PHYSICAL REVIEW A VOLUME 18, NUMBER 2 AUGUST 1978

Quantum theory of a one-dimensional optical cavity with output coupling. II. Thermal radiation field and the fluctuation-dissipation theorem

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(Received 16 February 1978)

The normally ordered correlation function and the power spectrum (cross spectral tensor) of a thermal radiation field existing inside and outside a one-dimensional optical cavity which transmits radiation at one of its end surfaces are calculated on the basis of a multimode formulation of the field developed in a previous paper. The correlation function is found to consist of a series of trigamma functions resulting from retarded correlations due to reflections at the cavity end surfaces. The power spectrum is a product of the Planck distribution and the mode functions of the universe which have a periodic factor with the period being equal to the cavity mode separation. For the purpose of obtaining a description of the cavity which allows one to deduce its thermal noise property, a form of fluctuation-dissipation theorem is derived which connects, in a one-dimensional space with arbitrary structure, the normally ordered correlation function with the admittance for the electric field probed by a current. Application of this theorem to the model cavity leads to an explicit and simultaneous description of the fluctuation and dissipation associated with all the cavity modes. The dissipation is shown to be due to radiation of electromagnetic energy into the free space. The above relation is reduced, under the assumption of high cavity quality factor and of single-cavity-mode selection, to that of a Markovian noise associated with a single decay constant. The consistency of these results with the field commutation relation is examined.

I. INTRODUCTION

In this paper the properties of the thermal radiation field in a space including an optical cavity with output coupling are considered by use of a model of the cavity. The main purpose of this paper is to bridge, by way of an example, the coherence theory of the blackbody radiation and the theory of thermal noise in a laser with particular attention to a rigorous treatment of the output coupling of the cavity and to the description of the exact spatial behavior of the radiation field.

The coherence properties of the blackbody radiation have been the subject of numerous papers.[1] Among these, of particular interest to us are those which considered the consequence of a nonuniform geometry of the space on the coherence,[2,3] and those which discussed the coherence on the basis of the theory of optical measurement in terms of normally ordered correlation functions.[4] Also of interest to us is the one which extended the concept of the power spectrum to the one including two space variables (cross spectral tensor[5]). However, there has been no treatment of an optical cavity with partial transmission.

On the other hand, there has been much effort to describe the radiation field in a laser cavity[6-8] or, more generally, the radiation field undergoing dissipation[9] without violating the commutation relation, which requires the introduction of thermal noise. Although thermal noise is quantitatively insignificant in the optical region, its proper treatment constitutes a part of exact quantum-mechanical theory of the laser. Most laser theories introduced the thermal noise via a heat bath made up of loss oscillators[6-8] or absorbing atoms,[10] which by virtue of their assumed local flatness of the absorption spectrum lead to a Markovian noise and a single decay constant for a single idealized mode of the laser cavity. We shall call such a theory a quasimode theory.[11] These approaches to thermal noise have the disadvantage of failing to describe properly an ideal laser cavity for which the only loss comes from the output coupling where the cavity loss should appear without introducing any absorbing particles, and where the thermal noise can be attributed to the thermal radiation itself which penetrates the cavity through the coupling. Also, they fail to describe the laser output, i.e., the field coupled out of the cavity. The former point was improved by Lang *et al.* by use of a multimode formulation of the field of the universe in which the cavity is enbedded.[11,12] They showed that a Markovian noise associated with a decay constant can be derived from the initial thermal radiation field. We have developed a multimode theory of the cavity[13] and of the laser[14-16] that can overcome both of the above disadvantages. In Ref. 16 (hereafter referred to as II) we obtained the quantum-mechanical coherence function of the laser output in the subthreshold region as a function of both space and time variables, and gave some discussion on the range of validity of the Markovian approximation of the thermal noise.

In this paper we investigate, by way of an example, the coherence properties of the thermal radiation field in a space including an optical cavity in terms of a normally ordered correlation function[17] and the power spectrum (cross spectral tensor[5]). Using this knowledge we investigate the noise characteristics of the cavity as a laser resonator. For this purpose we describe the cavity or the space outside in terms of the admittance probed by a current and derive a form of the fluctuation-dissipation theorem which connects the normally ordered correlation function of the thermal radiation field with the admittance. Application of the theorem to our model cavity gives a relation which describes the fluctuation of all the cavity modes simultaneously. This relation is a generalization of the simplest form, i.e., a Markovian noise associated with a single decay constant for a single cavity mode, assumed in quasimode theories.[6-10] The former relation is shown to give the latter simplified relation when the cavity has a high quality factor and the noise is considered within a band of width equal to the cavity-mode separation and centered at a cavity-mode frequency. In the following we use the model of the one-dimensional optical cavity which we analyzed in Ref. 13 (hereafter referred to as I).

II. DESCRIPTION OF THE QUANTIZED RADIATION FIELD

In this section we summarize the quantum-mechanical formulation of the field developed in I. We consider a one-dimensional space where physical quantities are functions only of spatial variable z and time t. The universe is bounded at $z=-d$ and at $z=L$ by perfectly conducting walls. The region $-d<z<0$ is occupied by a slab of a nonmagnetic dielectric with the dielectric constant ϵ^1 which constitutes the optical cavity. The region $0<z<L$ is a vacuum with the dielectric constant ϵ^0. We limit our consideration to the field component in a particular direction, e.g., in the x direction. Then the vector potential and the electric field of our radiation field can be expanded in terms of the normal modes of the universe as

$$A(z,t)=\sum_j Q_j(t)U_j(z), \quad (1)$$

$$E(z,t)=-\sum_j P_j(t)U_j(z), \quad (2)$$

where

$$Q_j(t)=(\hbar/2\omega_j)^{1/2}[a_j(t)+a_j^\dagger(t)], \quad (3)$$

$$P_j(t)=-i(\tfrac{1}{2}\hbar\omega_j)^{1/2}[a_j(t)-a_j^\dagger(t)], \quad (4)$$

and

$$[a_i,a_j]=[a_i^\dagger,a_j^\dagger]=0, \quad (5a)$$

$$[a_i,a_j^\dagger]=\delta_{ij}. \quad (5b)$$

The normal modes have different expressions for inside and outside of the cavity:

$$U_j(z)=(2/\epsilon^1 L)^{1/2}N_j u_j(z), \quad (6a)$$

$$N_j=(1-K\sin^2 k_j^1 d)^{-1/2} \quad (6b)$$

$$u_j(z)=\sin k_j^1(z+d), \quad -d<z<0 \quad (6c)$$

$$=(c^0/c^1)\cos k_j^1 d\sin k_j^0 z+\sin k_j^1 d\cos k_j^0 z, \quad 0<z<L \quad (6d)$$

where c^0 and c^1 are the light velocities for outside and inside of the cavity, respectively, and $k_j^0=\omega_j/c^0$ and $k_j^1=\omega_j/c^1$. The factor K is equal to $4r/(1+r)^2$, where r is the reflectivity at the coupling surface given by $(c^0-c^1)/(c^0+c^1)$. The normal modes have positive frequencies and the density of modes per unit angular frequency is given by

$$\rho(\omega)=L/c^0\pi\equiv\rho. \quad (7)$$

In Eqs. (6b) and (7) the quantity L is assumed to be large and quantities that are small compared with it are omitted.

The modes (quasimodes) of our cavity have the frequencies

$$\Omega_m=\omega_{cm}-i\gamma_c, \quad (8a)$$

$$\omega_{cm}=(2m+1)\pi/\tau_c, \quad (8b)$$

$$\gamma_c=(c^1/2d)\ln(1/r), \quad (8c)$$

$$\tau_c=2d/c^1. \quad (8d)$$

The quantity τ_c is the cavity round-trip time. The cavity mode separation is

$$\Delta\omega_c=\pi c^1/d. \quad (8e)$$

III. QUANTUM-MECHANICAL COHERENCE FUNCTION

The state of the field corresponding to that of blackbody radiation or thermal radiation can be described by the density operator in the photon-number representation[8]

$$\vec{\rho}=P_E|\Psi_E\rangle\langle\Psi_E|, \quad (9a)$$

$$P_E=\prod_j(1-e^{-\beta\hbar\omega_j})e^{-n_j\beta\hbar\omega_j}, \quad (9b)$$

$$\beta=1/kT, \quad (9c)$$

$$|\Psi_E\rangle=\prod_j|n_j\rangle=|\{n_j\}\rangle. \quad (9d)$$

We assume that the thermal radiation field with the above property is prepared at $t=0$ and the field oscillates freely afterwards. We will work in the Heisenberg picture so that we understand that the density operator in Eq. (9a) does not change with time t. We use the following notation to denote the ensemble average of any operator $O(t)$ with respect to the thermal radiation field at $t=0$:

$$\langle O(t)\rangle = \mathrm{Tr}[\bar{\rho}\, O(t)]. \tag{10}$$

Then it is easy to show that

$$\langle E_T^+(z,t)\rangle = 0, \tag{11}$$

where

$$E_T^+(z,t) = \sum_j i\left(\frac{\hbar\omega_i}{2}\right)^{1/2} U_j(z) a_j(0) e^{-i\omega_j t} \tag{12}$$

is the positive frequency part of the electric field operator. The suffix T denotes the thermal field. This quantity is equal to the noise term $F_1(z,t)$ in Eq. (6a) of II. The quantum-mechanical coherence function may be defined as the average of the normally ordered correlation function[17]

$$G(z',t',z,t) = \langle E_T^-(z',t')E_T^+(z,t)\rangle \tag{13}$$

where E^- is the negative frequency part of the electric field E having only creation operators. Calculating the trace we have

$$G(z',t',z,t) = \sum_j \frac{\hbar\omega_j}{2}\langle n_j\rangle U_j(z')U_j(z) e^{-i\omega_j(t-t')}, \tag{14}$$

where

$$\langle n_j\rangle = \langle a_j^\dagger(0)a_j(0)\rangle = (e^{\beta\hbar\omega_j}-1)^{-1}. \tag{15}$$

The right-hand side of Eq. (14) depends on the time variables only through the difference $t-t'$ showing the stationarity of the field. We have the symmetry properties

$$G(z,t',z',t) = G(z',t',z,t), \tag{16a}$$

$$G(z',t,z,t') = G^*(z',t',z,t). \tag{16b}$$

For the calculation of explicit expressions for the correlation function, we expand the product of the mode functions in Eq. (14) in a Fourier series[13,14]:

$$U_j(z')U_j(z) = \frac{2c^0}{c^1}\left(\frac{2}{\epsilon^1 L}\right) \times\left[\sum_{n=0}^{\infty}\frac{1}{1+\delta_{0,n}}(-r)^n \cos 2nk_j^1 d\right] \times u_j(z')u_j(z). \tag{17}$$

Since the product of cosine functions and $u_j(z')u_j(z)$ gives terms of the form $\exp(i\omega_j\tau)$, formula (14) results in a series of integrals of the form

$$\int_0^\infty d\omega_j\, \rho(\omega_j)\left(\frac{\hbar\omega_j \exp[i\omega_j(\tau-t+t')]}{\exp(\beta\hbar\omega_j)-1}\right) = \frac{\rho}{\beta^2\hbar}\Psi'\left(1-i\frac{t-t'-\tau}{\beta\hbar}\right), \tag{18}$$

where Ψ' is the trigamma function.[18] Define a function ϕ by

$$\phi(u) = \Psi'(1+iu/\beta\hbar). \tag{19}$$

Then we have, using Eqs. (17)–(19) and considering the stationarity of the field,

$$G(z',0,z,t) = \frac{(kT)^2}{4\pi\epsilon^1 c^1\hbar}\left(\phi(t\pm\tau_{10}) + \phi(t\pm\tau_{30}) + \sum_{n=1}^{\infty}(-r)^n[\phi(t\pm\tau_{1n}) + \phi(t\pm\tau_{2n}) - \phi(t\pm\tau_{3n}) - \phi(t\pm\tau_{4n})]\right),$$
$$-d<z'<0,\ -d<z<0 \tag{20a}$$

$$= \frac{(kT)^2}{4\pi\epsilon^0 c^0\hbar}\left(\phi(t\pm\tau_{50}) - r\phi(t\pm\tau_{60}) - (1-r^2)\sum_{n=1}^{\infty}(-r)^{n-1}\phi(t\pm\tau_{6n})\right),\quad 0<z',\ 0<z \tag{20b}$$

$$= \frac{(1+r)(kT)^2}{4\pi\epsilon^1 c^1\hbar}\sum_{n=0}^{\infty}(-r)^n[\phi(t\pm\tau_{7n}) - \phi(t\pm\tau_{8n})],\quad -d<z'<0,\ 0<z, \tag{20c}$$

where we used the notation

$$\phi(t\pm\tau) = \phi(t+\tau) + \phi(t-\tau), \tag{20d}$$

and

$$\tau_{1n}=\frac{z-z'}{c^1}+n\tau_c,\quad \tau_{2n}=\frac{z'-z}{c^1}+n\tau_c,\quad \tau_{3n}=\frac{z+z'+2d}{c^1}+n\tau_c,\quad \tau_{4n}=-\frac{z+z'+2d}{c^1}+n\tau_c,$$

$$\tau_{50}=\frac{z-z'}{c^0},\quad \tau_{6n}=\frac{z+z'}{c^0}+n\tau_c,\quad \tau_{7n}=\frac{z}{c^0}-\frac{z'}{c^1}+n\tau_c,\quad \tau_{8n}=\frac{z}{c^0}+\frac{z'+2d}{c^1}+n\tau_c, \tag{20e}$$

where τ_c is the cavity round-trip time which appeared in Eq. (8d). In Eqs. (20a)–(20c) the function ϕ represents the form of the correlation which might be obtained in a free one-dimensional space. The infinite number of terms with decaying amplitudes represent the retarded correlations owing to reflections at the cavity end surfaces. In particular, the term containing τ_{50} is the correlation function corresponding to that of free space, terms containing τ_{41} and τ_{60} represent the effect of a single reflection at the coupling surface at $z=0$. Terms where $n\geq 1$ represent the retarded correlations owing to multiple reflections at the cavity end surfaces. This is understood by noting the appearance of $n\tau_c$ in the retardation times in Eq. (20e). In Eq. (20b) the factor $1-r^2$ is the product of the transmission coefficients at $z=0$ by which the wave amplitude changes as the wave goes into and out of the cavity. In the limit $r\rightarrow 0$ $(c^1\rightarrow c^0)$ the three functions reduce to the same form having only two terms with $\tau_{50}=(z-z')/c^0$ and $\tau_{61}=(z+z'+2d)/c^0$. The former term represents the propagation in free space and the latter the reflection at $z=-d$, respectively. In the limit $r\rightarrow 1$ $(c^1\rightarrow 0)$, Eq. (20b) reduces to the above form with $d=0$, and Eq. (20c) vanishes. The function Φ defined by $\Phi(y)=\phi(t)$, where $y=t/\beta\hbar$ is depicted in Fig. 1. This function represents the correlation in a free one-dimensional space and its spread gives the measure of nonwhiteness of the thermal radiation owing to the quantum nature of the field. We see that the time spread of ϕ is of the order of $\beta\hbar$ and the corresponding bandwidth is $kT/\hbar$. It can be shown that

$$\mathrm{Re}\Phi(y)=\frac{\pi^2}{2}\frac{1-y^2\,\mathrm{cosec}^2 y}{y^2}, \tag{21a}$$

so that

$$\int_{-\infty}^{\infty}\mathrm{Re}\Phi(y)\,dy=\pi. \tag{21b}$$

On the other hand, the imaginary part of Φ has no compact expression, but has an asymptotic form[18]

$$\lim_{y\rightarrow\infty}\mathrm{Im}\Phi(y)=-\frac{\partial}{\partial y}\ln y=-\frac{1}{y}. \tag{21c}$$

Since $\mathrm{Im}\Phi(y)$ is antisymmetric in y, as can be seen from Eqs. (18) and (19),

$$\int_{-\infty}^{\infty}\mathrm{Im}\Phi(y)\,dy=0. \tag{21d}$$

Thus, for large y or in the classical limit $(h\rightarrow 0)$

$$\Phi(y)\rightarrow\pi\delta(y)-iP/y=-i\zeta^*(y), \tag{22a}$$

so that

$$\phi(t)\rightarrow -i\beta\hbar\zeta^*(t), \tag{22b}$$

where the function ζ is given by[19]

$$\zeta(t)=\frac{P}{t}-i\pi\delta(t)=-i\int_0^{\infty}e^{i\omega t}\,d\omega. \tag{23}$$

As can be seen by Fourier transforming this equation, $\phi(t)$ represents a white noise in the positive frequency region in the classical limit.

Higher-order correlation functions may be given in terms of the above first-order correlation functions since the quasiprobability distribution for the field amplitude of the thermal radiation field can be shown to be Gaussian.[20]

IV. POWER SPECTRUM

Next we consider the power spectrum. For the sake of convenience in later discussions, we define it as follows. Consider the spatial correlation, that is, the correlation function of Sec. III with $t=t'$ (or $t=0$). Then the contribution to this quantity from a unit frequency band may be defined

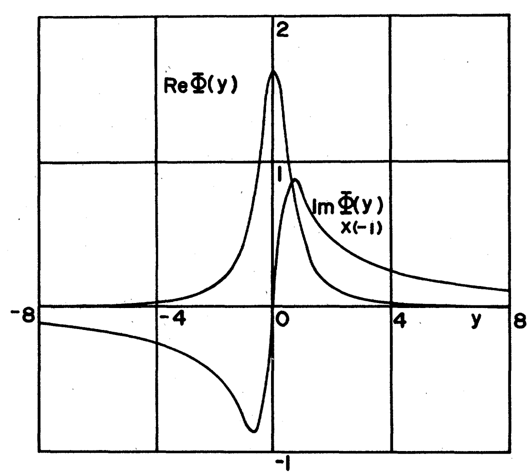

FIG. 1. Real part and the imaginary part (multiplied by -1) of the function Φ (y). These curves are drawn by numerical differentiation of the digamma function tabulated in Ref. 18.

as the power spectrum in a wide sense. We have from Eq. (14), using (7),

$$G(z',t,z,t)=\int_0^\infty \rho(\omega)\frac{\hbar\omega}{2}\langle n_\omega\rangle U_\omega(z')U_\omega(z)\,d\omega\,. \quad (24)$$

Thus by definition the power spectrum is

$$I(z',z,\omega)=\rho(\omega)\frac{\hbar\omega}{2}\langle n_\omega\rangle U_\omega(z')U_\omega(z)H(\omega)\,, \quad (25)$$

where

$$H(\omega)=\begin{cases}1, & \omega>0\\ \frac{1}{2}, & \omega=0\\ 0, & \omega<0.\end{cases} \quad (26)$$

Obviously, $I(z,z,\omega)\geq 0$, that is, the power spectrum in the usual sense is non-negative at any spatial point as expected. It has the symmetry property

$$I(z,z',\omega)=I(z',z,\omega)\,. \quad (27)$$

If we use Eqs. (7) and (14), it is easy to show that

$$I(z',z,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(z',0,z,t)e^{i\omega t}\,dt\,. \quad (28)$$

that is, this power spectrum is the Fourier transform of the correlation function. In this sense our power spectrum is equivalent to the cross spectral tensor of Mehta and Wolf.[5] The inverse relation is

$$G(z',0,z,t)=\int_0^\infty I(z',z,\omega)e^{-i\omega t}\,d\omega\,. \quad (29)$$

We see that the power spectrum has the envelope

$$\hbar\omega\langle n_\omega\rangle=\hbar\omega[\exp(\beta\hbar\omega)-1]^{-1},$$

which has the value $\beta^{-1}=kT$ for small values of ω and decreases monotonically with increasing value of ω falling to $\frac{1}{2}kT$ at $\omega=1.26kT/\hbar$.

The explicit expression for the power spectrum may be obtained by merely substituting the mode functions of Eqs. (6c) and (6d) into Eq. (25). Instead, we use Eq. (28) and Fourier transform each term in Eqs. (20a)–(20c) and rearrange the terms. After some calculation we have

$$I(z',z,\omega)=\frac{1}{2\pi\epsilon^1c^1}\hbar\omega\langle n_\omega\rangle\left[\cos\left(\omega\frac{z-z'}{c^1}\right)-\cos\left(\omega\frac{z+z'+2d}{c^1}\right)\right]\frac{1-r^2}{1+r^2+2r\cos\omega\tau_c}H(\omega),$$
$$-d<z'<0,\ -d<z<0 \quad (30)$$

$$=\frac{1}{2\pi\epsilon^0c^0}\hbar\omega\langle n_\omega\rangle\left[\cos\left(\omega\frac{z-z'}{c^0}\right)-\cos\left(\omega\frac{z+z'}{c^0}+\theta\right)\right]H(\omega),\quad 0<z',\ 0<z \quad (31a)$$

$$\cos\theta=\frac{2r+(1+r^2)\cos\omega\tau_c}{1+r^2+2r\cos\omega\tau_c}, \quad (31b)$$

$$\sin\theta=\frac{(1-r^2)\sin\omega\tau_c}{1+r^2+2r\cos\omega\tau_c}, \quad (31c)$$

$$I=(z',z,\omega)=\frac{\hbar\omega\langle n_\omega\rangle}{2\pi\epsilon^1c^1}\frac{1+r}{(1+r^2+2r\cos\omega\tau_c)^{1/2}}\left\{\cos\left[\omega\left(\frac{z}{c^0}-\frac{z'}{c^1}\right)-\theta'\right]-\cos\left[\omega\left(\frac{z}{c^0}+\frac{z'+2d}{c^1}\right)-\theta'\right]\right\}H(\omega),$$
$$-d<z'<0,\ 0<z \quad (32a)$$

$$\cos\theta'=\frac{1+r\cos\omega\tau_c}{(1+r^2+zr\cos\omega\tau_c)^{1/2}}, \quad (32b)$$

$$\sin\theta'=\frac{r\sin\omega\tau_c}{(1+r^2+2r\cos\omega\tau_c)^{1/2}}. \quad (32c)$$

For the limiting cases of $r\to 0$ or $r\to 1$, similar arguments can be given as those for the correlation functions given below Eq. (20). We see that the power spectrum has a periodic structure with the period $2\pi/\tau_c$ which is equal to the separation of the cavity modes $\Delta\omega_c$ implying that the power spectrum is made up of contributions from the cavity modes. Indeed we can obtain an expression which explicitly shows the contributions of individual cavity modes to the power spectrum. We note that in the general expression (25) the factor $N_\omega^2=(1-K\sin^2k_\omega^1d)^{-1}$ included in $U_\omega(z')U_\omega(z)$ has simple poles at $\omega=\omega_{cm}\pm i\gamma_c$, where ω_{cm} is given by Eq. (8b), but here m takes all integer values. This is easily verified by use of Eqs. (8b) and (8c) and the definition of K given below Eq. (6d). Thus, from the Mittag-Leffler theorem[21]

$$(1-K\sin^2 k_\omega^1 d)^{-1}=\sum_{m=-\infty}^{\infty}\frac{c^0\gamma_c/d}{\gamma_c^2+(\omega-\omega_{cm})}. \tag{33}$$

Substituting this into Eq. (25) we have

$$I(z',z,\omega)=\sum_{m=0}^{\infty}\frac{\hbar\omega\langle n_\omega\rangle}{\pi\epsilon^1 d}\frac{\gamma_c}{\gamma_c^2+(\omega-\omega_{cm})^2}\times u_\omega(z')u_\omega(z)H(\omega). \tag{34}$$

Here we ignored the terms of negative values of m, since a negative value of m gives a negative ω_{cm} which has no real physical meaning. Also, a term with negative ω_{cm} is relatively small and can be ignored for sufficiently high frequency ω in which we are interested. We see that each cavity mode contributes a term with a Lorentzian profile with a width of $2\gamma_c$. In the limit $r\to 1$ ($\gamma_c\to 0$), we have a series of impulse functions located at ω_{cm} for inside the cavity. (For outside, we can show that the power spectrum is governed by $\hbar\omega\langle n_\omega\rangle$ in this limiting case.)

V. ADMITTANCE AND THE FLUCTUATION-DISSIPATION THEOREM

The laser may be considered to consist of the optical cavity and the current sources which represent the atoms. The current sources excite the laser field which, in turn, drive the sources. The former process may be described in terms of suitably defined admittance of the space including the cavity. For the definition of the admittance, consider a current source located at $z=z_0$ which has a sinusoidal time dependence $J\exp(-i\omega t)\delta(z-z_0)$ and is coupled to the field at $t=0$. We assume an interaction linear in the field amplitude and in the current. The location of the source may be either inside or outside of the cavity. If outside, the source represents a laser amplifier. We define the admittance as the asymptotic ratio of the field induced at z to the current as the time t goes to infinity. More precisely, we define

$$Je^{-i\omega t}Y(z,z_0,\omega)=\lim_{t\to\infty}\langle E^+(z,t)\rangle \quad (J\text{: at } z_0 \text{ and coupled at } t=0), \tag{35}$$

where the thermal average is taken in order to extract only systematic motions. We understand that J should not depend explicitly on the field operators. If the field-atom interaction in a laser begins at $t=0$ and if the current has a density $\mathcal{J}(z,\omega)$ in the space and the frequency domain, the laser field for large t may be written

$$E^+(z,t)=\int dz_0\int_{-\infty}^{\infty}d\omega\, Y(z,z_0,\omega)\mathcal{J}(z_0,\omega)e^{-i\omega t}+E_T^+(z,t), \tag{36}$$

where $E_T^+(z,t)$ is the thermal fluctuation of the field given by Eq. (12) which exists in the absence of the sources. Here we are assuming that the thermal field is not disturbed by the introduction of current sources, which implies that Eq. (36) is subject to a perturbation approximation. The current $\mathcal{J}$ may be related to $E^+(z,t)$ by the atomic equation of motion and may contain atomic noise operators which are associated with the pumping and damping of the atoms and are responsible for the quantum noise.

The use of admittance as the description of the laser cavity is advantageous, in view of the presence of the fluctuation-dissipation theorem, in studying thermal noise of the laser cavity or of the laser field. The remainder of this section will be devoted to the calculation of and discussion on the admittance and related fluctuation-dissipation theorem. In order to obtain the expression for the admittance we calculate the field excited by the current $J\exp(-i\omega t)\delta(z-z_0)$ using the Heisenberg equation of motion. For the present purpose J may be considered to be a classical quantity. The total Hamiltonian reads

$$H=H_0+H_{\rm int}, \tag{37a}$$

where[13]

$$H_0=\sum_k\hbar\omega_k a_k^\dagger a_k, \tag{37b}$$

and the interaction Hamiltonian is given by[22]

$$H_{\rm int}=-\int_{-d}^{L}A(z,t){\rm Re}[J\exp(-i\omega t)\delta(z-z_0)]\,dz, \tag{37c}$$

which reads by Eqs. (1) and (3)

$$H_{\rm int}=-\sum_k\left(\frac{\hbar}{2\omega_k}\right)^{1/2}(a_k+a_k^\dagger)U_k(z_0){\rm Re}(Je^{-i\omega t}). \tag{37d}$$

We can solve for each of the field operators a_k and construct $E^+(z,t)$ using Eqs. (2) and (4). The exact solution is, using Eq. (12),

$$E^+(z,t)=\sum_k\left(-\frac{i}{2}\right)U_k(z)U_k(z_0)\left(\frac{1-e^{i(\omega-\omega_k)t}}{\omega-\omega_k}Je^{-i\omega t}-\frac{1-e^{-i(\omega+\omega_k)t}}{\omega+\omega_k}J^*e^{i\omega t}\right)+E_T^+(z,t). \tag{38a}$$

Since we are interested in the response of $E^+(z,t)$ to the current $J\exp(-i\omega t)$ we ignore the second term. This amounts to a rotating wave approximation and, since $\omega_k > 0$, frequency ω should be positive and large for this approximation to be justified. This is consistent with the approximation made in deriving Eq. (34). Then we have

$$E^+(z,t)=\sum_k\left(-\frac{i}{2}\right)U_k(z)U_k(z_0)\frac{1-e^{i(\omega-\omega_k)t}}{\omega-\omega_k}Je^{-i\omega t}+E_T^+(z,t). \tag{38b}$$

By the definition Eq. (35) we have, using Eq. (11),

$$Y(z,z_0,\omega)=-\frac{i}{2}\sum_k U_k(z)U_k(z_0)\zeta(\omega-\omega_k), \tag{39}$$

where the function ζ is given by Eq. (23). Since $\omega_k > 0$, this is nonvanishing for $\omega > 0$ as required. Of course, this expression could be obtained by the standard linear-response theory,[23] and this can be shown to be equal to that part of the usual admittance for the total electric field E which is nonvanishing for positive frequency ω.[24] We have, using Eqs. (7), (25), and (23)

$$\mathrm{Re}Y(z,z_0,\omega)=-\tfrac{1}{2}\pi\rho(\omega)U_\omega(z)U_\omega(z_0)H(\omega) \tag{40a}$$

$$=-\pi I(z,z_0,\omega)/\hbar\omega\langle n_\omega\rangle, \tag{40b}$$

$$\mathrm{Im}Y(z,z_0,\omega)=\frac{P}{\pi}\int_0^\infty\frac{\mathrm{Re}Y(z,z_0,\omega')}{\omega-\omega'}d\omega', \tag{40c}$$

where the function H is given by Eq. (26). We have the symmetry property

$$Y(z,z',\omega)=Y(z',z,\omega), \tag{41}$$

that is, the field at z induced by a current source at z' is equal to the field at z' induced by the same source located at z.

From Eqs. (28) and (40b) we see that the real part of the admittance is related to the correlation function by

$$\hbar\omega\langle n_\omega\rangle\,\mathrm{Re}Y(z,z',\omega)=-\tfrac{1}{2}\int_{-\infty}^{\infty}G(z',0,z,t)e^{i\omega t}\,dt. \tag{42}$$

This is a fluctuation-dissipation theorem.[23,25] This implies that if we know the normally ordered correlation function of the thermal radiation field we can obtain the description of the cavity using this equation and Eq. (40c). Conversely, if we know the admittance we can obtain the correlation function by an inverse Fourier transform of Eq. (42). If we admit that the distribution of the field amplitude is Gaussian for the thermal field we can say that either one of these two knowledges is sufficient to determine the thermal noise completely. We note that Eq. (42) is valid if we replace the thermal distribution (9b) by any other stationary and well-behaved distribution.

The term dissipation requires some interpretation since we are not assuming any loss oscillators nor absorbing atoms, but our space is assumed to be bounded by perfectly conducting walls. We now show that the dissipation associated with the real part of the admittance comes from the electromagnetic energy flow to a great distance in the positive z direction. First we calculate the average energy desnity $W(z,t)$ for large t at z induced by the current source $J\exp(-i\omega t)\delta(z-z_0)$. Using Eq. (36) for single frequency ω and Eq. (11) and their Hermitian adjoints we have

$$W(z,t)=\langle\epsilon(z)[E^-(z,t)E^+(z,t)-E_T^-(z,t)E_T^+(z,t)]\rangle$$
$$=\epsilon(z)\,|J|^2Y^*(z,z_0,\omega)Y(z,z_0,\omega), \tag{43}$$

where $\epsilon(z)$ stands for ϵ^0 for $z>0$ and ϵ^1 for $-d<z<0$. This is independent of time t. However, the total energy W_{tot} delivered to the space by the source does not saturate. To show this we calculate

$$W_{tot}(t)=\int_{-d}^{L}W(z,t)\,dz. \tag{44}$$

Using Eqs. (38b) and (11) we have

$$W_{tot}(t)=\int_{-d}^{L}dz\,\epsilon(z)\,\frac{|J|^2}{4}\sum_{j,k}U_j(z)U_j(z_0)U_k(z)U_k(z_0)\,\frac{(e^{i\omega t}-e^{i\omega_j t})(e^{-i\omega t}-e^{-i\omega_k t})}{(\omega_j-\omega)(\omega_k-\omega)}. \tag{45a}$$

Using the orthonormality of U's [Eq. (43) in I] we can convert the double sum into a single sum and get

$$W_{tot}(t)=\frac{|J|^2}{4}\sum_k[U_k(z_0)]^2\,\frac{\sin^2[\frac{1}{2}(\omega_k-\omega)t]}{[\frac{1}{2}(\omega_k-\omega)]^2}. \tag{45b}$$

Thus, for large t[19]

$$\frac{d}{dt}W_{tot}(t)=\frac{\pi\rho(\omega)}{2}|J|^2[U_\omega(z_0)]^2$$
$$=|J|^2|\mathrm{Re}Y(z_0,z_0,\omega)|. \tag{46}$$

The second form was obtained by use of Eq. (40a). This form for the power dissipation is well known in electrical engineering. Therefore, for large t Eqs. (43) and (46) show that the energy density induced by the current source is constant anywhere but the total energy delivered is proporitonal to time t. This discrepancy can be solved only by considering that the energy is flowing to the "deep space." This can be confirmed by calculating the Poynting vector for the induced field at a distance from the cavity and from the source J considered above. The induced Poynting vector at z may be defined[26]

$$S(z,t)=\langle E^-(z,t)\times H^+(z,t)-E_T^-(z,t)\times H_T^+(z,t)\rangle\,. \quad (47)$$

For calculation of H^+ we note that in our one-dimensional space[13]

$$H^+(z,t)=\frac{1}{\mu}\frac{\partial}{\partial z}A^+(z,t), \quad (48)$$

where μ is the magnetic permiability of the vacuum. Comparing Eqs. (1)–(4) and retracing the procedure from (37a) to (39) to obtain the admittance for E^+, we can easily show that the admittance for A^+ is merely the admittance in Eq. (39) divided by $i\omega$. With this in mind and proceeding as in Eq. (43), we have for large t

$$S(z,t)=\frac{1}{i\omega\mu}|J|^2Y^*(z,z_0,\omega)\frac{\partial}{\partial z}Y(z,z_0,\omega). \quad (49a)$$

Using Eq. (52c) below for a source located inside the cavity and Eq. (52b), with $z>z_0$, for outside we have $(\partial/\partial z)Y=(i\omega/c^0)Y$. Thus

$$S(z,t)=(|J|^2/\mu c^0)\,|Y(z,z_0,\omega)|^2\,, \quad (49b)$$

which reads, by Eqs. (52c) and (52b),

$$S(z,t)=\begin{cases}\dfrac{|J|^2}{\epsilon^1c^0}\dfrac{\sin^2k_\omega^1(z_0+d)}{1-K\sin^2k_\omega^1d}\,, & -d<z_0<0 \quad (49c)\\[2ex] \dfrac{|J|^2}{\epsilon^1c^0}\dfrac{[(c^0/c^1)\cos k_\omega^1d\sin k_\omega^0z_0+\sin k_\omega^1d\cos k_\omega^0z_0]^2}{1-K\sin^2k_\omega^1d}\,, & 0<z_0. \quad (49d)\end{cases}$$

Comparing this with Eq. (6) we have for any location of the source z_0

$$S(z,t)=\tfrac{1}{2}\pi\rho|J|^2[U_\omega(z_0)]^2,\quad z>\max(z_0,0) \quad (50)$$

where ρ is given by Eq. (7). This is equal to the power dissipation in (46):

$$S(z,t)=\frac{d}{dt}W_{\rm tot}(t)\,. \quad (51)$$

Therefore, we conclude that our power dissipation is owing to electromagnetic radiation into the free space. This is an irreversible process. That the power flow is constant for large t and that no return flow appears in spite of the assumed presence of the wall at $z=L$ are convenient consequences of our formulation of the field where we made L to be infinitely large and eliminated it in advance except for the ones appearing in the normalization factor N_l of the mode functions and in the density of modes ρ. Under this formulation the largest time t required in the calculation of any physical process can still be smaller than the time required for light to reach the boundary at $z=L$ and to come back.

The explicit forms of the admittance read

$$Y(z,z',\omega)=-\frac{1}{2\epsilon^1c^1}\left[\exp i\omega\frac{|z-z'|}{c^1}-\exp i\omega\frac{z+z'+2d}{c^1}-\frac{4r\exp(2i\omega d/c^1)\sin\omega(z+d)/c^1\sin\omega(z'+d)/c^1}{1+r\exp(2i\omega d/c^1)}\right],$$
$$-d<z'<0,\ -d<z<0 \quad (52a)$$

$$=-\frac{1}{2\epsilon^0c^0}\left[\exp i\omega\frac{|z-z'|}{c^0}-r\exp i\omega\frac{z+z'}{c^0}-(1-r^2)\frac{\exp i\omega[(z+z')/c^0+2d/c^1]}{1+r\exp(2i\omega d/c^1)}\right],$$
$$0<z',\ 0<z \quad (52b)$$

$$=i\frac{1+r}{\epsilon^1c^1}\sin\omega\frac{z'+d}{c^1}\frac{\exp[i\omega(z/c^0+d/c^1)]}{1+r\exp(2i\omega d/c^1)}\,,\quad -d<z'<0,\ 0<z. \quad (52c)$$

In the derivation of the above results[27] we used formulas (40b) and (40c), the explicit forms of the power spectrum given by Eqs. (30)–(32), and a Fourier expansion similar to the one in Eq. (17). In using Eq. (40c) we made the approximation that

$$P\int_0^\infty \frac{\cos a\omega'}{\omega-\omega'}d\omega' = P\int_{-\infty}^\infty \frac{\cos a(x+\omega)}{(-x)}dx\,. \qquad (53)$$

This approximation is fairly accurate as long as we restrict ourselves to high frequencies. This restriction is consistent with the rotating-wave approximation made in Eq. (38b). In conclusion we note that formulas (39)–(41) for the admittance, the fluctuation-dissipation theorem (42), and the formulas concerning power dissipation (43), (46), and (49a), do not depend on our particular cavity model but hold for arbitrary, one-dimensional "lossless" structure with coupling to the free space. The derivation of Eq. (51), which was made by use of our model, is to be regarded as a special proof of the energy-conservation law under the presence of a current source and an optical discontinuity.

VI. CAVITY NOISE CHARACTERISTICS

To begin with we note that the general feature of the noise characteristics of the cavity is given by the fluctuation-dissipation theorem (42) and formula (39) for the admittance. In this section we give a formulation which is useful when we consider a particular cavity mode, and using this formulation we derive the usual form of the fluctuation-dissipation theorem assumed in most quasimode theories of the laser.

We define the field amplitude of a particular cavity mode of which the real part of the frequency is ω_c by

$$\tilde{E}^+(\omega_c,z,t)=E^+(z,t)e^{i\omega_c t}\,. \qquad (54)$$

Obviously

$$\langle \tilde{E}^+(\omega_c,z,t)\rangle = 0\,. \qquad (55)$$

We introduce new definitions for the correlation function, the power spectrum, and the admittance associated with the above mode amplitude [cf. Eqs. (13), (28), and (35)]:

$$G(\omega_c,z',t',z,t)=\langle \tilde{E}^-(\omega_c,z',t')\tilde{E}^+(\omega_c,z,t)\rangle\,, \qquad (56)$$

$$I(\omega_c,z',z,\omega)=\frac{1}{2\pi}\int_{-\infty}^\infty G(\omega_c,z',0,z,t)e^{i\omega t}dt\,, \qquad (57)$$

$$Je^{-i\omega t}Y(\omega_c,z,z_0,\omega)=\lim_{t\to\infty}\langle\tilde{E}^+(\omega_c,z,t)\rangle\,. \qquad (58)$$

In Eq. (58) a probing current

$$J\exp[-i(\omega_c+\omega)t]\delta(z-z_0)$$

is assumed which is coupled to the field at $t=0$. The correlation function is given by Eq. (14) multiplied by $\exp[i\omega_c(t-t')]$ and the power spectrum and the admittance are given by Eqs. (25) and (39), respectively, if we replace ω in these equations by $\omega_c+\omega$. Then we have [cf. Eqs. (40b) and (40c)]

$$\mathrm{Re}Y(\omega_c,z,z_0,\omega)=-\frac{\pi I(\omega_c,z,z_0,\omega)}{\hbar(\omega_c+\omega)\langle n_{\omega_c+\omega}\rangle}\,, \qquad (59)$$

$$\mathrm{Im}Y(\omega_c,z,z_0,\omega)=P\int_{-\omega_c}^\infty \frac{d\omega'}{\pi}\frac{\mathrm{Re}Y(\omega_c,z,z_0,\omega')}{\omega_c+\omega-\omega'}\,. \qquad (60)$$

In particular, Eq. (34) is rewritten

$$I(\omega_c,z,z',\omega)=\frac{\hbar(\omega_c+\omega)\langle n_{\omega_c+\omega}\rangle}{\pi\epsilon^1 d}\times\sum_{m=0}^\infty \frac{\gamma_c}{\gamma_c^2+(\omega+\omega_c-\omega_{cm})^2}u_{\omega_c+\omega}(z')\times u_{\omega_c+\omega}(z)H(\omega_c+\omega)\,. \qquad (61)$$

The fluctuation-dissipation theorem (42) reads

$$\hbar(\omega_c+\omega)\langle n_{\omega_c+\omega}\rangle\mathrm{Re}Y(\omega_c,z,z',\omega) = -\frac{1}{2}\int_{-\infty}^\infty G(\omega_c,z',0,z,t)e^{i\omega t}dt\,. \qquad (62)$$

Next we show that our thermal fluctuation leads, under certain conditions, to a Markovian noise. We consider the thermal field inside the cavity. As seen from Eq. (61) the power spectrum has contributions from individual cavity modes each having a width of $2\gamma_c$, so that if γ_c is much smaller than the cavity-mode separation $\Delta\omega_c$, the power spectrum in the region $|\omega-\omega_c|\leqslant\frac{1}{2}\Delta\omega_c$ can be approximately given by a single term that corresponds to ω_c. Thus

$$I(\omega_c,z',z,\omega)=\frac{\hbar\omega_c\langle n_{\omega c}\rangle}{\pi\epsilon^1 d}\frac{\gamma_c}{\gamma_c^2+\omega^2}u_{\omega_c}(z')u_{\omega_c}(z)H(\omega_c+\omega),\quad |\omega-\omega_c|<\tfrac{1}{2}\Delta\omega_c \qquad (63)$$

provided

$$\gamma_c \ll \Delta\omega_c\,, \qquad (64)$$

where we replaced $u_{\omega_c+\omega}(z')u_{\omega_c+\omega}(z)$ by $u_{\omega_c}(z')u_{\omega_c}(z)$ since under the inequality (64) the inequality $|\omega|<\gamma_c\ll\Delta\omega_c$ holds for important frequency components, so that inside the cavity, these components have essentially the same spatial waveform as that of ω_c. Also, we replaced $\hbar(\omega_c+\omega)\langle n_{\omega_c+\omega}\rangle$ by $\hbar\omega_c\langle n_{\omega_c}\rangle$. This is allowable as long as $|\omega|\ll kT/\hbar$ holds for important frequency components. If we take the inequality $|\omega|<\gamma_c$ into account the last condition is satisfied provided

$$\gamma_c \ll kT/\hbar\,. \qquad (65)$$

Now we further assume that Eq. (63) is true for any frequency ω, that is, we ignore contributions from all the cavity modes other than ω_c.

$$I(\omega_c, z', z, \omega) = [\text{same as (63)}], \quad -\omega_c < \omega < \infty \,. \quad (66)$$

Evidently this is valid on a time scale greater than the reciprocal-cavity-mode separation $\Delta\omega_c^{-1}$. Then substituting Eq. (66) into (59) and (59) into (60)[28] we have

$$Y(\omega_c, z', z, \omega) = -\frac{1}{\epsilon^1 d}\frac{1}{\gamma_c - i\omega} u_{\omega_c}(z') u_{\omega_c}(z) \,. \quad (67)$$

The corresponding correlation function is given by Eq. (62). It follows under condition (65) that

$$G(\omega_c, z', t', z, t) = \frac{\hbar\omega_c \langle n_{\omega_c}\rangle}{\epsilon^1 d} u_{\omega_c}(z') \times u_{\omega_c}(z) e^{-\gamma_c |t-t'|} \,. \quad (68)$$

Under the inequality (64) this function is slowly varying on a time scale greater than $\Delta\omega_c^{-1}$, which is consistent with the condition stated below Eq. (66). Noting that the amplitude ratio r of the neighboring impulsive functions ϕ's separated by τ_c in the exact correlation function Eq. (20a) is equal to $\exp(-\gamma_c \tau_c)$ [see Eqs. (8c) and (8d)], we can say that the above correlation function is, roughly, the envelope of the exact correlation function. If the cavity mode is excited by a current distribution $\mathcal{J}(z,t)$ in the cavity which has the Fourier components $\mathcal{J}(z,\omega)$ given by

$$\mathcal{J}(z,t) = \int_{-\infty}^{\infty} \mathcal{J}(z,\omega) e^{-i\omega t} d\omega \,, \quad (69)$$

we have, using Eqs. (58) and (67), for large t

$$\langle \tilde{E}^+(\omega_c, z, t)\rangle = \int_{-d}^{0} dz_0 \int_{-\infty}^{\infty} d\omega' \left(-\frac{1}{\epsilon^1 d}\right) \frac{1}{\gamma_c - i\omega'} u_{\omega_c}(z_0) u_{\omega_c}(z) \times H(\omega_c + \omega') \mathcal{J}(z_0, \omega_c + \omega') e^{-i\omega' t} \,. \quad (70)$$

Differentiation with respect to time t changes the integrand by a factor $-i\omega' = -\gamma_c + (\gamma_c - i\omega')$, so that we have

$$\frac{d}{dt}\langle \tilde{E}^+(\omega_c, z, t)\rangle = -\gamma_c \langle \tilde{E}^+(\omega_c, z, t)\rangle - \int_{-d}^{0} \frac{dz_0}{\epsilon^1 d} u_{\omega_c}(z_0) u_{\omega_c}(z) \mathcal{J}(z_0, t) e^{i\omega_c t} \,, \quad (71)$$

where we used Eq. (69) in the second term. Thus we should have

$$\frac{d}{dt}\tilde{E}^+(\omega_c, z, t) = -\gamma_c \tilde{E}^+(\omega_c, z, t) - \int_{-d}^{0} \frac{dz_0}{\epsilon^1 d} u_{\omega_c}(z_0) u_{\omega_c}(z) \times \mathcal{J}(z_0, t) e^{i\omega_c t} + f(z, t) \,, \quad (72)$$

with

$$\langle f(z,t)\rangle = 0 \,. \quad (73)$$

We should have, in the absence of the source,

$$\frac{d}{dt}\tilde{E}^+(\omega_c, z, t) = -\gamma_c \tilde{E}^+(\omega_c, z, t) + f(z, t) \,. \quad (74)$$

Then it follows that

$$\langle f^\dagger(z', t') f(z, t)\rangle = \left\langle \left[\left(\frac{d}{dt'} + \gamma_c\right)\tilde{E}^-(\omega_c, z', t')\right] \times \left[\left(\frac{d}{dt} + \gamma_c\right)\tilde{E}^+(\omega_c, z, t)\right]\right\rangle = \left[\frac{\partial}{\partial t'}\frac{\partial}{\partial t} + \gamma_c\left(\frac{\partial}{\partial t'} + \frac{\partial}{\partial t}\right) + \gamma_c^2\right] \times G(\omega_c, z', t', z, t) \,. \quad (75)$$

Using Eq. (68) we can show that

$$\langle f^\dagger(z', t') f(z, t)\rangle = 2\gamma_c G(\omega_c, z', t, z, t)\delta(t - t') \,. \quad (76)$$

Since $G(\omega_c, z', t, z, t)$ is constant in time, we have a Markovian noise for the field amplitude of the cavity mode ω_c. The set of equations (73), (74), and (76) gives the usual form of the description of the thermal noise in a quasimode theory of the laser.[6,7] (Usually the mode functions and the space variables are omitted.) Approximations used in deriving these equations are the inequalities (64) and (65). Also, contributions from other cavity modes than ω_c were neglected. It is to be noted that a high cavity quality factor is not a sufficient condition for a Markovian noise, but that it is necessary that the contributions from other cavity modes than the one under consideration be neglected. Physically, the latter condition corresponds to a situation where a single mode of the cavity is selected by some means internal to the cavity or where the noise is detected with a limited bandwidth. When we consider the cavity to be coupled with a current source which is nonlinear in nature, the latter situation cannot be used as the interpretation of the mathematical model, since the source may mix the contributions from different cavity modes and the mixed contributions cannot be filtered out in the detection stage. Thus the former situation, physical selection of a single mode within the cavity, is more appropriate as the interpretation of the model that leads to a Markovian noise. It should also be noted that to have a thermal radiation field is sufficient, under the prescribed conditions, to derive a Markovian noise but it is not necessary. It can be seen that the necessary condition is that we have a field with stationary, well-behaved distribution around the cavity mode under consideration.

In II we had a thermal driving term for a single-cavity mode which is Markovian on a time scale greater than the reciprocal-cavity half width γ_c^{-1} [cf. Eq. (15) in II] and than the reciprocal-cavity-mode separation $\Delta\omega_c^{-1}$ [see below Eq. (20) in II].

The latter condition is consistent with the condition stated below Eq. (66), but the inequality (64) was not required in II. This is because, in a laser, the linewidth $\Delta\omega$, which is the effective cavity half width under the presence of the laser active medium, can be much smaller than the cavity half width γ_c^{-1}, so that γ_c need not necessarily be small.

VII. PRESERVATION OF COMMUTATION RULES

Here we examine the consistency of our field fluctuation with the field comlutation relation. First we consider the fluctuating part of the field $E^+(z,t)=E_T^+(z,t)$ given by Eq. (36) with $\mathcal{J}=0$ the explicit expression of which is given by Eq. (12). This is freely oscillating by assumption. Therefore, it is evident that in this case the field obeys the correct commutation relation. To confirm this we calculate

$$\langle[E_T(z',t'),E_T(z,t)]\rangle = G(z',t',z,t)-G(z,t,z',t') + G^\dagger(z',t',z,t)-G^\dagger(z,t,z',t') . \quad (77)$$

Here

$$G^\dagger(z',t',z,t) = \langle E_T^+(z',t')E_T^-(z,t)\rangle = \sum_j \frac{\hbar\omega_j}{2}[\langle n_j\rangle+1]U_j(z')U_j(z)e^{-i\omega_j(t'-t)} , \quad (78)$$

where in the second line we used the commutation relations (5a) and (5b), and definitions (12) and (15). Using Eqs. (14) and (78), the symmetry properties (16a) and (16b), and similar relations for $G^\dagger$ we have

$$\langle[E_T(z',t'),E_T(z,t)]\rangle = \sum_j i\hbar\omega_j U_j(z')U_j(z)\sin\omega_j(t-t') . \quad (79)$$

The right-hand side is just the commutator for a freely oscillating field, of which a special example for $-d<z'<0$ and $0<z$ was given in Eq. (64) and below in I. We note that the proof of Eq. (79) does not depend on the special form of the distribution (9b): it can be proved by use of Eqs. (5a) and (5b) even if we do not know anything concerning the initial distribution.

Next we examine the field given by the approximate expression (74). Integrating Eq. (74) and its Hermitian adjoint and using Eq. (73) and similar equation for the corresponding adjoint we have

$$\langle[\tilde{E}^+(\omega_c,z',t),\tilde{E}^-(\omega_c,z,t)]\rangle = \langle[\tilde{E}^+(\omega_c,z',0),\tilde{E}^-(\omega_c,z,0)]\rangle e^{-2\gamma_c t} + \int_0^t dt'\int_0^t dt'' e^{\gamma_c(t'+t''-2t)} \times\langle[f(z',t'),f^\dagger(z,t'')]\rangle . \quad (80)$$

We use Eq. (76) and the analogous equation

$$\langle f(z,t)f^\dagger(z',t')\rangle = 2\gamma_c G^\dagger(\omega_c,z,t,z',t)\delta(t-t') , \quad (81)$$

the pair of identities (14) and (78), and the symmetry property (16a) for G and analogous equation for $G^\dagger$ to obtain

$$\langle[\tilde{E}^+(\omega_c,z',t),\tilde{E}^-(\omega_c,z,t)]\rangle = (\hbar\omega_c/\epsilon^1 d)\,u_{\omega_c}(z')u_{\omega_c}(z) . \quad (82)$$

This equation can be consistent with the commutation relations (5a) and (5b) as long as we write

$$\tilde{E}^+(\omega_c,z,t) = \sum_j i(a_j e^{i\omega_c t})\left(\frac{\hbar\omega_c}{2}\right)^{1/2} \times\left(\frac{2}{\epsilon^1 d}\right)^{1/2} M_j u_{\omega_c}(z) , \quad (83)$$

and choose M_j such that $\sum_j M_j^2=1$. Obviously this is consistent with Eq. (55). To be consistent with the power spectrum (66) which was the starting point towards the expression (74) we should have

$$\langle a_j^\dagger a_j\rangle = (e^{\beta\hbar\omega_c}-1)^{-1} \quad (84)$$

and

$$M_j^2 = \frac{1}{\rho}\frac{\gamma_c/\pi}{\gamma_c^2+(\omega_j-\omega_c)^2} , \quad (85)$$

where ρ is given by Eq. (7). Thus in this second case we cannot preserve the original commutation relation given by Eq. (79) since we abandoned the use of exact-mode functions, but we can retian the respective commutation relations (5a) and (5b) for each freedom (mode) of the field.

In the limit of infinitely large cavity quality factor, that is, in the limit that $\gamma_c\to 0$, we may have from Eq. (85)

$$M_j = \delta_{j,\omega_c} , \quad (86)$$

where we have, recalling the form of the mode function (6c) for inside the cavity,

$$E^+(z,t) = ia\left(\frac{\hbar\omega_c}{2}\right)^{1/2}\left(\frac{2}{\epsilon^1 d}\right)^{1/2}\sin\omega_c\frac{z+d}{c^1} , \quad (87)$$

with

$$[a,a^\dagger]=1 , \quad (88)$$

which is equavalent to the usual expression for the field of a single quasimode of the cavity. In this limiting case we, of course, have no thermal noise affecting the coherence of the field. In fact we have from Eq. (76), for $\gamma_c\to 0$,

$$\langle f^\dagger(z',t')f(z,t)\rangle \to 0 . \quad (89)$$

For a single-cavity mode, therefore, our cavity is noiseless if we have no output coupling. As was mentioned below Eq. (34), a perfect cavity has discrete modes each of which oscillates in an exactly sinusoidal manner, i.e., without fluctuation nor damping.

Thermal noise appears when output coupling is introduced which allows ambient thermal radiation to penetrate the cavity and at the same time causes a systematic motion in the cavity to be damped by radiation to the outside. This is the content of our fluctuation-dissipation theorem (42) or (76).

ACKNOWLEDGMENT

The author wishes to thank Dr. M. Natori for advice in numerical calculation of the trigamma function.

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[25]Equation (3.13) of Ref. 3 seems to give correlation spectrum which is larger by a factor of exp $(\beta\hbar\omega)$ than that of ours even if the differences in the probes for the admittance and in the definitions of the Fourier transform are taken into account.

[26]C. L. Mehta and E. Wolf, Phys. Rev. 157, 1183 (1967).

[27]Exactly the same results as Eqs. (52a)–(52c) can be obtained by a classical wave analysis, i.e., without using the multimode expansion and by solving the Maxwell's equations for $E(z)\exp(-iwt)$ with the source $J\exp(-i\omega t)\delta(z-z_0)$ introduced, discarding the boundary condition at $z=L$, and assuming that only outgoing waves exist for large z. Thus if we are given the fluctuation-dissipation theorem (42), we can calculate the quantum-mechanical correlation function of first order without quantum-mechanical analysis. This is Agarwal's point of view stated in Ref. 3.

[28]On using Eq. (60) we neglected $H(\omega_c+\omega)$ and extended the lower limit of integration to minus infinity. See Eq. (53) and below.