}{(2\pi)^3}i\tilde{K}(\vec{k}\vec{r}P)\tilde{G}(\vec{r}\vec{q}P)\right)$$
$$\to\frac{\phi(\vec{k})\phi^*(\vec{q})}{P^0-P_n^0} \text{ as } P^0\to P_n^0\,, \tag{2.10a}$$

$$[P^0-E_1(k)-E_2(k)]\phi(\vec{k}) = \int\frac{d^3q}{(2\pi)^3}i\tilde{K}(\vec{k}\vec{q}P)\phi(\vec{q})\,. \tag{2.10b}$$

Using methods described in I, we obtain the orthonormality conditions

$$\int\frac{d^3kd^3q}{(2\pi)^6}\tilde{G}(\vec{r}\vec{k}P)W(\vec{k}\vec{q}PP_n)\phi_n(\vec{q}) = \frac{\phi_n(\vec{r})}{P^0-P_n^0}$$
$$\Rightarrow\int\frac{d^3kd^3q}{(2\pi)^6}\phi_m^*(\vec{k})W(\vec{k}\vec{q}P_mP_n)\phi_n(\vec{q}) = \delta_{nm}\,, \tag{2.11}$$

where

$$W(\vec{k}\vec{q}P_mP_n) = (2\pi)^3\delta^3(\vec{k}-\vec{q}) - \frac{i\tilde{K}(\vec{k}\vec{q}P_m)-i\tilde{K}(\vec{k}\vec{q}P_n)}{P_m^0-P_n^0}\,.$$

Perturbation theory for this equation also follows as in I. Let $\phi_n^0(\vec{k})$ be the eigenfunction and ϵ_n^0 the eigenvalue ($P^0\equiv m_1+m_2+\epsilon$) of (2.10b) with kernel $\tilde{K}_0$. Then if $\tilde{G}_0$ is the corresponding Green's function (2.10a), the perturbed energies and wave functions when $\tilde{K}=\tilde{K}_0+\delta\tilde{K}$ are given by

$$\epsilon_n = \epsilon_n^0 + (\phi_n^{0*}i\delta\tilde{K}\phi_n^0)\left[1+\left(\phi_n^{0*}\frac{\partial}{\partial\epsilon}i\delta\tilde{K}\phi_n^0\right)\right]_{\epsilon=\epsilon_n^0}$$
$$+\left[\phi_n^{0*}i\delta\tilde{K}\left(\tilde{G}_0-\frac{\phi_n^0\phi_n^{0*}}{\epsilon-\epsilon_n^0}\right)i\delta\tilde{K}\phi_n^0\right]\Big|_{\epsilon=\epsilon_n^0} + O(\delta\tilde{K}^3)$$
$$\phi_n\propto\phi_n^0\left[1+\left(\phi_n^{0*}\frac{\partial}{\partial\epsilon}i\delta\tilde{K}\phi_n^0\right)\right]_{\epsilon=\epsilon_n^0}$$
$$+\left(\tilde{G}_0-\frac{\phi_n^0\phi_n^{0*}}{\epsilon-\epsilon_n^0}\right)i\delta\tilde{K}\phi_n^0\Big|_{\epsilon=\epsilon_n^0} + O(\delta\tilde{K}^2)\,,$$

where the momentum integrations are implicit. Note that these formulas are also valid when ϕ, ϕ^*, $\delta\tilde{K}$, and $\tilde{G}$ are replaced by ψ, $\bar{\psi}$, $\delta\bar{K}$, and $\bar{G}$, respectively.

III. UNPERTURBED PROBLEM IN QED

Equation (2.10b) is rendered more tractable by multiplying on both sides by $N(k)/\sqrt{N(k)}$ where[12]

$$N(k)=\frac{[P_0+E_1(k)+E_2(k)]\{P_0^2-[E_1(k)-E_2(k)]^2\}}{2P_0[P_0^2-(m_1-m_2)^2]}$$
$$\simeq 1+\frac{k^2}{4m^2}-\frac{3}{4}\frac{\vec{k}^2}{m_1m_2}-\frac{\epsilon}{2(m_1+m_2)}+\cdots \quad |\vec{k}|\ll m \tag{3.1}$$

and

$$m\equiv m_1m_2/(m_1+m_2)$$

is the reduced mass. The resulting equation is

$$\left(\tilde{\epsilon}-\frac{\vec{k}^2}{2\tilde{m}}\right)\frac{\phi(\vec{k})}{\sqrt{N(k)}}$$
$$=\int\frac{d^3q}{(2\pi)^3}[N(k)N(q)]^{1/2}i\tilde{K}(\vec{k}\vec{q}P)\frac{\phi(\vec{q})}{\sqrt{N(q)}}. \tag{3.2}$$

This is just a Schrödinger equation for an effective particle with "binding energy" and "mass"

$$\tilde{\epsilon}=\frac{P_0^2-(m_1+m_2)^2}{2P_0}\simeq\epsilon-\frac{\epsilon^2}{2(m_1+m_2)}+\cdots,$$
$$\tilde{m}=\frac{P_0^2-(m_1-m_2)^2}{4P_0}\simeq m+\frac{\epsilon}{2}\left(1-\frac{2m}{m_1+m_2}\right)+\cdots.$$

We emphasize that this equation is exact and equivalent to (2.10b).

For QED bound states, the choice of zeroth-order kernel is now obvious

$$i\tilde{K}_0(\vec{k}\vec{q}P)=(-e^2/|\vec{k}-\vec{q}|^2)\{1/[N(k)N(q)]^{1/2}\}, \tag{3.3}$$

as then (2.10b) reduces to the Schrödinger-Coulomb equation

$$\left(\tilde{\epsilon}-\frac{\vec{k}^2}{2\tilde{m}}\right)\frac{\phi(\vec{k})}{\sqrt{N(k)}}=\int\frac{d^3q}{(2\pi)^3}\frac{-e^2}{|\vec{k}-\vec{q}|^2}\frac{\phi(\vec{q})}{\sqrt{N(q)}}. \tag{3.4}$$

The eigenfunctions are simply related to the (normalized) nonrelativistic Schrödinger wave functions with m replaced by $\tilde{m}$

$$\phi(\vec{k})=\left(\frac{N(k)}{1+\alpha^2/4n^2}\right)^{1/2}\phi_{\rm Sch}(\vec{k};\tilde{m}),\quad n=1,2,\ldots,$$
$$\Rightarrow\psi(\vec{k})=\frac{u^{(1)}(\vec{k})u^{(2)}(-\vec{k})}{[4E_1(k)E_2(k)]^{1/2}}\phi(\vec{k}).$$

The normalization is fixed by Eq. (2.11). Note that $\phi(\vec{x}=0)\propto\int d^3k\phi(k)$ is always finite in the unperturbed problem. The unperturbed energy levels follow by solving

$$\frac{P_0^2-(m_1+m_2)^2}{2P_0}=-\frac{\alpha^2}{2n^2}\left(\frac{P_0^2-(m_1-m_2)^2}{4P_0}\right)$$
$$\Rightarrow P_0=(m_1+m_2)\left(1-\frac{\alpha^2}{n^2+\frac{1}{4}\alpha^2}\frac{m}{m_1+m_2}\right)^{1/2}$$
$$\simeq m_1+m_2-\frac{\alpha^2m}{2n^2}+\frac{\alpha^4m}{8n^4}\left(1-\frac{m}{m_1+m_2}\right)$$
$$+O(\alpha^6).$$

It is readily demonstrated that the remaining $O(\alpha^4)$ terms are due to the following static kernels (in Coulomb gauge)[13]:

(a) Relativistic corrections to single Coulomb exchange [Fig. 3(a)]:

$$i\delta\bar{K}_c=\frac{-e^2}{|\vec{k}-\vec{q}|^2}\gamma_0^{(1)}\gamma_0^{(2)}-i\bar{K}_0$$
$$\Rightarrow i\delta\tilde{K}_c=\frac{-e^2}{|\vec{k}-\vec{q}|^2}\left\{\frac{\vec{k}\cdot\vec{q}}{4m^2}-\frac{\vec{k}^2+\vec{q}^2+4\vec{k}\cdot\vec{q}}{8m_1m_2}\right.$$
$$-\frac{\epsilon}{2(m_1+m_2)}$$
$$\left.+\frac{i\vec{k}\times\vec{q}\cdot\vec{\sigma}_1}{4m_1^2}+\frac{i\vec{k}\times\vec{q}\cdot\vec{\sigma}_2}{4m_2^2}\right\}. \tag{3.5a}$$

(b) Single transverse photon exchange [Fig. 3(b)]:

$$i\delta\bar{K}_T=\frac{e^2}{|\vec{k}-\vec{q}|^2}\gamma_i^{(1)}\gamma_j^{(2)}\delta_\perp^{ij}(\vec{k}-\vec{q})$$
$$\Rightarrow i\delta\tilde{K}_T\simeq\frac{e^2}{|\vec{k}-\vec{q}|^2}\left\{\frac{(\vec{k}\cdot\vec{q})^2-\vec{k}^2\vec{q}^2}{m_1m_2|\vec{k}-\vec{q}|^2}\right.$$
$$-\frac{i\vec{k}\times\vec{q}\cdot(\vec{\sigma}_1+\vec{\sigma}_2)}{2m_1m_2}$$
$$\left.+\frac{(\vec{k}-\vec{q})\times\vec{\sigma}_1\cdot(\vec{k}-\vec{q})\times\vec{\sigma}_2}{4m_1m_2}\right\}. \tag{3.5b}$$

(c) Single photon annihilation [positronium only; Fig. 3(c)]:

$$i\delta\bar{K}_A=[\gamma_F\cdot\gamma_I/(P^0)^2]e^2$$
$$\Rightarrow i\delta\bar{K}_A\simeq\frac{e^2}{8m_1m_2}(3+\sigma_1\cdot\sigma_2),\quad m_1=m_2, \tag{3.5c}$$

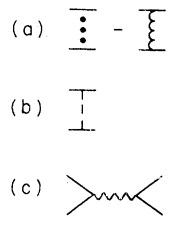

FIG. 3. Kernels contributing to $O(\alpha^4)$ energy-level corrections.

Only the dominant parts of each kernel have been exhibited.

These kernels are important for the analysis presented in Sec. IV. We will also require the ground-state ($n=1$) wave function

$$\phi_0(\vec{k})=\left(\frac{N(k)}{1+\frac{1}{4}\alpha^2}\right)^{1/2}\left(\frac{\gamma^3}{\pi}\right)^{1/2}\frac{8\pi\gamma}{(\vec{k}^2+\gamma^2)^2}\chi^{(1)}\chi^{(2)}$$
$$\simeq\left(\frac{\gamma^3}{\pi}\right)^{1/2}\frac{8\pi\gamma}{(\vec{k}^2+\gamma^2)^2}\chi^{(1)}\chi^{(2)}\,|\vec{k}|\ll m_1,m_2\,, \quad (3.6)$$
$$\gamma=\alpha\tilde{m}\simeq\alpha m\,,$$

where $\chi^{(1)}$, $\chi^{(2)}$ are two-component spinors.

We now examine the Green's function $\tilde{G}_0$ for kernel $\tilde{K}_0$. We require

$$\lim_{\epsilon\to\epsilon^0}[\tilde{G}_0-\phi_0\phi_0^*/\epsilon-\epsilon^0]\,,$$

for second-order perturbation theory (2.12). Applied to (2.10a), the arguments used above lead to a simple relationship between $\tilde{G}_0$ and the nonrelativistic Schrödinger-Coulomb propagator[14]

$$\tilde{G}_0(\vec{k}\vec{q}P)=[N(k)N(q)]^{1/2}G_{\rm Sch}(\vec{k}\vec{q}\tilde{\epsilon};\tilde{m})$$
$$\simeq G_{\rm Sch}(\vec{k}\vec{q}\epsilon;m)\;\;|\vec{k}|\,,|\vec{q}|\ll m_1,m_2\,. \quad (3.7)$$

Though analytic expressions exist for $G_{\rm Sch}$ in coordinate space,[15] we find it convenient to use an expression in momentum space due to Schwinger[16]

$$G_{\rm Sch}(\vec{k}\vec{q}\epsilon;m)=\frac{(2\pi)^3\delta^3(\vec{k}-\vec{q})}{\epsilon-\vec{k}^2/2m}-\frac{1}{\epsilon-\vec{k}^2/2m}\,\frac{e^2}{|\vec{k}-\vec{q}|^2}\,\frac{1}{\epsilon-\vec{q}^2/2m}$$
$$-\frac{e^2}{\epsilon-\vec{k}^2/2m}\int_0^1 d\rho\;\frac{i\eta\rho^{-i\eta}}{|\vec{k}-\vec{q}|^2\rho-m/2\epsilon(\epsilon-\vec{k}^2/2m)(\epsilon-\vec{q}^2/2m)(1-\rho)^2}\,\frac{1}{\epsilon-\vec{q}^2/2m}$$

where $i\eta=\alpha m/\sqrt{-2m\epsilon}$. The first two terms are just the zero and one Coulomb terms in the Born series. Integrating by parts and taking $i\eta\to 1$ we can isolate and remove the ground-state pole, and perform the ρ integration. The resulting (exact) expression is[17]

$$\lim_{\tilde{\epsilon}\to\tilde{\epsilon}_0}\left(G_{\rm Sch}(\vec{k}\vec{q}\tilde{\epsilon};\tilde{m})-\frac{\phi^0_{\rm Sch}(\vec{k})\phi^0_{\rm Sch}(\vec{q})^*}{\tilde{\epsilon}-\tilde{\epsilon}_0}\right)$$
$$=\frac{-64\pi}{\alpha\gamma^4}\left(\frac{\pi^2\gamma^5\delta^3(\vec{k}-\vec{q})}{4(\vec{k}^2+\gamma^2)}+\frac{\gamma^6}{4(\vec{k}^2+\gamma^2)|\vec{k}-\vec{q}|^2(\vec{q}^2+\gamma^2)}\right.$$
$$\left.+\tilde{R}(\vec{k}\vec{q})\right)\,. \quad (3.8a)$$

$\tilde{R}(\vec{k}\vec{q})$ represents all contributions due to exchange of two or more Coulomb photons and is given by

$$\tilde{R}(\vec{k}\vec{q})=\frac{\gamma^8}{(\vec{k}^2+\gamma^2)^2(\vec{q}^2+\gamma^2)^2}\left(\frac{5}{2}-\frac{4\gamma^2}{\vec{k}^2+\gamma^2}-\frac{4\gamma^2}{\vec{q}^2+\gamma^2}+\frac{1}{2}\ln A\right.$$
$$+\frac{2A-1}{(4A-1)^{1/2}}$$
$$\left.\times\tan^{-1}(4A-1)^{1/2}\right)\,, \quad (3.8b)$$

$$A=(\vec{k}^2+\gamma^2)(\vec{q}^2+\gamma^2)/4\gamma^2|\vec{q}-\vec{k}|^2\,.$$

It is convenient when computing $O(\alpha^6)$ hfs to isolate the zero and one Coulomb terms as these result in one- and two-loop kernels [when inserted into (2.12)] which are most easily computed with all other kernels of the same order. In Sec. IV we compute all terms involving the remainder $\tilde{R}$, i.e., the kernels which arise when we substitute[18]

$$\lim_{\epsilon\to\epsilon^0}(\tilde{G}_0-\phi_0\phi_0^*/\epsilon-\epsilon^0)\to(-64\pi/\alpha\gamma^4)\tilde{R}\,,$$

in (2.12). As mentioned earlier these are the only relevant kernels having three or more loops, aside from the $O(\alpha^3\Delta E_0)$ which are calculable in Dirac theory.

IV. HFS IN MUONIUM AND POSITRONIUM

All kernels, except one, contributing to $O(\alpha^6)$ ground-state splitting in this formalism are displayed in Fig. 4. These have been expressed in terms of the BS kernel $K_{\rm BS}$ (Fig. 5), the unperturbed kernel $\tilde{K}_0$, and $\tilde{R}$ [Eq. (3.8(b)], by combining expansions (2.5), (2.12), and (3.8). Only those parts of $K_{\rm BS}$ need be retained in Fig. 4(a) as result in diagrams with two or fewer loops.

The only contribution not appearing in Fig. 4 is

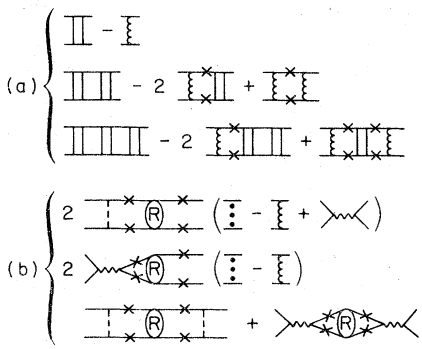

FIG. 4. Kernels contributing to $O(\alpha^6)$ hfs. A double line represents the two-particle irreducible BS kernel (Fig. 5).

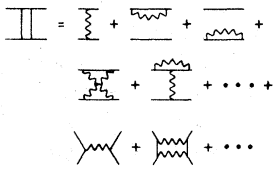

FIG. 5. Two-particle irreducible BS kernel.

trivially computed

$$\delta E = E_0\left[\phi_0^*\frac{\partial}{\partial\epsilon}i\delta\tilde{K}_C\phi_0\right]_{\epsilon=\epsilon_0} = \begin{cases} \frac{7}{96}\alpha^6 m_e \\ \dfrac{4\gamma^3\alpha^3}{3(m_e+m_\mu)^2}. \end{cases}$$

In what follows, we compute the hfs due to the kernels in Fig. 4(b). To exhibit the mass dependences, calculations are for muonium when it is appropriate. The corresponding results for positronium are found simply by setting $m_\mu = m_e$. Note that only the dominant perturbation kernels [Eq. (3.5)] must be treated in second-order perturbation theory, and then only in the region of nonrelativistic momentum. Note also that $i\delta\tilde{K}_T$ and $i\delta\tilde{K}_A$ alone contain spin-spin interactions, and thus we need only consider pairs of interactions which include one or the other of these kernels.

To illustrate the procedure, we consider the term containing $i\delta\tilde{K}_C$ and $i\delta\tilde{K}_T$:

$$\delta E_{CT} = 2\int\frac{d^3r\,d^3p}{(2\pi)^6}\frac{d^3k\,d^3q}{(2\pi)^6}\phi_0^*(\vec{r})i\delta\tilde{K}_C(\vec{r}\vec{k})\left(\frac{-64\pi}{\alpha\gamma^4}\right)\tilde{R}(\vec{k}\vec{q}) \times i\delta\tilde{K}_T(\vec{q}\vec{p})\phi_0(\vec{p})\Big|_{J=0}^{J=1}.$$

The spherical symmetry of the wave functions allows us to drop spin dependent terms in $i\delta\tilde{K}_C$ and for hfs to replace $i\delta\tilde{K}_T$ by

$$i\delta\tilde{K}_T \to (e^2/6m_e m_\mu)\vec{\sigma}_e\cdot\vec{\sigma}_\mu$$

We find

$$\delta E_{CT} = E_F\alpha^2\frac{64\pi^2}{\gamma^6}\int\frac{d^3p}{(2\pi)^3}\frac{8\pi\gamma}{(\vec{p}^2+\gamma^2)^2}\int\frac{d^3r}{(2\pi)^3}\frac{8\pi\gamma}{(\vec{r}^2+\gamma^2)^2} \times\int\frac{d^3k\,d^3q}{(2\pi)^6}\left(\frac{2\vec{p}\cdot\vec{q}}{|\vec{p}-\vec{q}|^2}-\frac{\vec{p}^2+\vec{q}^2+4\vec{p}\cdot\vec{q}+4m\epsilon}{|\vec{p}-\vec{q}|^2} \times\frac{m}{m_e+m_\mu}\right)\tilde{R}(\vec{q}\vec{k}),$$

where E_F is the hfs in lowest order (Fermi splitting)

$$E_F = \frac{2}{3}\frac{\gamma^3\alpha}{m_e m_\mu}\langle\vec{\sigma}_e\cdot\vec{\sigma}_\mu\rangle\Big|_{J=0}^{J=1} = \frac{8}{3}\frac{\gamma^3\alpha}{m_e m_\mu},$$

($\to\frac{1}{3}\alpha^4 m_e$, in positronium). The r and p integrations are easily performed (using Feynman parameters for the latter), leaving

$$\delta E_{CT} = \frac{E_F\alpha^2}{\pi^4}\int\frac{d^3k\,d^3q}{\gamma^6}\left[2\left(\frac{\gamma}{q}\tan^{-1}\frac{q}{\gamma}-\frac{\gamma^2}{q^2+\gamma^2}\right)-\frac{m}{m_e+m_\mu}\left(\frac{6\gamma}{q}\tan^{-1}\frac{q}{\gamma}-\frac{8\gamma^2}{q^2+\gamma^2}+1\right)\right]\tilde{R}(\vec{q}\vec{k})$$
$$= E_F\left[\left(\frac{9}{4}-\frac{\pi^2}{6}\right)\alpha^2-\left(\frac{17}{2}-\frac{\pi^2}{2}\right)\frac{\gamma^2}{m_e m_\mu}\right].$$

The last integrals were evaluated numerically[19]; the analytic results quoted here agree to at least five significant figures with the numerical results. The term $(\frac{9}{4}-\frac{1}{6}\pi^2)\alpha^2 E_F$ when combined with similar contributions from Fig. 4(a) results in the usual Breit-Dirac correction $\frac{3}{2}\alpha^2 E_F$. The remaining term is a new recoil correction.

A similar analysis has been performed for each term in Fig. 4(b). The results are summarized in Table I. δE_{AA} agrees with the value computed in Ref. 20.

We list here the final integrals for each case. Again these were evaluated numerically to one part in 10^5 or better. As the spin-spin part of

TABLE I. $O(\alpha^6)$ hfs from second-order perturbation theory involving kernels with three or more loops.

	Muonium Coefficient of $[\alpha^2 m_e m_\mu/(m_e+m_\mu)^2]E_F$	Positronium Coefficient of $\alpha^6 m_e$
δE_{CT}	$[(m_e+m_\mu)^2/m_e m_\mu](\frac{9}{4}-\frac{1}{6}\pi^2)-\frac{17}{2}+\frac{1}{2}\pi^2$	$\frac{1}{24}-\frac{1}{72}\pi^2$
δE_{TT}	$11-\frac{1}{72}-\frac{2}{3}\pi^2$	$\frac{791}{864}-\frac{1}{18}\pi^2$
δE_{AT}	...	$\frac{7}{16}-\frac{1}{24}\pi^2$
δE_{CA}	...	$\frac{1}{32}-\frac{1}{96}\pi^2$
δE_{AA}	...	$-\frac{3}{16}$
Sum	$[(m_e+m_\mu)^2/m_e m_\mu]\,0.6051+0.8412$	0.038 99

$\delta\tilde{K}_A$ is $\frac{3}{4}$ that of $\delta\tilde{K}_T$, we find $\delta E_{CA} = \frac{3}{4}\delta E_{CT}(m_\mu \to m_e)$ for these contributions.

$$\delta E_{TT} = \frac{E_F}{\pi^4}\frac{\gamma^2}{m_e m_\mu}\left\{8\int\frac{d^3k\,d^3q}{\gamma^6}\left(\frac{\gamma}{k}\tan^{-1}\frac{k}{\gamma}-\frac{\gamma^2}{\vec{k}^2+\gamma^2}+\frac{3}{16}\right)\tilde{R}(\vec{k}\vec{q})-\frac{1}{2}\int\frac{d^3k\,d^3q}{\gamma^6}\frac{\tilde{R}(\vec{k}\vec{q})}{\vec{k}^2\vec{q}^2}\left[\frac{(\vec{k}\cdot\vec{q})^2}{\vec{k}^2\vec{q}^2}\left(\frac{3}{2}+\frac{\vec{k}^2}{\gamma^2}-\frac{3}{2}\frac{\vec{k}^2+\gamma^2}{\gamma k}\tan^{-1}\frac{k}{\gamma}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{3}{2}+\frac{\vec{q}^2}{\gamma^2}-\frac{3}{2}\frac{\vec{q}^2+\gamma^2}{\gamma q}\tan^{-1}\frac{q}{\gamma}\right)+2\left(\frac{3}{4}-\frac{3}{4}\frac{\vec{k}^2+\gamma^2}{\gamma k}\tan^{-1}\frac{k}{\gamma}+\frac{\vec{k}^2}{\gamma^2}\right)\left(-\frac{1}{2}+\frac{1}{2}\frac{\vec{q}^2+\gamma^2}{\gamma q}\tan^{-1}\frac{q}{\gamma}\right)\right]\right\}$$
$$= E_F\frac{\gamma^2}{m_e m_\mu}\left(11-\frac{1}{72}-\frac{2}{3}\pi^2\right),$$

$$\delta E_{AT} = \frac{\alpha^6 m_e}{2\pi^4}\int\frac{d^3k\,d^3q}{\gamma^6}\left(\frac{\gamma}{k}\tan^{-1}\frac{k}{\gamma}-\frac{\gamma^2}{\vec{k}^2+\gamma^2}-\frac{1}{6}\right)\tilde{R}(\vec{k}\vec{q}) = \alpha^6 m_e\left(\frac{7}{16}-\frac{\pi^2}{24}\right),$$

$$\delta E_{CA} = \frac{3}{4}\delta E_{CT}(m_\mu \to m_e) = \alpha^6 m_e\left(\frac{1}{32}-\frac{\pi^2}{96}\right),$$

$$\delta E_{AA} = -\frac{\alpha^6 m_e}{8\pi^4}\int\frac{d^3k\,d^3q}{\gamma^6}\tilde{R}(\vec{k}\vec{q}) = -\frac{3}{16}\alpha^6 m_e .$$

Perturbations of δE_{TT}, δE_{AT}, and δE_{AA} are independent to this order of the details of the bound-state formalism used. The same results should occur in most any analysis and in particular they appear in a BS treatment or in the formalism of I. On the other hand, $i\delta K_C$ is very dependent upon the nature of the propagator and of the unperturbed kernel used. Thus the formalism described in I gives the following results:

$$\delta E_{CT} = E_F\frac{\gamma^2}{m_e m_\mu}\left[\frac{\pi^2}{6}-\frac{7}{2}+\frac{m_e}{m_\mu}\left(\frac{5}{2}-\frac{\pi^2}{6}\right)\right],$$

$$\delta E_{CA} = -\tfrac{1}{16}\alpha^6 m_e .$$

V. CONCLUSIONS

In this paper we have described further alternatives to the traditional BS analysis of bound states in field theory.[21] These novel methods are well suited to computational QED as analytic solutions of great simplicity can be found for a zeroth-order interaction containing the basic physics. The corrections to this basic interaction are then unambiguously specified by a simple perturbation theory.

Applying these results, we have computed new $O(\alpha^6)$ terms in the ground-state splitting of muonium and positronium which require an all orders treatment of the binding potential. Of the terms still to be computed before theory matches experiment in precision, only those of $O(\alpha^3 E_F)$ present a major conceptual problem. Evaluation of the remaining terms [Fig. 4(a)] is straightforward though perhaps tedious.

Note added in proof All terms in Table I other than δE_{TT} have now been evaluated analytically using the general identity

$$\frac{1}{\pi^4}\int\frac{d^3k\,d^3q}{\gamma^6}\tilde{R}(\vec{k}\vec{q})f(k) = -\frac{4}{\pi}\int_0^\infty\frac{\gamma k^2 dk}{(k^2+\gamma^2)^2}f(k)\left[\ln 2-\frac{5}{2}+\frac{\gamma}{k}\tan^{-1}\left(\frac{k}{\gamma}\right)-\frac{1}{2}\ln\left(1+\frac{k^2}{\gamma^2}\right)+\frac{4\gamma^2}{k^2+\gamma^2}\right].$$

The results in Table I are correct.

ACKNOWLEDGMENTS

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APPENDIX A: COMMENTS ON RENORMALIZATION

The solutions presented in Sec. III contain none of the divergences associated with the short-distance (high-energy) behavior of QED. In particular the wave functions at the origin

$$\psi(\vec{x}=0) = \int\frac{d^3k}{(2\pi)^3}\psi(k)$$

are finite. Here we illustrate how this property allows us to disentangle the low-energy features of the field theory (e.g., bound states) from the high-energy features (e.g., UV divergences). This is most desirable as the first must be analyzed to all orders, while the latter are most conveniently handled in perturbation theory.

The wave functions in Sec. III are finite at $\vec{x}=0$ only because $\bar{K}^0$ falls faster for large momenta than does the true one-photon interaction. Indeed the BS wave function ψ_{BS} for the exact one-photon interaction [Fig. 6(a)] is mildly divergent at the origin (just like solutions of the Dirac-Coulomb equation). This divergence causes problems only when evaluating the energy shift due to one-photon

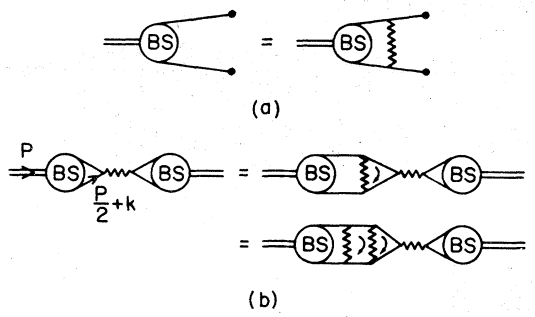

FIG. 6. Divergent subdiagrams in the expectation value of the one-photon annihilation kernel with wave function ψ_{BS}.

annihilation [Fig. 6(b)] and similar kernels. Since the annihilation kernel is independent of relative momentum k, the perturbation is proportional to $|\psi_{BS}(x=0)|^2 \sim |\int d^4k\psi(k)|^2 = \infty$. This expectation value contains an infinity of divergent vertex subdiagrams, as is evident when the wave function is iterated [Fig. 6(b)]. Thus the energy shift has the form

$$\Delta E(\Lambda) = K\alpha^4[1 + \alpha f_1(\Lambda) + \alpha^2 f_2(\Lambda) + \cdots],$$

if k is cutoff at Λ. The functions $f_i(\Lambda)$ all diverge as $\Lambda \to \infty$ and these divergences are removed only by an all orders vertex renormalization. Note that the leading order contribution is finite $[= |\psi_{NR}(0)|^2 \mathfrak{M}(e\bar{e} \to e\bar{e})$ where ψ_{NR} is the non-relativistic wave function and $\mathfrak{M}$ is the annihilation amplitude at threshold]. The divergence is a relativistic effect and as such is suppressed by a factor $\langle v/c \rangle \approx \alpha$.

The infinity in $\psi_{BS}(0)$ is spurious insofar as it is removed by a (complicated) renormalization when computing one-photon annihilation terms, and cutoff by momenta of $O(m)$ in other terms. Using the more convergent wave functions ψ from Sec. III, UV divergences only appear within the kernels themselves. These are removed order by order in just the way they are removed from on-shell amplitudes. No further infinities can be introduced when evaluating the corresponding expectation values since $\psi(\vec{x}=0)$ is finite. Thus the perturbation due to lowest-order annihilation [Fig. 3(c)] is finite for these wave functions, and agrees in lowest order with $\Delta E(\Lambda)$. The divergent parts of $\Delta E(\Lambda)$ appear here, one at a time, in higher-order kernels. For example, the first-order vertex correction (Fig. 7) arises from terms in the second line of Fig. 4(a). The kernel in Fig. 7(a) diverges as loop momentum $k \to \infty$, and reproduces the lowest-order divergence in ΔE [i.e., $K\alpha^5 f_1(\Lambda)$]. This divergence is exactly cancelled by the usual (lowest-order) renormalization counter-term [Fig. 7(b)] for all finite external momenta q. The q-integration must then converge because $\int d^3q\psi(q)$ does. Thus the energy shift due to the kernels in Fig. 3(c), 7 is completely finite.

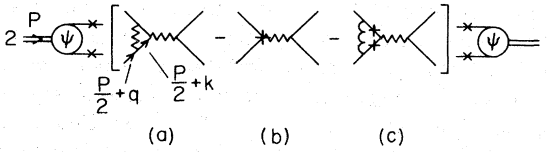

FIG. 7. First-order radiative corrections to one-photon annihilation kernel using solutions from Sec. III.

Finally, we note that the Green's functions and kernels discussed in Sec. II are all unrenormalized (though masses and charges in $\bar{K}_0$, ψ *are* physical). Overall multiplicative constants, such as Z_2, cannot shift the locations of bound state poles in the Green's function. It is obvious from the derivation of perturbation theory (see Ref. 4) that such constant factors cancel in the final expression for the perturbed energy. As mentioned above, the masses and charges appearing in the unperturbed interaction (K_0) and wave functions (Sec. III) are the physical quantities. Consequently all renormalization is due to counterterms appearing in the kernels (Fig. 4) of the bound-state perturbation theory [Eq. (2.12)]. In particular it is *not* correct to replace the unperturbed wave function ψ by $Z_2\psi$ when calculating radiative corrections. The factors of $\sqrt{Z_2}$ required for charge renormalization already occur in the kernel. To illustrate this, consider the first-order radiative corrections on the electron line using the BS wave function de-

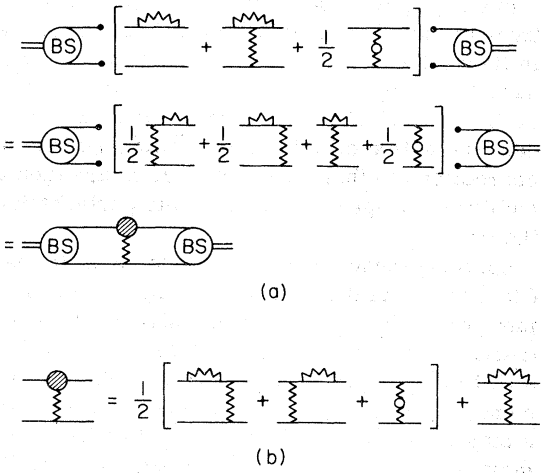

FIG. 8. (a) Perturbation due to first-order radiative corrections related to the electron. Similar terms must be included for the muon (or positron). Renormalization counterterms are implicit. (b) Definition of the "effective vertex".

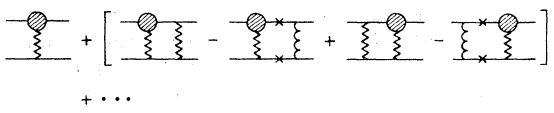

FIG. 9. Terms from Fig. 4 due to first-order radiative corrections as rewritten in terms of the "effective vertex" defined in Fig. 8(b).

picted in Fig. 6(a). By iterating the wave function, we can express these corrections in terms of a single "effective vertex" (Fig. 8). Clearly charge is properly renormalized. Similar rearrangements of perturbation theory can be obtained beginning with the solutions in Sec. III. The lowest-order radiative corrections in Fig. 4 can readily be rewritten in terms of the same "effective vertex" (Fig. 9).

An advantage of grouping terms as in Fig. 9 is that Z_1 and Z_2 cancel explicitly because of Ward's identity (QED). These are very complicated momentum dependent renormalization factors in Coulomb gauge and it is fortunate that they need not be computed. The vacuum polarization is gauge invariant in QED and as such it (and Z_3) can be computed in Feynman gauge (or any other gauge one might prefer).

A detailed application of renormalization theory is described in Ref. 22 for bound states in Dirac theory. Most of that discussion applies to two particle bound-state theory as well.

APPENDIX B: RELATION TO THE BETHE-SALPETER FORMALISM

At a bound-state energy P_n^0, the complete two-particle Green's function has a pole

$$G_T(kqP) \to [-i\phi_{BS}(k)\bar{\phi}_{BS}(q)/(P_0 - P_n^0)]. \tag{B1}$$

Here ϕ_{BS} is the truncated BS wave function

$$\psi_{BS}(k) = \frac{i}{\tau_1 \not{P} + \not{k} - m_1} \frac{i}{\tau_2 \not{P} - \not{k} - m_2} \phi_{BS}(k)$$

$$= \frac{i}{\tau_1 \not{P} + \not{k} - m_1} \frac{i}{\tau_2 \not{P} - \not{k} - m_2} \times \int \frac{d^4q}{(2\pi)^4} K_{BS}(kqP)\psi_{BS}(q). \tag{B2}$$

Substituting (1) into (2.1) and evaluating at the pole we find (momentum integrations implicit)

$$\phi_{BS} = K(P)\Lambda S(P)\phi_{BS}, \tag{B3}$$

where Λ and $S(P)$ are arbitrary. Defining a new wave function $\psi = \Lambda S(P)\phi_{BS}$, we obtain

$$S^{-1}(P)\psi = \Lambda K(P)\psi. \tag{B4}$$

This is simply the effective bound-state equation of the formalism defined by Λ and S [Eq. (2.7)]. Thus given the solutions ψ of (4), the BS wave function is just $\phi_{BS} = K(P)\psi$.

Specializing to the formalism in Sec. III, we see that when K_{BS} is static (independent of k^0) the truncated BS wave function is

$$\phi_{BS}(\vec{k}) = \int \frac{d^3q}{(2\pi)^3} i\bar{K}(\vec{k}\vec{q}P)\psi(\vec{q}). \tag{B5}$$

This is true only when K_{BS} is static, as only then is ϕ_{BS} independent of k^0 [Eq. (3)]. In the general case, $\bar{K}$ must be redefined to include the k^0 behavior of K_{BS} and its iterates. Whether or not K_{BS} is static, the following relation is valid:

$$\psi(\vec{q}) = \frac{\Lambda_+^{(1)}(\vec{k})\Lambda_+^{(2)}(-\vec{k})}{P^0 - E_1(k) - E_2(k)} \phi_{BS}(k^0 = 0, \vec{k}).$$

It has recently been suggested that high-order computations be performed in two stages.[21] First the BS wave function is determined for the fully relativistic (static) Coulomb interaction using a perturbative expansion. This wave function is then used in BS perturbation theory to compute contributions from transverse photons, cross graphs, etc. The basic difference between this approach and that described in this paper is that we abandon the BS formalism completely. All perturbations, static or otherwise, are treated in the same Schrödinger-like theory, avoiding the need for two separate perturbation series. Note, however, that the methods described in this paper (or in I) together with (5) can be used to determine the BS wave function to any level of accuracy for a static kernel. Thus, they are of use even if the two stage approach is adopted.

Finally we note that if $\psi(\vec{k})$ is a solution of Eq. (2.7) for some kernel $\bar{K}(\vec{k}\vec{q}P)$, then wave function

$$\phi_{BS}(\vec{k}) \equiv (P^0 - E_1(k) - E_2(k))\psi(\vec{k}),$$

is an exact (truncated) solution of the BS equation with kernel

$$K_{BS}(\vec{k}\vec{q}P) = \Lambda_+^{(1)}(\vec{k})\Lambda_+^{(2)}(-\vec{k})\bar{K}(\vec{k}\vec{q}P)\Lambda_+^{(1)}(\vec{q}) \times \Lambda_+^{(2)}(-\vec{q})\gamma_0^{(1)}\gamma_0^{(2)}.$$

Thus, the wave functions of Sec. III are also exact solutions of the BS equation with this kernel ($\bar{K} \to \bar{K}_0$). It is possible to restate all of the analysis in this paper in terms of the BS formalism, using these as the unperturbed BS wave functions. However such an approach is awkward (a) because it obscures the simple connection with nonrelativistic Schrödinger theory, and (b) because the wave functions ψ_{BS} [Eq. (2)] still depend upon relative time (or energy).

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[6]See summary in Sec. V of Ref. 4; V. K. Cung, A. Devoto, T. Fulton, and W. W. Repko, Phys. Lett. B 68, 474 (1977). The *naive* order for any kernel is α^3 (for the wave function squared) times one α for each photon in the graph. Threshold singularities, if present, increase the contribution over this naive estimate.

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[10]This choice of k^0 and q^0 results in constituent energies $E_1=\tau_1 P_0$, $E_2=\tau_2 P_0$. This is but one of many possible alternatives. Another is

$$E_1=\frac{P_0^2-m_2^2+m_1^2}{2P_0},\quad E_2=\frac{P_0^2-m_1^2+m_2^2}{2P_0}$$

which is symmetric in the masses and which restricts the constituents to their mass shell when P_0 is above threshold. Thus $\bar{G}$ becomes the usual scattering amplitude when $P_0 \geq m_1+m_2$. This last property also follows when one constituent is put on its mass shell

$$E_1=(\vec{k}^2+m_1^2)^{1/2},\quad E_2=P_0-E_1.$$

This choice also results in a limited form of gauge invariance (see Sec. V of Ref. 4). None of the results of this paper is altered if either of these alternatives is adopted.

[11]It is conceivable that the residue vanishes and thus $\bar{G}$ may contain fewer bound states than G_T. This is not the case for nonrelativistic QED systems.

[12]This method of rationalizing the propagator is suggested in G. Bodwin and D. R. Yennie, Hyperfine Structure in Positronium and Muonium (unpublished).

[13]The Coulomb gauge seems to be optimal for atomic physics insofar as it incorporates the most physics in the simplest graphs. For example, an infinite number of kernels is required to $O(\alpha^4)$ in the Feynman gauge.

[14]The wave-function normalization is obtained immediately by equating the residues of the bound-state poles on each side of (3.7).

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[17]Note that $\tilde{R}$, like $\tilde{G}_0$, is independent of particle spins.

[18]There are other terms due to the additional factor in (3.7) and due to the nonlinear relationship between ϵ and $\tilde{\epsilon}$. These terms are easily computed but are of higher order in α and have been omitted.

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