Effect of space charge on free-electron-laser gain

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The first-order correction due to space-charge effects to the small-signal gain of a free-electron laser is calculated classically. For a current density of 10^2 A/cm^2, an interaction length of 1 m and an electron beam energy of 20 MeV, the maximum small-signal gain is reduced by about 0.5% while for 10^3 Amp/cm^2, it is reduced by about 5%. The reduction is proportional to $(\omega_p L/c)^2/\gamma_0{}^3 \equiv S$, where ω_p is the electron plasma frequency, L is the interaction length, c is the velocity of light, and $mc^2\gamma_0$ is the initial electron energy. $S = 0.092$ in the former case and 0.92 in the latter.

I. INTRODUCTION

In a recent paper[1] Kroll has given a small-signal classical theory of a free-electron amplifier and has included the effects of space charge in a plane-wave approximation. In this paper the theory will be extended to the case of finite-length interaction region in the small-cavity limit[2] in which the width of the electron-momentum distribution function is small compared with the cavity linewidth. This limit is appropriate for the recent Stanford experiments.[3,4] The theory presented here is valid in the large-signal–large-space-charge regime, but the gain reduction owing to space charge will be calculated in the small-signal–small-space-charge regime only.

In order to obtain higher-output power from the free-electron laser, it has been proposed[3] that a storage ring be used in order to increase the electron density. In this paper we shall obtain the limit of electron densities at which the Coulomb repulsion between electrons cannot be ignored. We obtain the lowest-order correction to the gain owing to space charge. It should be noted that this calculation does not take into account the phenomena of wave-wave interaction. Stimulated scattering processes have been examined in the recent work of Kwan, Lin, and Dawson.[5]

II. THEORY

The analysis follows closely the theory in Ref. 6 with the addition of space charge. We neglect all dependence on the transverse variables x and y so that the transverse canonical momentum is a constant of the motion. Since transverse-momentum spread is negligible compared with the longitudinal-momentum spread, we take the transverse canonical momentum to be zero. Thus the transverse mechanical momentum is

$$\vec{p}_T = m\gamma\vec{v}_T = -e\vec{A}_T(z,t)\,, \tag{1}$$

where e and m are the electron charge and mass, respectively, $\vec{v}_T$ is the transverse electron velocity, $\gamma = (1-\beta^2)^{-1/2}$, and $\vec{A}_T$ is the transverse vector potential of the circularly polarized dc magnetic field plus the circularly polarized scattered-radiation field. Under the Weizacker-Williams approximation[6] we replace the dc magnetic field with an approximately equivalent plane wave. Thus,

$$\vec{A}_T(z,t) = \hat{e}_-[A_i e^{-i(\omega_i t + k_i z)} + A_s(z,t) \times e^{-i(\omega_s t - k_s z)}] + (\text{c.c.})\,, \tag{2}$$

where

$$\hat{e}_\pm = (\hat{x} \pm i\hat{y})/\sqrt{2}\,. \tag{3}$$

If B_0 is the dc magnetic induction and $\lambda_0 = 2\pi/k_0$ is the dc field period, then we have for the equivalent plane wave[7]

$$k_i A_i = B_0/(1+\beta_0) \approx \tfrac{1}{2}B_0\,, \tag{4}$$

$$k_0 = k_i(1+\beta_0) \approx 2k_i\,. \tag{5}$$

β_0 is the incident electron velocity divided by the velocity of light and is assumed to be near unity. A_s is the scattered-radiation field in the forward direction and we have[7]

$$\gamma_0^2 \cong k_s/4k_i = \lambda_0/2\lambda_s\,. \tag{6}$$

Thus, for fixed λ_0, λ_s may be varied by varying the electron energy, which is given by $\gamma_0 mc^2$.

The relativistic equation of motion for the z component of the electron momentum is given by

$$\frac{dp_z}{dt} = eE_z - \frac{e^2}{2m\gamma}\frac{\partial}{\partial z}(\vec{A}_T^2)\,, \tag{7}$$

where E_z is the electric field owing to the electron Coulomb repulsion. If we make the slowly varying amplitude and phase approximation for $A_s(z,t)$, from (2) and (7) it follows that

$$\frac{dp_z}{dt} \cong -\frac{ie^2k}{m\gamma}(A_i^* A_s e^{i\xi} - A_i A_s^* e^{-i\xi}) + eE_z\,, \tag{8}$$

where

$$k \equiv k_s + k_i\,,\quad \Delta\omega \equiv \omega_s - \omega_i\,,\quad \xi \equiv kz - \Delta\omega t\,. \tag{9}$$

Under the slowly varying amplitude and phase approximation for $A_s(z,t)$, the wave equation for $\vec{A}_T$ reduces to

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)A_s(z,t)=\frac{i\mu_0 F(\vec{J}_T\cdot\hat{e}_+)e^{i(\omega_s t-k_s z)}}{2k_s}\,, \tag{10}$$

where $\vec{J}_T$ is the transverse-current density and F is a filling factor given roughly by the ratio of the electron-beam cross section to the cavity cross section. We have used the relations

$$\omega_i = ck_i \text{ and } \omega_s = ck_s \tag{11}$$

in obtaining (10).

The space-charge field obeys Poisson's equation

$$\frac{\partial E_z}{\partial z}(z,t)=F\frac{\rho}{\epsilon_0}\,, \tag{12}$$

where ρ is the charge density.

The electron distribution function $h(z, p_z, t)$ obeys the Vlasov equation

$$\frac{\partial h}{\partial t}+v_z\frac{\partial h}{\partial z}+\frac{dp_z}{dt}\frac{\partial h}{\partial p_z}=0\,, \tag{13}$$

where the electron charge density is given by

$$\rho(z,t)=e\int_{-\infty}^{\infty}dp_z\,h(z,t,p_z)\,, \tag{14}$$

and the transverse-current density is given by

$$\vec{J}_T(z,t)=e\int_{-\infty}^{\infty}\vec{v}_T h\,dp_z=\frac{-e^2\vec{A}_T(z,t)}{m}\int_{-\infty}^{\infty}\frac{h}{\gamma}\,dp_z\,, \tag{15}$$

where we used (1).

If we use (15) and (2), (10) becomes

$$\frac{\partial A_s}{\partial t}+\frac{1}{c}\frac{\partial A_s}{\partial z}=-\frac{ie^2F}{2k_s mc^2\epsilon_0}(A_i e^{i\xi}+A_s)\int_{-\infty}^{\infty}\frac{h}{\gamma}\,dp_z\,. \tag{16}$$

If we multiply both sides of (16) by A_s^* and its conjugate by A_s and add, we obtain

$$\left(\frac{\partial}{\partial t}+\frac{1}{c}\frac{\partial}{\partial z}\right)|A_s|^2 =\frac{ie^2F}{2k_s mc^2\epsilon_0}(A_i A_s^* e^{-i\xi}-A_i^* A_s e^{i\xi})\int_{-\infty}^{\infty}\frac{h}{\gamma}\,dp_z\,. \tag{17}$$

If we use (8) in (13), we obtain for the distribution function

$$\frac{\partial h}{\partial t}+v_z\frac{\partial h}{\partial z}+\left(eE_z-\frac{ie^2k}{m\gamma}(A_i^*A_s e^{i\xi} -A_i A_s^* e^{-i\xi})\right)\frac{\partial h}{\partial p_z}=0\,. \tag{18}$$

Finally, if we use the charge density given by (14), Poisson's equation becomes

$$e\frac{\partial E_z}{\partial z}=\frac{Fe^2}{\epsilon_0}\int_{-\infty}^{\infty}dp_z\,h(z,t,p_z)\,. \tag{19}$$

The coupled equations (17) for the field intensity, (18) for the distribution function, and (19) for the space-charge field are the equations which determine the gain in the presence of space charge.

In passing, one should note that for the self-consistency, one would have to satisfy the equation of continuity. One can show that under the operating conditions for the free-electron laser, it will be satisfied automatically within a negligibly small error when (17)–(19) are satisfied.

III. SMALL-SIGNAL-SMALL-SPACE-CHARGE SOLUTION

From the equation of motion (7) or (8), one sees that the "incident" and scattered fields form an interference field, or trapping "potential," that propagates with velocity[4,8]

$$v_w = \Delta\omega/k \equiv (\omega_s-\omega_i)/(k_s+k_i)\approx \omega_s/k_s = c\,, \tag{20}$$

where we used (6) when $\gamma_0^2 \gg 1$. The electrons that are traveling slower than v_w are speeded up and those traveling faster are slowed down. This bunching causes us to look for solutions in which the space-charge field has solutions of the form

$$eE_z \cong \psi(z,t)e^{i\xi}+\psi^*(z,t)e^{-i\xi}+\cdots\,. \tag{21}$$

In addition, to lowest order when the field is absent, (18) reduces to

$$\frac{\partial h^{(0)}}{\partial t}+v_z\frac{\partial h^{(0)}}{\partial z}=0\,. \tag{22}$$

Let us consider the case in which electrons are injected at a constant rate so there are n_0 electrons/m^3 in the interaction region. A solution of (22) is then given by

$$h^{(0)}=n_0\delta(p_z-p_0) \tag{23}$$

under the small-cavity limit in which the electron-momentum distribution is very narrow compared with the "cavity" linewidth.

Next, since space-charge waves that propagate as (21) will interact most strongly with the interference field, it is plausible to subtract the effects of the background charge density. That is, we replace Poisson's equation by[1]

$$e\frac{\partial E_z}{\partial z}=\frac{Fe^2}{\epsilon_0}\left(\int_{-\infty}^{\infty}h\,dp_z-n_0\right). \tag{24}$$

If we now expand h to lowest order in e^2A_s and e^4A_s, viz.,

$$h=h^{(0)}+h^{(1)}+h^{(2)}+\cdots\,, \tag{25}$$

then (18) becomes

$$\frac{\partial h^{(1)}}{\partial t}+v_z\frac{\partial h^{(1)}}{\partial z}=\frac{ie^2k}{m\gamma}(A_i^*A_s e^{i\xi}-A_iA_s^*e^{-i\xi})\frac{\partial h^{(0)}}{\partial p_z}, \quad (26)$$

while (24) becomes

$$e\frac{\partial E_z^{(2)}}{\partial z}=\frac{Fe^2}{\epsilon_0}\int_{-\infty}^{\infty}h^{(1)}\,dp_z. \quad (27)$$

That is, eE_z is of order e^4A_s since $h^{(1)}$ is of order e^2A_s.

We may solve (26) easily by a Green's function and find

$$h^{(1)}=\frac{\epsilon_0\omega_p^2k}{c\gamma_0}\frac{\partial\delta(p_z-p_0)}{\partial p_z}[A_i^*A_se^{i\xi}\Lambda(\mu,z)+(\text{c.c.})] \quad (28)$$

plus higher-order terms, where

$$\mu\equiv k-\Delta\omega/v_z=k(1-v_w/v_z), \quad (29)$$

$$\Lambda(\mu,z)\equiv(1-e^{-i\mu z})/\mu, \quad (30)$$

and

$$\omega_p^2\equiv n_0e^2/m\epsilon_0. \quad (31)$$

Also we have that

$$\gamma_0=[1+(p_0/mc)^2]^{1/2}. \quad (32)$$

If we substitute (21) and (28) in (27) and make the slowly varying amplitude and phase approximation on $\psi(z,t)$ we find that

$$\psi(z,t)=\frac{ie^2\omega_p^2A_i^*A_skF}{mc^2\gamma_0^4}\frac{\partial\Lambda(\mu_0,z)}{\partial\mu_0}, \quad (33)$$

where we used the result that

$$\left.\frac{\partial\mu}{\partial p_z}\right|_{p_z=p_0}\cong\frac{k}{mc\gamma_0^3} \quad (34)$$

when integrating (27) by parts. Also we used (20).

To order e^4A_s, (18) now becomes

$$\frac{\partial h^{(2)}}{\partial t}+v_z\frac{\partial h^{(2)}}{\partial z}=-(\psi e^{i\xi}+\psi^*e^{-i\xi})\frac{\partial h^{(0)}}{p_z}. \quad (35)$$

When we use (33) and (23), the Green's-function solution of (35) is given by

$$h^{(2)}=-\frac{\epsilon_0\omega_p^4A_i^*A_sk}{c^3\gamma_0^4}\frac{\partial\delta(p_z-p_0)F}{\partial p_z}\times\left(ie^{i\xi}\frac{\partial}{\partial\mu_0}\int_0^z dz'\,\Lambda(\mu_0,z')e^{i\mu(z'-z)}+(\text{c.c.})\right) \quad (36)$$

when $v_z\approx c$.

When we use (28) and (36) in (17), and integrate over dp_z by parts, we obtain

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\ln|A_s|^2=\left(\frac{\omega_p}{c}\right)^2\frac{e^2|A_i|^2}{m^2c^2}\frac{kF}{\gamma_0^5}\left[\frac{\sin\mu_0z-\mu_0z\cos\mu_0z}{\mu_0^2}-\left(\frac{\omega_p}{c}\right)^2\frac{1}{\gamma_0^3}\times\left(\frac{(6\mu_0z-\mu_0^3z^3)\cos\mu_0z-(6-3\mu_0^2z^2)\sin\mu_0z}{6\mu_0^4}\right)\right]. \quad (37)$$

Since the right-hand side is independent of the time, we obtain the steady-state gain G,

$$G=\int_0^L dz\,\frac{\partial}{\partial z}\ln|A_s|^2=g_{\max}[A(\chi)-SB(\chi)], \quad (38)$$

where L is the interaction length,

$$g_{\max}=\left(\frac{\omega_p}{c}\right)^2L^2\left(\frac{e|A_i|}{mc}\right)^2\frac{k_sLF}{\gamma_0^5}(0.135\,02) \quad (39)$$

is the maximum small-signal gain[6] at $\chi\cong2.6$, where

$$\chi=\mu_0L\cong k_sL(1-v_w/v_0), \quad (40)$$

$$A(\chi)=\frac{2-2\cos\chi-\chi\sin\chi}{\chi^3 0.135\,02}, \quad (41)$$

$$B(\chi)=\frac{(24-6\chi^2)\cos\chi+(18\chi-\chi^3)\sin\chi-24}{6\chi^5 0.135\,02}, \quad (42)$$

and the space-charge reduction parameter S is given by

$$S=\left(\frac{\omega_p}{c}\right)^2\frac{L^2}{\gamma_0^3}=\frac{c}{mc^3\epsilon_0}\frac{JL^2}{\gamma_0^3}, \quad (43)$$

where J is the current density.

Note that as $v_w\cong c\to v_0$, $\chi\to0$ and $A(0)=B(0)\to0$, that is, there is no gain when the electrons travel in synchronism with the interference wave. Gain and loss exactly cancel in this case. Also note that we have normalized A so that $A_{\max}=A(2.6)\cong1.0$.

In Fig. 1 we have plotted the small-signal gain without space charge, $A(\chi)$, as open circles, the space-charge reduction $SB(\chi)$, as Y's and the net gain, $A(\chi)-SB(\chi)$, as X's for a current density of 10^2 A/cm^2, an interaction length of one meter, and a beam energy of 20 MeV. $S=0.092$ in this case. The reduction in gain is about 0.5% and is

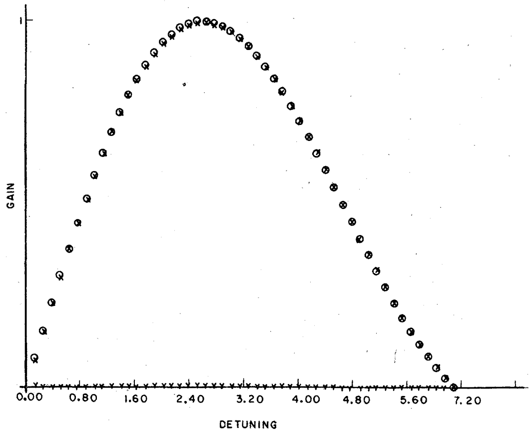

FIG. 1. Plot of the small-signal gain without space charge (open circles), space-charge reduction (Y's) and the gain including space-charge reduction (X's) for $S = 0.092$.

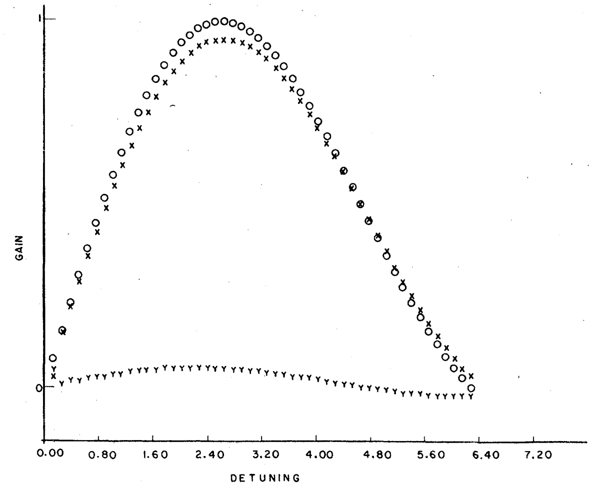

FIG. 2. Plot of the small-signal gain without space charge (open circles), space-charge reduction (Y's), and the gain including space-charge reduction (X's) for $S = 0.92$.

barely visible. In Fig. 2 we repeated for a current density of 10^3 A/cm^2 and found approximately a 5% reduction in the gain. We took the filling factor to be unity.

IV. CONCLUSIONS

The first-order correction of the small-signal gain owing to space charge has been obtained for the free-electron laser. The gain is reduced by about 0.5% when $S = 0.092$ and by about 5% when $S = 0.92$. The excitation of the electrostatic field appears to be the source of the gain reduction.

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