

Quantum statistics of oscillatory superfluorescence†

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A generalized master equation previously proposed to describe oscillatory superfluorescence is numerically solved in the case of strong ringing. When the atomic system is fully excited, anomalous fluctuations and large atom-field correlations make the time evolution of the mean quantities (e.g., number of photons, total population inversion) quantitatively different from those found in the semiclassical treatment. In particular, because of fluctuations, the minima of the radiation output are notably larger than zero. On the contrary, when the atomic system exhibits a sizable initial macroscopic polarization (superradiance), we find a remarkable non-Markoffian effect consisting of a sizable delay in the radiation emission. However, in such conditions all quantum effects disappear, as in the case of pure superradiance.

I. INTRODUCTION

Superfluorescence is cooperative spontaneous emission of radiation from a fully excited atomic system. Since this radiation process is started by normal spontaneous emission, quantum effects can be relevant. A semiclassical treatment¹ of superfluorescence reduces the problem to a suitable damped-pendulum equation. As a result one sees that the radiation output is proportional to N^2 (N being the number of excited atoms) when

$$K^{-1} \lesssim \tau_R \ll T_2^*,$$

where (i) K^{-1} is the transit time of photons in the active region; (ii) τ_R is the characteristic time of cooperative spontaneous emission;² and (iii) T_2^* is the inverse of the inhomogeneous linewidth.

Furthermore, when $K\tau_R \gg 1$ (pure or Markoffian superfluorescence), the output consists in a single pulse of radiation, whereas for $K\tau_R \sim 1$ (oscillatory or non-Markoffian superfluorescence) the first pulse is followed by a few pulses of decreasing height. This nomenclature arises from the fact that in the case of pure superfluorescence the radiation field follows the motion of the atoms adiabatically so that one can perform a Markoff approximation in the generalized master equation (ME) for the atomic density operator.

The quantum effects in pure superfluorescence have been numerically analyzed.³ The main results are that (a) these quantum effects are important not only at the beginning of the emission but also during the whole time evolution, and (b) they do not disappear for very large N . More specifically, the macroscopically relevant quantities, e.g., the mean photon number or the mean total population inversion, exhibit sizable fluctua-

tions especially in correspondence to the peak of the radiation output. Moreover, these fluctuations are *not* "normal," since they do not scale as $N^{1/2}$ but as N . In Ref. 3 it is shown that these features are intimately connected with the presence of large atom-atom correlations which are the very cause of cooperative emission.

The numerical analysis of Ref. 3 has been analytically substantiated.⁴⁻⁸ These analyses have also shed light on the connection between the quantum and the semiclassical results. From Refs. 3-8 it appears that the presence of large fluctuations is due to the fact that in superfluorescence the pendulum starts from the very vicinity of the unstable equilibrium point, and this gives rise to a large delay in the radiation emission. Anomalous fluctuations around an unstable equilibrium point have been pointed out in Ref. 9.

In the present paper we study the quantum effects in oscillatory superfluorescence, i.e., the impact of quantum effects in the non-Markoffian case. To this aim, we solve numerically the ME for superfluorescence deduced in Ref. 1. This equation gives a full macroscopic and microscopic description of this phenomenon. In fact, from its solution we get, on the one hand, the time evolution of the mean values of the macroscopic quantities, which can be compared with the results of the semiclassical pendulum equation, and on the other hand, we also get the time evolution of the atomic and field fluctuations, of the atom-field correlations, and of the probability distribution.

The superfluorescence ME is solved for a number of atoms up to 20, which seems already to reproduce the macroscopic limit. The main results are the following. One finds anomalous fluctuations and large atom-field correlations which af-

fect the mean time evolution of the macroscopic quantities. In fact, as in the case of pure superfluorescence³ the pulses are appreciably lower and broader than the semiclassical ones. Furthermore, the effect of fluctuations on ringing (which is the main non-Markoffian effect) is that the minima of the radiation output are appreciably larger than zero. Correspondingly, the minima of the total population inversion are appreciably larger than $-N$. If, on the contrary, the atomic system has initially a sizable macroscopic polarization (superradiance), the quantum effects disappear, exactly as in the case of pure superradiance.³ However, we find a remarkable delay in the radiation output with respect to the case of pure superradiance in which the peak of the pulse occurs at $t=0$. This is a non-Markoffian retardation effect.

Our results agree with the numerical analysis recently performed in Ref. 10 on the basis of a suitable phase-space description of oscillatory superfluorescence.

In Sec. II we recall the operator ME used for the description of superfluorescence and illustrate the c -number equations that we solve numerically. The numerical results are given and commented on in Sec. III.

II. THE EQUATIONS

We consider a system of N two-level atoms contained in a "pencil-shaped" region of length L and volume V . The one-mode model deduced in Ref. 1 describes the interaction between these atoms whose transition frequency is ω_0 and a single longitudinal field mode resonant with them. Let $W(t)$ be the statistical operator of the whole system (atoms plus resonant mode) in the interaction picture. It obeys the following ME:

$$\begin{aligned} \frac{dW(t)}{dt} &= -iL_{AF}W(t) + \Lambda_F W(t), \\ L_{AF}W(t) &= \hbar^{-1}[H_{AF}, W(t)], \\ H_{AF} &= i\hbar g_0(AR^+ - A^\dagger R^-), \\ \Lambda_F W(t) &= K\{[A, W(t)A^\dagger] + [AW(t), A^\dagger]\}, \end{aligned} \quad (2.1)$$

where (i) g_0 is the coupling constant in the dipole and rotating-wave approximations,

$$g_0 = (\pi\omega_0\mu^2/V\hbar)^{1/2}, \quad (2.2)$$

with μ being the modulus of the dipole moment of the atomic transition; (ii) K^{-1} is the round-trip transit time of photons in the cavity,

$$K = c/2L; \quad (2.3)$$

(iii) R^\pm are collective dipole operators, and the set of operators R^+ , R^- , and R_3 , where R_3 is half

the total population inversion operator, obey the angular-momentum commutation relations

$$[R^+, R^-] = 2R_3, \quad [R^\pm, R_3] = \mp R^\pm; \quad (2.4)$$

and (iv) A is the annihilation operator for photons, which obeys the commutation relation

$$[A^\dagger, A] = -1. \quad (2.5)$$

Let us consider the basis $|n\rangle$ in the field Hilbert space:

$$A^\dagger A |n\rangle = n |n\rangle, \quad n = 0, 1, 2, \dots$$

and the Dicke-state basis $|m\rangle$ in the atomic Hilbert space:

$$\begin{aligned} R_3 |m\rangle &= m |m\rangle, \\ R^\pm |m\rangle &= \{(\frac{1}{2}N \mp m)(\frac{1}{2}N \pm m + 1)\}^{1/2} |m \pm 1\rangle, \\ m &= -\frac{1}{2}N, -(\frac{1}{2}N - 1), \dots, \frac{1}{2}N - 1, \frac{1}{2}N. \end{aligned} \quad (2.6)$$

Of course, Eq. (2.1) is equivalent to a set of linear c -number equations for the matrix elements

$$\begin{aligned} W_{n,m,n',m'}(t) &= \langle n, m | W(t) | n', m' \rangle, \\ |n, m\rangle &= |n\rangle \otimes |m\rangle. \end{aligned}$$

For initial conditions such that the field is initially in the vacuum state, one has

$$W_{n,m,n',m'}(t) = 0 \text{ for } n, n' > N,$$

since the atomic system can emit at most N photons. Hence, Eq. (2.1) is equivalent to a number of linear equations of the order of N^4 . Therefore, we prefer to follow the method devised in Refs. 1, 11, and 13, which leads to a set of linear c -number equations of the order of N^3 . To this aim, we consider the operators

$$N_{p,q}(t) = \text{Tr}_F[(A^\dagger)^p A^q W(t)], \quad p, q \geq 0 \quad (2.7)$$

where Tr_F means partial trace on the field Hilbert space. $N_{p,q}(t)$ are operators on the atomic Hilbert space alone. In particular, $N_{0,0}(t)$ is the reduced statistical operator for the atomic system alone, so that

$$\text{Tr}_A N_{0,0}(t) = 1,$$

where Tr_A means partial trace on the atomic Hilbert space.

On the other hand, $N_{p,q}(t)$ with $p+q > 0$ allow us to calculate the mean values of the field operators and the atom-field correlation functions. In fact, if O_A is a generical operator of the atomic system alone, the mean value of $(A^\dagger)^p A^q O_A$ is given by

$$\langle (A^\dagger)^p A^q O_A \rangle(t) = \text{Tr}_A [O_A N_{p,q}(t)]. \quad (2.8)$$

In particular, Eq. (2.8) shows that $N_{p,q}(t)$ is normalized as follows:

$$\text{Tr}_A N_{p,q}(t) = \langle (A^\dagger)^p A^q \rangle(t). \quad (2.9)$$

For initial conditions such that the field is initially in the vacuum state, one has

$$N_{p,q}(t) = 0 \text{ for } p, q > N. \quad (2.10)$$

By using Eqs. (2.5), (2.7), and the relation

$$\begin{aligned} \text{Tr}_F[(A^\dagger)^p A^q \Lambda_F W(t)] \\ = -n(p+q) \text{Tr}_F[(A^\dagger)^p A^q W(t)], \end{aligned} \quad (2.11)$$

one easily gets the following time evolution equations for $N_{p,q}(t)$:

$$\begin{aligned} \frac{d}{dt} N_{p,q}(t) = -K(p+q)N_{p,q}(t) \\ + g_0 \{ [R^+, N_{p,q+1}(t)] - [R^-, N_{p+1,q}(t)] \\ - pN_{p-1,q}(t)R^+ - qR^- N_{p,q-1}(t) \}. \end{aligned} \quad (2.12)$$

$$\begin{aligned} \frac{\partial}{\partial t} \mathfrak{X}_{p,q}(m, t) = -K(p+q)\mathfrak{X}_{p,q}(m, t) + g_0 [-qg^{1/2}(p+m+1)\mathfrak{X}_{p,q-1}(m+1, t) - pg^{1/2}(q+m+1)\mathfrak{X}_{p-1,q}(m+1, t) \\ + g^{1/2}(p+m)\mathfrak{X}_{p,q+1}(m-1, t) - g^{1/2}(q+m+1)\mathfrak{X}_{p,q+1}(m, t) \\ + g^{1/2}(q+m)\mathfrak{X}_{p+1,q}(m-1, t) - g^{1/2}(p+m+1)\mathfrak{X}_{p+1,q}(m, t)], \end{aligned} \quad (2.15)$$

where we have put

$$g(m) = \left(\frac{1}{2}N + m\right)\left(\frac{1}{2}N - m + 1\right). \quad (2.16)$$

Equations (2.15) are recurrence relations that can be quite easily solved by means of a computer.

The number of quantities $\mathfrak{X}_{p,q}(m, t)$ is of the order of N^3 , as it follows from (2.10) and (2.13).

In Ref. 1 the finite hierarchy (2.15) has been tentatively truncated by neglecting all the quantities $\mathfrak{X}_{p,q}(m, t)$ with $p > 1$ or $q > 1$.¹² An analogous truncation has been performed in the phase-space description of the phenomenon, too.¹³ We must stress that both truncations are incorrect. Therefore one must consider the full set of equations (2.15). We have solved them numerically for a number of atoms up to 20. The results are compared with the solution of the semiclassical damped pendulum equation

$$\ddot{\varphi} + K\dot{\varphi} = g_0^2 N \sin \varphi. \quad (2.17)$$

We have considered the case $K\tau_R = 0.1$, where

$$\tau_R = K/g_0^2 N, \quad (2.18)$$

and we have solved Eqs. (2.15) for two different initial conditions:

(a) Fully excited atomic system (superfluorescence), i.e.,

$$\begin{aligned} W(0) = |0, \frac{1}{2}N\rangle \langle \frac{1}{2}N, 0|, \\ \mathfrak{X}_{p,q}(m, 0) = \delta_{p,0} \delta_{q,0} \delta_{m,N/2}. \end{aligned} \quad (2.19)$$

Let us now introduce the c -number quantities

$$\mathfrak{X}_{p,q}(m, t) = \langle m+p | N_{p,q}(t) | m+q \rangle, \quad (2.13)$$

with

$$\begin{aligned} m = -\frac{1}{2}N - \min(p, q), -\frac{1}{2}N - \min(p, q) - 1, \dots, \\ \frac{1}{2}N - \max(p, q), \end{aligned}$$

where $\min(p, q) = p$ if $p < q$ and $\min(p, q) = q$ if $q < p$, and, analogously, $\max(p, q) = p$ if $p > q$ and $\max(p, q) = q$ if $q > p$. Note that $\mathfrak{X}_{0,0}(m, t) \equiv p(m, t)$ is the probability of occupation of the Dicke state $|m\rangle$. One has immediately the relation

$$\mathfrak{X}_{p,q}(m, t) = \mathfrak{X}_{q,p}^*(m, t). \quad (2.14)$$

Using Eqs. (2.6), (2.12), and (2.13), one obtains

Equations (2.19) imply that $\langle R_z \rangle(0) = \frac{1}{2}N$ and $\langle R^* R^- \rangle(0) = N$. The corresponding initial condition for the pendulum equation, which simulates the action of normal fluorescence at the beginning of the motion, is¹

$$\varphi(0) = \arccos\left(\frac{\frac{1}{2}N - \frac{1}{2}}{\frac{1}{2}N + \frac{1}{2}}\right), \quad \dot{\varphi}(0) = 0. \quad (2.20)$$

(b) System with an initial macroscopic polarization, i.e.,

$$\begin{aligned} W(0) = |0, 0\rangle \langle 0, 0|, \\ \mathfrak{X}_{p,q}(m, 0) = \delta_{p,0} \delta_{q,0} \delta_{m,0}. \end{aligned} \quad (2.21)$$

Equations (2.21) imply that $\langle R_z \rangle(0) = 0$ and $\langle R^* R^- \rangle(0) = \frac{1}{2}N(\frac{1}{2}N + 1)$. The corresponding initial condition for the pendulum equation is

$$\varphi(0) = \frac{1}{2}\pi, \quad \dot{\varphi}(0) = 0. \quad (2.22)$$

The quantities $\mathfrak{X}_{p,q}(m, t)$ are appropriately suited to calculate in a straightforward way the most relevant physical quantities. We enlist here the formulas which show how these quantities can be calculated from $\mathfrak{X}_{p,q}(m, t)$ and from φ . We use an index Q for the quantities calculated through the quantum-mechanical equations (2.15) and an index SC for the quantities calculated through the semiclassical pendulum equation (2.17).

$$\langle A^\dagger A \rangle_Q = \text{Tr}_F A N_{1,1}(t) = \sum_m \mathfrak{X}_{1,1}(m, t), \quad (2.23a)$$

$$\langle R_3 \rangle_Q = \sum_m m p(m, t), \quad (2.23b)$$

$$\langle (A^\dagger)^2 A^2 \rangle_Q = \text{Tr}_A N_{2,2}(t) = \sum_m \mathfrak{N}_{2,2}(m, t), \quad (2.23c)$$

$$\langle R_3^2 \rangle_Q = \sum_m m^2 p(m, t),$$

$$\langle A^\dagger A R_3 \rangle_Q = \text{Tr}_A [R_3 N_{1,1}(t)] = \sum_m (m+1) \mathfrak{N}_{1,1}(m, t), \quad (2.23d)$$

$$\begin{aligned} \langle A^\dagger A R^* R \rangle_Q &= \text{Tr}_A [R^* R N_{1,1}(t)] \\ &= \sum_m g(m+1) \mathfrak{N}_{1,1}(m, t), \end{aligned} \quad (2.23e)$$

$$\begin{aligned} \langle A A R^* \rangle_Q &= \text{Tr}_A [R^* R N_{0,2}(t)] \\ &= \sum_m g^{1/2}(m+2) g^{1/2}(m+1) \mathfrak{N}_{0,2}(m, t), \end{aligned} \quad (2.23f)$$

$$\langle A^\dagger A \rangle_{SC} = (N \tau_R / 4K) \dot{\phi}^2, \quad (2.23g)$$

$$\langle R_3 \rangle_{SC} = \frac{1}{2} N \cos \phi. \quad (2.23h)$$

III. NUMERICAL RESULTS

Let us review the results. Unless otherwise explicitly specified, we refer to the case $N = 20$.

A. Initial condition (2.19)

(i) In Fig. 1 we have plotted the time evolution of $\langle A^\dagger A \rangle_Q$ (quantum-mechanical result) and $\langle A^\dagger A \rangle_{SC}$ (semiclassical result). The time is scaled by τ_R .

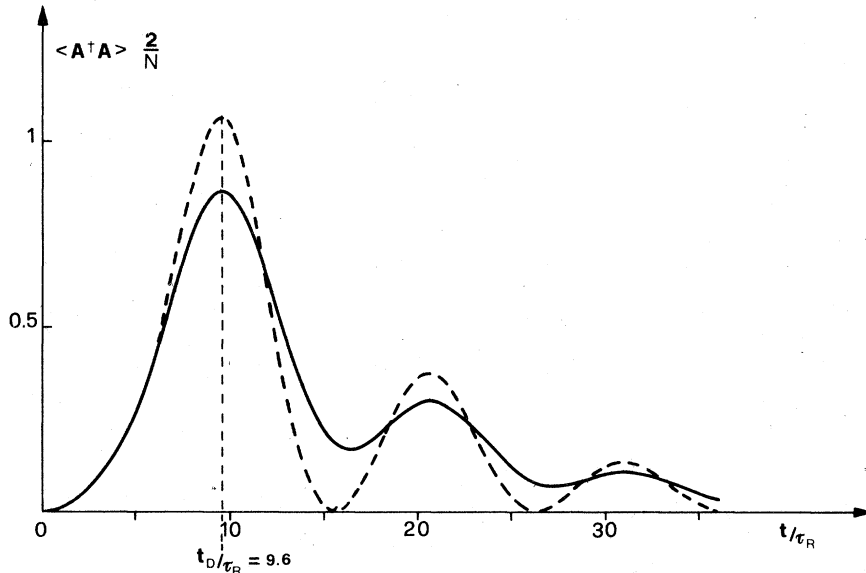


FIG. 1. Fully excited system: time evolution of the scaled mean photon number $\langle A^\dagger A \rangle / 2/N$, both in the semiclassical (dotted line) and in the quantum treatment (solid line). In this figure and in the following ones, we show the results for $N = 20$ unless otherwise specified.

The delay time t_D (i.e., the time at which the first maximum occurs) and the ringing ratio R (i.e., the ratio between the height of the first two maxima) are practically the same in both cases ($t_D \sim 9.6$, $R \sim 2.9$). But the peaks of the quantum-mechanical curve are considerably lower, while the minima are appreciably higher than the corresponding semiclassical ones. In particular, $\langle A^\dagger A \rangle_Q$ does not vanish at the minima. Therefore the plot of $\langle A^\dagger A \rangle_Q$ bears a stronger similarity to the experimental curves.^{14,15} The nonvanishing of $\langle A^\dagger A \rangle$ at any finite time is an obvious quantum-mechanical effect. In fact, $\langle A^\dagger A \rangle = 0$ if and only if the field is exactly in the vacuum state. But since the field exhibits observable fluctuations [see (iii) below], the mean value $\langle A^\dagger A \rangle$ is appreciably higher than zero.

(ii) Figure 2 compares the time evolution of $\langle R_3 \rangle_Q$ and $\langle R_3 \rangle_{SC}$. The initial value of $\langle R_3 \rangle_{SC}$ is slightly smaller than $\frac{1}{2}N$ because the initial condition (2.3) for the pendulum equation corresponds to

$$\langle R_3 \rangle(0) = \frac{1}{2}N(1 - 1/N).$$

The most remarkable feature is that $\langle R_3 \rangle_Q$ is larger than $-\frac{1}{2}N$ at the minima. Again this is due to quantum fluctuations [see (iv) below]. In fact, $\langle R_3 \rangle = -\frac{1}{2}N$ if and only if the atomic system is exactly in the ground state $|m = -\frac{1}{2}N\rangle$.

(iii) Figure 3 shows the time evolution of the field-intensity fluctuations. More precisely, Fig. 3(a) gives the plot of¹⁶

$$\sigma^2(I) = [\langle (A^\dagger A)^2 \rangle_Q - \langle A^\dagger A \rangle_Q^2] 15/N^2,$$

while Fig. 3(b) gives the plot of the normally ordered fluctuations

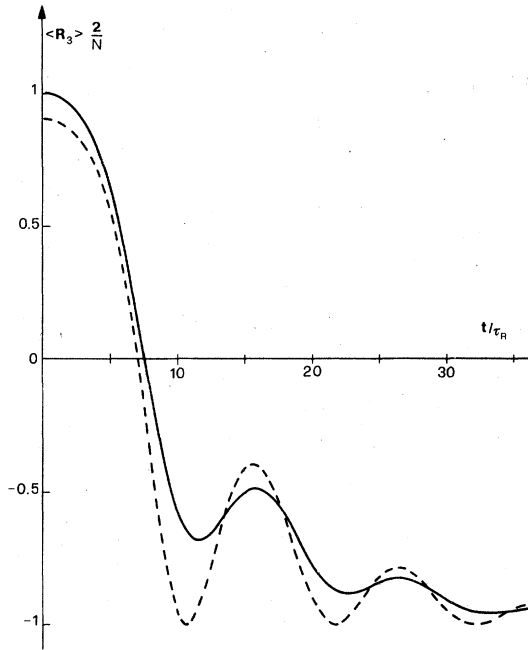


FIG. 2. Fully excited system: time evolution of the scaled total population inversion $\langle R_3 \rangle / 2N$, both in the semiclassical (dotted line) and in the quantum case (solid line).

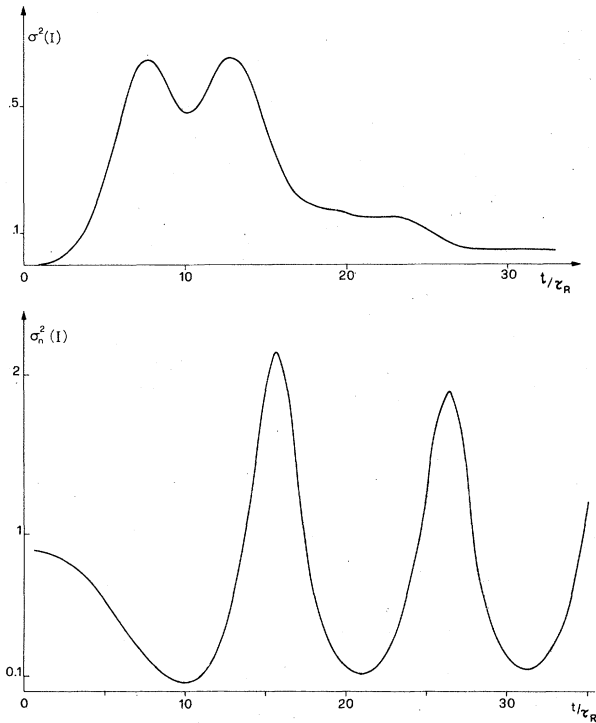


FIG. 3. (a) Fully excited system: time evolution of the scaled field intensity fluctuations $\sigma^2(I) = [\langle (A^\dagger A)^2 \rangle_Q - \langle A^\dagger A \rangle_Q^2] 15/N^2$. (b) Fully excited system: time evolution of the normally ordered intensity fluctuations $\sigma_n^2(I) = [\langle (A^\dagger)^2 A^2 \rangle_Q - \langle A^\dagger A \rangle_Q^2] / \langle A^\dagger A \rangle_Q^2$.

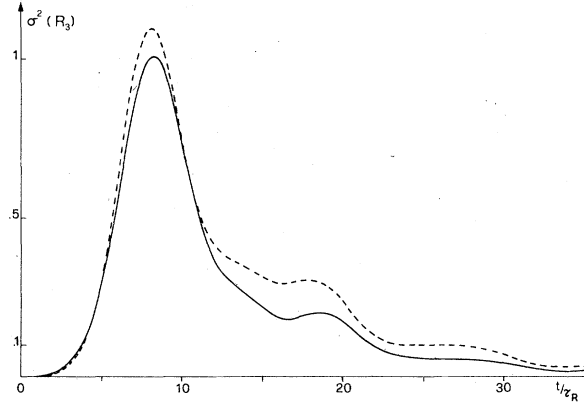


FIG. 4. Fully excited system: time evolution of the scaled atomic fluctuations $\sigma^2(R_3) = (\langle R_3^2 \rangle_Q - \langle R_3 \rangle_Q^2) 10/N^2$ for $N=10$ (broken line) and $N=20$ (solid line). In the case $N=10$, the time axis has been shifted in such a way that the delay times t_D for $N=10$ and $N=20$ coincide.

$$\sigma_n^2(I) = \frac{\langle (A^\dagger)^2 A^2 \rangle_Q - \langle A^\dagger A \rangle_Q^2}{\langle A^\dagger A \rangle_Q^2}$$

Note in Fig. 3(a) the dip in the neighborhood of the intensity peak.

(iv) In Fig. 4 one finds the time plot of the atomic fluctuations

$$\sigma^2(R_3) = (\langle R_3^2 \rangle_Q - \langle R_3 \rangle_Q^2) 10/N^2.$$

(v) Figure 5 shows the atom-field correlation function

$$C_{IR_3} = (\langle A^\dagger A \rangle_Q \langle R_3 \rangle_Q - \langle A^\dagger A R_3 \rangle_Q) 15/N^2.$$

This quantity turns out to be positive for all times.

(vi) An argument to explain the fact that the peak of $\langle A^\dagger A \rangle_{SC}$ is higher than the peak of $\langle A^\dagger A \rangle_Q$ is the following one. From energy conservation,

$$\frac{d}{dt} (\langle A^\dagger A \rangle + \langle R_3 \rangle) = -2K \langle A^\dagger A \rangle,$$

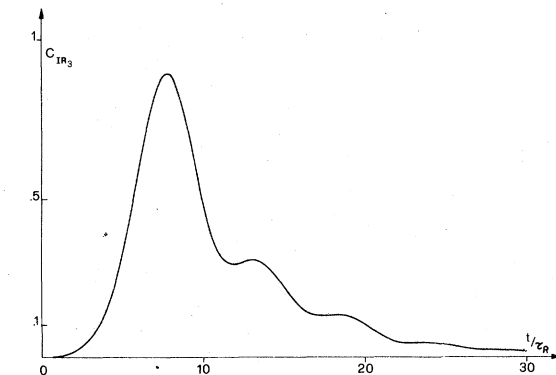


FIG. 5. Fully excited system: time evolution of the scaled atom-field correlation function $C_{IR_3} = (\langle A^\dagger A \rangle_Q \langle R_3 \rangle_Q - \langle A^\dagger A R_3 \rangle_Q) 15/N^2$.

we get

$$\langle A^\dagger A \rangle(t) = -\frac{1}{2K} \int_0^t ds e^{-K(t-s)} \langle \dot{R}_3 \rangle(s), \quad (3.1)$$

both in the quantum and in the semiclassical case. Hence

$$\begin{aligned} \langle A^\dagger A \rangle_{sc}(t) - \langle A^\dagger A \rangle_Q(t) \\ = \frac{1}{2K} \int_0^t ds e^{-K(t-s)} [\langle \dot{R}_3 \rangle_Q(s) - \langle \dot{R}_3 \rangle_{sc}(s)]. \end{aligned} \quad (3.2)$$

Now we have

$$\begin{aligned} \langle \dot{R}_3 \rangle_Q(t) - \langle \dot{R}_3 \rangle_{sc}(t) \\ = \int_0^t ds e^{-K(t-s)} \{ \langle R_3^2 \rangle_Q(s) - \langle R_3^2 \rangle_{sc}(s) \\ + [\langle A^\dagger A \rangle_{sc}(s) \langle R_3 \rangle_{sc}(s) - \langle A^\dagger A R_3 \rangle_Q(s)] \}. \end{aligned} \quad (3.3)$$

Therefore, if in the right-hand side of Eq. (3.3) we replace $\langle R_3 \rangle_{sc}$ and $\langle A^\dagger A \rangle_{sc}$ by $\langle R_3 \rangle_Q$ and $\langle A^\dagger A \rangle_Q$, respectively, and take into account remark (v), we have

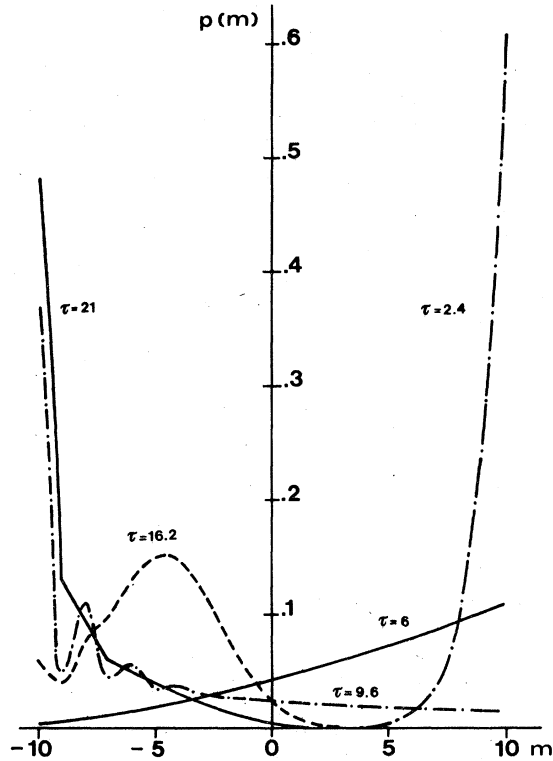


FIG. 6. Fully excited system: time evolution of the probability of occupation of the Dicke states $p(m, t)$. The normalized time τ is defined as $\tau = t/\tau_R$. $\tau = 6$ corresponds to half maximum of $\langle A^\dagger A \rangle$; $\tau = 9.6$ corresponds to the peak of $\langle A^\dagger A \rangle_Q$ (delay time); $\tau = 16.2$ corresponds to the first minimum of $\langle A^\dagger A \rangle_Q$; $\tau = 21$ corresponds to the second maximum of $\langle A^\dagger A \rangle_Q$.

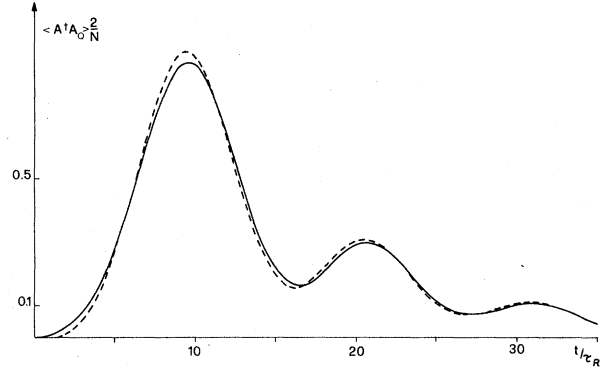


FIG. 7. Comparison between $\langle A^\dagger A \rangle_Q$ for $N=10$ (broken line) and $N=20$ (solid line) in the case of a fully excited system. For $N=10$ the time axis has been shifted in such a way that the delay times t_D for $N=10$ and $N=20$ coincide.

$$\langle \dot{R}_3 \rangle_Q(t) - \langle \dot{R}_3 \rangle_{sc}(t) \geq 0 \Rightarrow \langle A^\dagger A \rangle_{sc}(t) \geq \langle A^\dagger A \rangle_Q(t).$$

Now, the substitution of $\langle R_3 \rangle_{sc}$ and $\langle A^\dagger A \rangle_{sc}$ by $\langle R_3 \rangle_Q$ and $\langle A^\dagger A \rangle_Q$ gives a substantially correct estimate in the neighborhood of the first maximum, whereas it fails completely in the neighborhood of the first minimum.

(vii) The quantities $\langle A^\dagger A R^\dagger R \rangle$ and $\text{Re} \langle A A R^\dagger R \rangle$ are very close to each other during the whole time evolution, in accordance with the claim of Ref. 13.

(viii) Figure 6 shows the time evolution of the probability distribution $p(m, t)$ of the Dicke state $|m\rangle$. Notice that this distribution is broad during most of the time evolution. Notice also the oscillations in time. The probability of the fully excited state $p(\frac{1}{2}N, t)$ undergoes a practically exponential decay: for $N=20$ one has

$$p(10, t) \approx 1.8 \exp(-0.535t/\tau_R).$$

(ix) Figures 4 and 7 show that the scaled quantities exhibit only minor variations when N is changed from 10 to 20. This fact is of course essential to ensure that our results for the scaled quantities are meaningful even if the value of N is not very large.

Let us explain in this connection why $\langle A^\dagger A \rangle$ has been scaled to N rather than to N^2 even if we are describing superfluorescence. We have from Eqs. (2.17), (2.18), and (2.23g)

$$\langle A^\dagger A \rangle_{sc}(\tau) = \frac{N}{4K\tau_R} \left(\frac{d\varphi}{d\tau} \right)^2, \quad \tau = t/\tau_R, \quad (3.4)$$

$$\frac{d\varphi}{d\tau} = \int_0^\tau ds K\tau_R e^{-K\tau_R(\tau-s)} \sin\varphi(s).$$

Since $\tau_R \propto N^{-1}$, when $d\varphi/dt = O(1)$ we have that $\langle A^\dagger A \rangle \propto N^2$. But if, as we do, we vary N keeping $K\tau_R$ fixed, $\langle A^\dagger A \rangle$ varies proportionally to N .

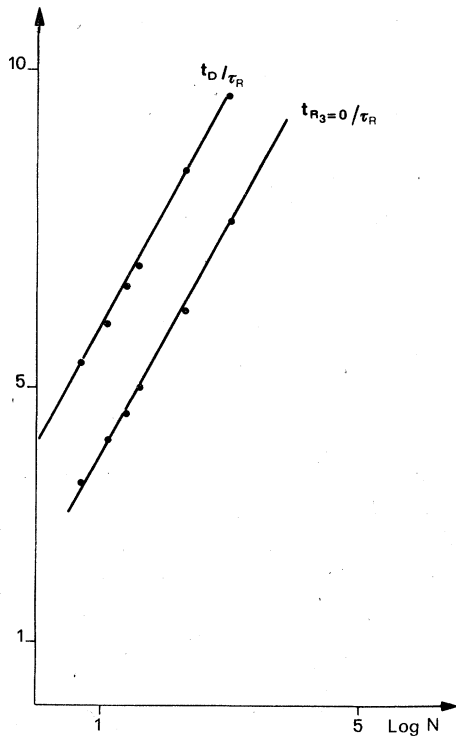


FIG. 8. Dependence of the delay time t_D and of the time $t_{R_3=0}$ (at which $\langle R_3 \rangle = 0$) on the number of atoms N for a fully excited system with fixed $K\tau_R = 0.1$,

(x) In Fig. 8 we have plotted, for fixed $K\tau_R = 0.1$, (a) the delay time t_D vs $\ln N$ and (b) the time $t_{R_3=0}$ (at which $\langle R_3 \rangle = 0$) vs $\ln N$. Roughly speaking, one finds the linear relation

$$t_D = \tau_R(1.82 \ln N + 4.1),$$

which agrees with the expression given in Ref. 10. One sees also that there is a systematic delay between $t_{R_3=0}$ and t_D . This is a non-Markoffian retardation effect. In fact, as it appears from Eq. (3.1), in the Markoffian case ($K\tau_R \gg 1$) one has $\langle A^\dagger A \rangle(t) = -\langle R_3 \rangle(t)/2K$, so that t_D coincides with the time at which $-\langle R_3 \rangle$ is maximum, which in turn coincides with $t_{R_3=0}$.

B. Initial condition (2.21)

In this case the quantum-mechanical fluctuations and all their effects disappear. This is clearly shown in Fig. 9: $\langle A^\dagger A \rangle_{SC}$ and $\langle A^\dagger A \rangle_Q$ practically coincide. The ringing ratio R remains essentially the same as in the fully-excited-state

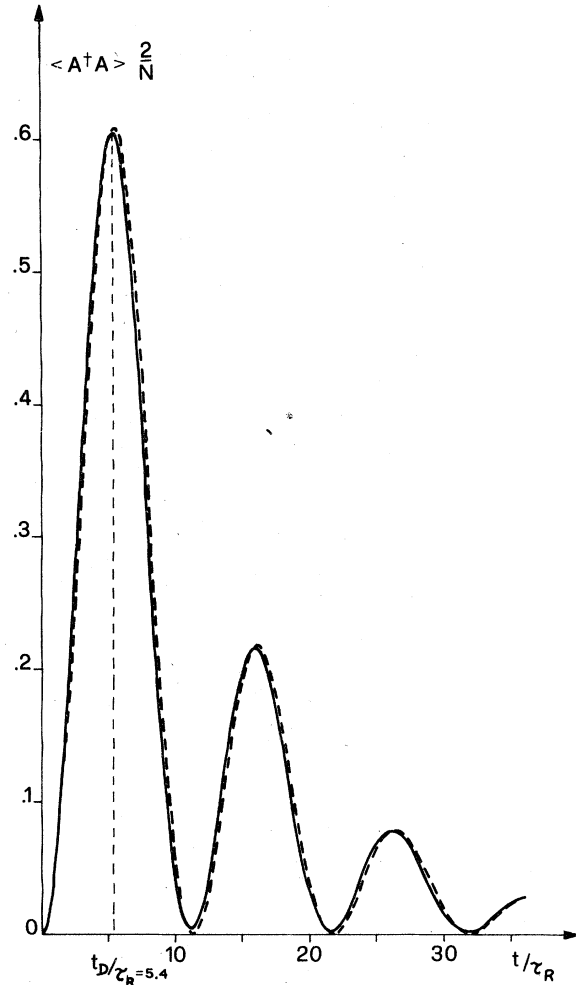


FIG. 9. System with an initial macroscopic polarization [$\langle R_3 \rangle(0) = 0$]: time evolution of the scaled mean photon number $\langle A^\dagger A \rangle / 2N$ both in the semiclassical (dotted line) and in the quantum case (solid line).

case; in fact, R is controlled by the quantity $K\tau_R$. Furthermore, we find a spectacular non-Markoffian effect consisting in the presence of a noticeable delay in the radiation emission. In fact, in the Markoffian case, t_D is zero for the initial condition (2.21).

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deduced from Eqs. (2.15) by eliminating the quantities $\mathcal{N}_{p,q}(m,t)$ with $p+q$ odd and taking into account the initial condition $\mathcal{N}_{p,q}(m,0)=0$ for $p+q>0$. The elimination is simply performed by integrating Eqs. (2.15) for $p+q$ odd, which gives

$$\begin{aligned} \mathcal{N}_{p,q}(m,t) = g_0 \int_0^t ds e^{-K(p+q)(t-s)} \\ \times [-qg^{1/2}(p+m+1)\mathcal{N}_{p,q-1}(m+1,s) \\ -pg^{1/2}(q+m+1)\mathcal{N}_{p-1,q}(m+1,s) \\ +g^{1/2}(p+m)\mathcal{N}_{p,q+1}(m-1,s) \\ -g^{1/2}(q+m+1)\mathcal{N}_{p,q+1}(m,s) \\ +g^{1/2}(q+m)\mathcal{N}_{p+1,q}(m-1,s) \\ -g^{1/2}(p+m+1)\mathcal{N}_{p+1,q}(m,s)]. \end{aligned}$$

In fact, if $p+q$ is odd, all the quantities $\mathcal{N}_{p',q'}(m,t)$ appearing on the right-hand side have $p'+q'$ even.

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