Effect of optical-cavity length on laser photon statistics

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The question of how the light output of a laser and its fluctuations depend on the optical-cavity length at a given frequency is investigated theoretically for a laser oscillating in a single Gaussian mode. The treatment is quantum mechanical, and is based on the Scully-Lamb laser model, except that a perturbation expansion is used and possible cooperative atomic effects are included. It is shown, with the help of some reasonable approximations, that the probability distribution of the photon-occupation number can be cast into a form that is similar to the Scully-Lamb formula, but with coefficients that depend on cavity length in a more complicated way. Curves are presented that illustrate the behavior and should lend themselves to experimental test.

I. INTRODUCTION

The question how the photon statistics of a laser in the steady state depend on the length of the optical cavity, with all other conditions remaining constant, has recently been examined.[1] In the treatment it was assumed that a single, axial, plane-wave mode of the cavity of definite frequency was excited, so that the geometry was as simple as possible, and that the active medium occupied only a portion of the cavity volume. The principal conclusions of that analysis were (a) that below the laser threshold the light intensity falls off with increasing cavity length, but that the relative intensity fluctuations do not change, and (b) that above threshold the light intensity does not change, but that the relative intensity fluctuations fall off with increasing cavity length. In other words, the optical field becomes increasingly coherent as the cavity length increases. Near the laser threshold an intermediate situation is encountered.

Although, in principle, these predictions should lend themselves to experimental test, in practice the situation is always more complicated because the laser mode is never a plane wave.[2,3] Most commonly the field has a Gaussian distribution in the radial direction, and a phase shift that also increases radially in a Gaussian manner.[3] Moreover, the spread of the Gaussian amplitude distribution varies with position along the laser axis. These features have the effect of complicating the length dependence of the photon statistics, as compared with a plane-wave laser.

In the following we reexamine the problem of how the photon statistics change with cavity length within the framework of the Scully-Lamb theory of the laser,[4] with the help of the orthogonal mode functions used by Kogelnik and Li.[3] We also include possible contributions from multiatom or cooperative interactions within the laser, and we show that these also have the effect of modifying the photon statistics and their dependence on cavity length. Unlike Scully and Lamb,[4] who made use of the Weisskopf-Wigner[5] procedure, we proceed by a perturbation expansion, so that our calculation is not applicable very far above threshold. However, our procedure should be valid up to pump parameter values of at least 10, and it is only in the neighborhood of the laser threshold that the photon statistics really reveal interesting features.

II. CAVITY MODE FUNCTIONS

We consider a typical laser of the form illustrated in Fig. 1, in which an optical cavity of length l is formed by a plane mirror and a concave mirror of radius R. Then the lowest-order or fundamental mode function $u_{00}(r,z)$ of the cavity depends on the radial distance r from the axis and on the axial distance z from the beam waist, which is the position of the plane mirror, and is given by[3]

$$u_{00}(r,z)=\left(\frac{2}{\pi w_0^2 l}\right)^{1/2}\frac{w_0}{w(z)}\exp\left[-i(kz-\psi(z))-r^2\left(\frac{1}{w^2(z)}+\frac{ik}{2R(z)}\right)\right]. \tag{1}$$

Here k is the wave number, w_0 is the radius of the beam waist, which is determined by the geometry,

$$w_0^2=(2/k)[l(R-l)]^{1/2}, \tag{2a}$$

and

$$w(z)=w_0[1+(2z/kw_0^2)^2]^{1/2}, \tag{2b}$$

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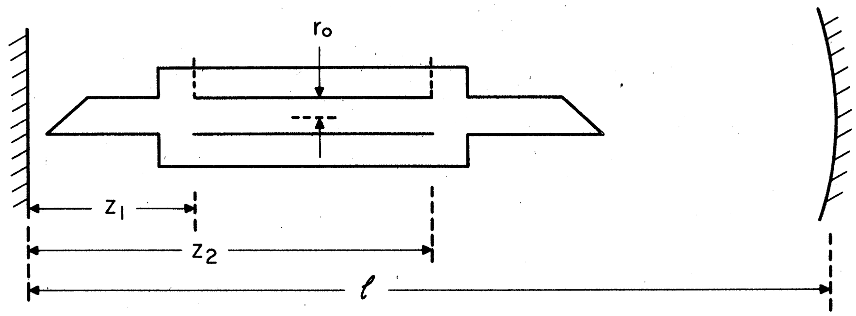

FIG. 1. Outline of the laser geometry.

$$R(z) = z[1 + (kw_0^2/2z)^2], \tag{2c}$$

$$\psi(z) = \tan^{-1}(2z/kw_0^2). \tag{2d}$$

The same expressions hold more generally for a cavity formed by two curved mirrors, except that Eq. (2a) for the beam waist has to be replaced by the more general relation

$$w_0^2 = \left(\frac{2}{k}\right)\frac{[l(R'-l)(R-l)(R+R'-l)]^{1/2}}{(R+R'-2l)},$$

where R' is the radius of curvature of the other mirror. The higher-order mode functions $u_{mn}(x, y, z)$ are related to $u_{00}(r, z)$, with $r = (x^2 + y^2)^{1/2}$, by[3]

$$u_{mn}(x, y, z) = \left(\frac{1}{2^m m!}\frac{1}{2^n n!}\right)^{1/2} H_m\left(\frac{\sqrt{2}\,x}{w(z)}\right) \times H_n\left(\frac{\sqrt{2}\,y}{w(z)}\right) u_{00}(r, z), \tag{3}$$

where $H_m(X)$ is the mth-order Hermite polynomial, and the different mode functions $u_{mn}(x, y, z)$ form an orthonormal set, so that

$$\int u^*_{mn}(x, y, z) u_{m'n'}(x, y, z)\, dx\, dy\, dz = \delta_{mm'}\delta_{nn'}. \tag{4}$$

The electromagnetic field in the cavity can then be given a representation in terms of the mode functions $u_{mn}(\vec{r})$. Thus, for the electric field vector $\hat{\vec{E}}(\vec{r}, t)$, we write (in mks units)[6]

$$\hat{\vec{E}}(\vec{r}, t) = \sum_\lambda \left(\frac{\hbar\omega_\lambda}{2\epsilon_0}\right)^{1/2} [\hat{a}_\lambda(t) u_\lambda(\vec{r})\vec{\epsilon}_\lambda + \text{H.c.}], \tag{5}$$

where λ is an abbreviated label for the mode index mn and a possible polarization index, $\hat{a}_\lambda(t)$ is the photon annihilation operator for an excitation of the mode λ, and $\vec{\epsilon}_\lambda$ is a unit polarization vector. In the following we shall, however, suppose that it is possible to treat the problem as if only a single mode, the fundamental, plays a role in the laser mechanism, so that we can dispense with the summation in Eq. (5) and drop the mode label λ.

We assume that the active laser medium is located in the cavity between planes $z = z_1$ and $z = z_2$, as shown, and that the medium extends radially sufficiently far from the axis that at the radial boundary $u_{00}(r_0, z) \equiv u(r_0, z)$ is very small compared with $u(0, z)$. Because of the Gaussian r dependence of $u(r, z)$, and the rapid falloff with r, these conditions are generally satisfied in practice when the medium is close to the beam waist.

III. EQUATIONS OF MOTION OF THE OPTICAL FIELD

We shall be interested in the question how the light output of the laser and its fluctuations are affected when the laser is oscillating in a single mode and the cavity length l is changed, with the resonant frequency and all the foregoing conditions preserved. Superficially it might seem that there would be little effect on the light. However, a closer examination of the problem shows that an effect is to be expected because the coupling constant between the atoms and the laser field involves the cavity length, and the rates of stimulated and spontaneous emission into the laser mode must therefore depend on the cavity length also.

In order to investigate this effect we adopt the simple model of the laser that was introduced by Scully and Lamb.[4] They supposed that the laser atoms could be treated as identical two-level quantum systems with an energy separation $\hbar\omega_0$, where ω_0 is equal or close to the frequency ω of the fundamental cavity mode, and that the atoms may decay nonradiatively out of the two levels $|1\rangle$ and $|2\rangle$ to various other states that do not concern us at rates γ_1 and γ_2, respectively. They modeled the laser-excitation and the laser-loss mechanisms by supposing that excited atoms in state $|2\rangle$ and unexcited atoms in state $|1\rangle$ were introduced into the laser cavity at rates R_2 and R_1, respectively, although, ultimately, the cavity Q factor was identified as being responsible for the loss mechanism.

The energy of the coupled system of a field and

N identical atoms then takes the form

$$\hat{H} = \hbar\omega_0 \sum_{i=1}^{N} \hat{R}_3^{(i)} + \hbar\omega \hat{a}^\dagger \hat{a} - 2 \sum_{i=1}^{N} \vec{\mu} \cdot \hat{\vec{E}}(\vec{r}_i, t) \hat{R}_1^{(i)} , \quad (6)$$

where $\hat{R}_1^{(i)}$, $\hat{R}_2^{(i)}$, $\hat{R}_3^{(i)}$ are the three Pauli spin operators for the ith atom, $\vec{\mu}$ is the transition dipole moment between levels $|1\rangle$ and $|2\rangle$ that we take to be real, and $\vec{r}_i$ is the position of the ith atom. The last term in Eq. (6) represents the interaction energy $\hat{H}_1$. We shall find it convenient to work in the interaction picture in which, with the help of Eq. (5), $\hat{H}_1(t)$ can be expressed in the form

$$\hat{H}_1(t) = -\left(\frac{\hbar\omega}{2\epsilon_0}\right)^{1/2} \vec{\mu} \cdot \vec{\epsilon} \sum_{i=1}^{N} [\hat{b}^{(i)} e^{-i\omega_0 t} + \hat{b}^{(i)\dagger} e^{i\omega_0 t}] \times [\hat{a} e^{-i\omega t} u(\vec{r}_i) + \hat{a}^\dagger e^{i\omega t} u^*(\vec{r}_i)] \quad (7)$$

where $\hat{b}^{(i)}$ and $\hat{b}^{(i)\dagger}$ are lowering and raising operators for the ith atom. If we discard terms oscillating at double the optical frequency on the grounds that their contributions over any measurable time interval are negligible, we can simplify Eq. (7) to

$$\hat{H}_1(t) = \hbar f \sum_{i=1}^{N} [\hat{a} \hat{b}^{(i)\dagger} e^{i(\omega_0 - \omega)t} u(\vec{r}_i) + \text{H.c.}] \quad (8)$$

with

$$\hbar f = -\vec{\mu} \cdot \vec{\epsilon} (\hbar\omega / 2\epsilon_0)^{1/2} . \quad (9)$$

It should be noted that, because of the $L^{-3/2}$ dimensions of the mode function $u(\vec{r})$, f is not a frequency but has the dimensions of (frequency) $\times L^{3/2}$.

In the Scully-Lamb theory[4] it is assumed that the atoms make their contributions to the field one at a time, and that the total rate of change of the field may be calculated as a "coarse-grained derivative," by multiplying the change produced by one atom by the rate at which atoms in the same initial state are being introduced. We shall adopt some of the same ideas, except that we allow for the possibility of some cooperative effects by treating the atoms collectively.

Suppose that at some instant t, N_2 laser atoms are in the excited state $|2\rangle$, and that the radiation field is in some state characterized by the density operator $\hat{\rho}_F(t)$. We suppose that at this moment the density operator $\hat{\rho}(t)$ of the coupled system may be factorized in the form

$$\hat{\rho}_2(t) \equiv \hat{\rho}_F(t) \prod_{i=1}^{N_2} |2_i\rangle\langle 2_i| . \quad (10)$$

Although we shall not explicitly allow for inhomogeneous broadening of the laser atoms, N_2 could depend on the atomic frequency ω_0, so that the possibility of inhomogeneous broadening is not excluded. N_2 would then be the number of excited atoms within a natural linewidth of the cavity frequency ω.

These atoms and the field now interact according to the interaction Hamiltonian $\hat{H}_1(t)$ given by Eq. (8) for a short time T_2 that we take to be of the order of the lifetime $1/\gamma_2$ of the excited state. From the equation of motion for $\hat{\rho}(t)$, the change $\delta\hat{\rho}_F(t)$ in the density operator of the field brought about by the interaction may be expressed by the perturbation expansion

$$\begin{aligned}\delta\hat{\rho}_F(t) = \mathrm{Tr}_A \Bigg(&\frac{1}{i\hbar} \int_t^{t+T_2} dt_1 [\hat{H}_1(t_1), \hat{\rho}_2(t)] + \frac{1}{(i\hbar)^2} \int_t^{t+T_2} dt_1 \int_t^{t_1} dt_2 [\hat{H}_1(t_1), [\hat{H}_1(t_2), \hat{\rho}_2(t)]] \\ &+ \frac{1}{(i\hbar)^3} \int_t^{t+T_2} dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 [\hat{H}_1(t_1), [\hat{H}_1(t_2), [\hat{H}_1(t_3), \hat{\rho}_2(t)]]] \\ &+ \frac{1}{(i\hbar)^4} \int_t^{t+T_2} dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \int_t^{t_3} dt_4 [\hat{H}_1(t_1), [\hat{H}_1(t_2), [\hat{H}_1(t_3), [\hat{H}_1(t_4), \hat{\rho}_2(t)]]]] + \cdots \Bigg), \end{aligned} \quad (11)$$

where Tr_A denotes the trace over atomic variables. The series is infinite in principle, but as is well known from the Lamb theory of the laser,[7] the fourth-order commutator already contains the essential nonlinearity required for a steady state, at least not too far above the laser threshold. We shall therefore terminate the series after the fourth term. We make one other simplification. If the natural linewidth is sufficiently small that we may take $|\omega - \omega_0| T_2 \ll 1$, then all the integrands in Eq. (11) are almost independent of the integration variables, and each integral yields a coefficient that is just some power of T_2. Finally, if the initial state $\hat{\rho}_2(t)$ given by Eq. (10) is reestablished at some rate R_2 by some pumping mechanism, then the average rate of change of the density operator of the optical field is taken to be given by $R_2 \delta\hat{\rho}_F(t)$, provided that the change $R_2 \delta\hat{\rho}_F(t) T_2$ brought about during the time T_2 is small.

The change of $\hat{\rho}_F(t)$ associated with the initially excited atoms is a manifestation of the laser gain mechanism. Losses can be introduced phenomenologically in a corresponding manner, as in the Scully-Lamb theory,[4] by supposing that there is also a group of N_1 atoms in the lower state $|1\rangle$ at time t, that serves to absorb some of the laser radiation. The corresponding density operator $\hat{\rho}_1(t)$ for the coupled system of these atoms and the field is given by

$$\hat{\rho}_1(t) \equiv \hat{\rho}_F(t) \prod_{j=1}^{N_1} |1_j\rangle\langle 1_j| , \tag{12}$$

and they produce a change in the density operator of the field in a time T_1, that we take to be of order of the lifetime $1/\gamma_1$ of the lower state, given by

$$\delta\hat{\rho}_F(t) = \mathrm{Tr}_A\left(\frac{1}{i\hbar}\int_t^{t+T_1} dt_1\,[\hat{H}_1(t_1), \hat{\rho}_1(t)] + \frac{1}{(i\hbar)^2}\int_t^{t+T_1} dt_1 \int_t^{t_1} dt_2\,[\hat{H}_1(t_1), [\hat{H}_1(t_2), \hat{\rho}_1(t)]]\right) . \tag{13}$$

We again take the integrands to be almost independent of t_1 and t_2, and we deliberately terminate the perturbation expansion after the second term, which is proportional to the intensity of the radiation, because losses associated with the optical cavity are generally proportional to the light intensity. If we multiply $\delta\hat{\rho}_F(t)$ given by Eq. (13) by the rate R_1 at which the initial state $\hat{\rho}_1(t)$ is reestablished, we obtain the average rate of change of the density operator $\hat{\rho}_F(t)$ owing to the loss mechanism. Finally we combine the rates of change associated with gains and losses to obtain the master equation

$$\begin{aligned}\frac{d\hat{\rho}_F(t)}{dt} = \mathrm{Tr}_A\Bigg(&\frac{R_2}{i\hbar}\int_t^{t+T_2} dt_1[\hat{H}_1(t_1), \hat{\rho}_2(t)] + \cdots \\ &+ \frac{R_2}{(i\hbar)^4}\int_t^{t+T_2} dt_1 \int_t^{t_1} dt_2 \int_t^{t_2} dt_3 \int_t^{t_3} dt_4\,[\hat{H}_1(t_1), [\hat{H}_1(t_2), [\hat{H}_1(t_3), [\hat{H}_1(t_4), \hat{\rho}_2(t)]]]] \\ &+ \frac{R_1}{i\hbar}\int_t^{t+T_1} dt_1[\hat{H}_1(t_1), \hat{\rho}_1(t)] + \frac{R_1}{(i\hbar)^2}\int_t^{t+T_1} dt_1 \int_t^{t_1} dt_2\,[\hat{H}_1(t_1), [\hat{H}_1(t_2)\hat{\rho}_1(t)]]\Bigg) .\end{aligned} \tag{14}$$

The first- and second-order commutators are readily evaluated with the help of Eqs. (8), (10), and (12). Since the operators associated with different atoms, and those associated with atoms and fields at the same time, commute, we obtain

$$[\hat{H}_1(t_1), \hat{\rho}_2(t)] = \hbar f \sum_{i=1}^{N_2} [-\hat{\rho}_F(t)\hat{a}\hat{b}^{(i)\dagger}u(\vec{r}_i) + \hat{a}^\dagger\hat{\rho}_F(t)\hat{b}^{(i)}u^*(\vec{r}_i)] \prod_{j=1,\, j\neq i}^{N_2} \hat{b}^{(j)\dagger}\hat{b}^{(j)} , \tag{15}$$

$$[\hat{H}_1(t_1), \hat{\rho}_1(t)] = \hbar f \sum_{i=1}^{N_1} [\hat{a}\hat{\rho}_F(t)\hat{b}^{(i)\dagger}u(\vec{r}_i) - \hat{\rho}_F(t)\hat{a}^\dagger\hat{b}^{(i)}u^*(\vec{r}_i)] \prod_{j=1,\, j\neq i}^{N_1} \hat{b}^{(j)}\hat{b}^{(j)\dagger} \tag{16}$$

when the time-dependent exponential factors are neglected, and

$$\begin{aligned}[\hat{H}_1(t_1), [\hat{H}_1(t_2), \hat{\rho}_2(t)]] = (\hbar f)^2 \sum_{\substack{k=1\\k\neq i}}^{N_2} \sum_{i=1}^{N_2} u(\vec{r}_i)u(\vec{r}_k)\Bigg(&-\hat{a}\hat{\rho}_F(t)\hat{a}\hat{b}^{(k)\dagger}\hat{b}^{(i)\dagger} \prod_{j=1,\, j\neq i}^{N_2} [\hat{b}^{(j)\dagger}\hat{b}^{(j)}] \\ &+ \hat{\rho}_F(t)\hat{a}^2\hat{b}^{(i)\dagger} \prod_{j=1,\, j\neq i}^{N_2} [\hat{b}^{(j)\dagger}\hat{b}^{(j)}]\hat{b}^{(k)\dagger}\Bigg) \\ + (\hbar f)^2 \sum_{k=1}^{N_2} \sum_{i=1}^{N_2} u^*(\vec{r}_k)u(\vec{r}_i)\Bigg(&-\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{b}^{(k)}\hat{b}^{(i)\dagger} \prod_{j=1,\, j\neq i}^{N_2} [\hat{b}^{(j)\dagger}\hat{b}^{(j)}] \\ &+ \hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{b}^{(i)} \prod_{j=1,\, j\neq i}^{N_2} [\hat{b}^{(j)\dagger}\hat{b}^{(j)}]\hat{b}^{(k)}\Bigg) + \mathrm{H.c.} ,\end{aligned} \tag{17}$$

$$[\hat{H}_1(t_1), [\hat{H}_1(t_2), \hat{\rho}_1(t)]] = (\hbar f)^2 \sum_{\substack{k=1\\k\neq i}}^{N_1} \sum_{i=1}^{N_1} u(\vec{r}_k) u(\vec{r}_i) \bigg(-\hat{a}\hat{\rho}_F(t)\hat{a}\hat{b}^{(i)\dagger} \prod_{j=1,\ j\neq i}^{N_1} [\hat{b}^{(j)}\hat{b}^{(j)\dagger}]\hat{b}^{(k)\dagger} + \hat{a}^2\hat{\rho}_F(t)\hat{b}^{(k)\dagger}\hat{b}^{(i)\dagger} \prod_{j=1,\ j\neq i}^{N_1} [\hat{b}^{(j)}\hat{b}^{(j)\dagger}] \bigg)$$
$$+ (\hbar f)^2 \sum_{k=1}^{N_1} \sum_{i=1}^{N_1} u^*(\vec{r}_k) u(\vec{r}_i) \bigg(\hat{a}^\dagger \hat{a}\hat{\rho}_F(t)\hat{b}^{(k)}\hat{b}^{(i)\dagger} \prod_{j=1,\ j\neq i}^{N_1} [\hat{b}^{(j)}\hat{b}^{(j)\dagger}] - \hat{a}\hat{\rho}_F(t)\hat{a}^\dagger \hat{b}^{(i)\dagger} \prod_{j=1,\ j\neq i}^{N_1} [\hat{b}^{(j)}\hat{b}^{(j)\dagger}]\hat{b}^{(k)} \bigg) + \text{H.c.} \tag{18}$$

These expressions simplify substantially when we trace over atomic variables, and they lead to

$$\mathrm{Tr}_A[\hat{H}_1(t_1), \hat{\rho}_2(t)] = 0 = \mathrm{Tr}_A[\hat{H}_1(t_1), \hat{\rho}_1(t)], \tag{19}$$

$$\mathrm{Tr}_A[\hat{H}_1(t_1), [\hat{H}_1(t_2), \hat{\rho}_2(t)]] = (\hbar f)^2 \sum_{i=1}^{N_2} |u(\vec{r}_i)|^2 [\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{\rho}_F(t) - 2\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}], \tag{20}$$

$$\mathrm{Tr}_A[\hat{H}_1(t_1), [\hat{H}_1(t_2), \hat{\rho}_1(t)]] = (\hbar f)^2 \sum_{i=1}^{N_1} |u(\vec{r}_i)|^2 [\hat{a}^\dagger\hat{a}\hat{\rho}_F(t) + \hat{\rho}_F(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_F(t)\hat{a}^\dagger]. \tag{21}$$

The higher-order commutators can be evaluated in a similar manner, except that the expressions get progressively longer. After some rather lengthy but straightforward calculations we find that the third-order commutator vanishes when traced over atomic variables, and that the fourth-order commutator yields

$$\mathrm{Tr}_A[\hat{H}_1(t_1), [\hat{H}_1(t_2), [\hat{H}_1(t_3), [\hat{H}_1(t_4), \hat{\rho}_2(t)]]]]$$
$$= (\hbar f)^4 \sum_{\substack{i=1\\i\neq k}}^{N_2} \sum_{k=1}^{N_2} |u(\vec{r}_i)|^2 |u(\vec{r}_k)|^2 [-8\hat{a}\hat{a}^{\dagger 2}\hat{\rho}_F(t)\hat{a} - 8\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}^2\hat{a}^\dagger - 4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a} - 4\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a} + 6\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger$$
$$+ \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t) + \hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + 12\hat{a}^{\dagger 2}\hat{\rho}_F(t)\hat{a}^2 + 2\hat{\rho}_F(t)\hat{a}^2\hat{a}^{\dagger 2} + 2\hat{a}^2\hat{a}^{\dagger 2}\hat{\rho}_F(t)]$$
$$+ (\hbar f)^4 \sum_{i=1}^{N_2} |u(\vec{r}_i)|^4 [-4\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a} - 4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a} + 6\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger + \hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + \hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)]. \tag{22}$$

When expressions (19) to (22) for the various commutators are substituted in Eq. (14) we arrive at the following master equation:

$$\frac{d\hat{\rho}_F(t)}{dt} = -\tfrac{1}{2}A[\hat{a}\hat{a}^\dagger\hat{\rho}_F(t) + \hat{\rho}_F(t)\hat{a}\hat{a}^\dagger - 2\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}] - \tfrac{1}{2}C[\hat{a}^\dagger\hat{a}\hat{\rho}_F(t) + \hat{\rho}_F(t)\hat{a}^\dagger\hat{a} - 2\hat{a}\hat{\rho}_F(t)\hat{a}^\dagger]$$
$$+ \tfrac{1}{8}B[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t) + \hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + 6\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger - 4\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a} - 4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}]$$
$$+ \tfrac{1}{8}D[\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t) + \hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a}\hat{a}^\dagger + 6\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger - 4\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}\hat{a}^\dagger\hat{a} - 4\hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}$$
$$+ 2\hat{a}^2\hat{a}^{\dagger 2}\hat{\rho}_F(t) + 2\hat{\rho}_F(t)\hat{a}^2\hat{a}^{\dagger 2} + 12\hat{a}^{\dagger 2}\hat{\rho}_F(t)\hat{a}^2 - 8\hat{a}^\dagger\hat{\rho}_F(t)\hat{a}^2\hat{a}^\dagger - 8\hat{a}\hat{a}^{\dagger 2}\hat{\rho}_F(t)\hat{a}], \tag{23}$$

in which we have introduced the following abbreviations:

$$A \equiv (fT_2)^2 R_2 \sum_{i=1}^{N_2} |u(\vec{r}_i)|^2, \tag{24}$$

$$C \equiv (fT_1)^2 R_1 \sum_{i=1}^{N_1} |u(\vec{r}_i)|^2, \tag{25}$$

$$B \equiv \tfrac{1}{3}(fT_2)^4 R_2 \sum_{i=1}^{N_2} |u(\vec{r}_i)|^4, \tag{26}$$

$$D \equiv \tfrac{1}{3}(fT_2)^4 R_2 \sum_{\substack{i=1\\ i\neq k}}^{N_2} \sum_{k=1}^{N_2} |u(\vec{r}_i)|^2 |u(\vec{r}_k)|^2 . \qquad (27)$$

As in the Scully-Lamb theory,[4] A and C play the roles of gain and loss coefficients, and C is ultimately related to the cavity loss parameter $\mathfrak{L}$ representing the fractional loss per cavity transit by putting

$$C = c\mathfrak{L}/2\pi l , \qquad (28)$$

where l is the cavity length. B is a small saturation parameter that determines the light intensity at the laser threshold. D is a new parameter that does not appear in the Scully-Lamb theory, which is evidently associated with cooperative atomic interactions since it vanishes if we allow the atoms to interact with the field one at a time. Whereas A and B are proportional to the number of atoms involved in the laser process, D is proportional to the square of the number of atoms, as expected for cooperative atomic interactions.

IV. PHOTON STATISTICS

If we calculate the matrix element of each term in Eq. (23) between Fock states $\langle n|$ and $|n\rangle$, and make use of the fact that $\langle n|\hat{\rho}_F(t)|n\rangle = p(n,t)$, the probability for n laser photons in the optical cavity,[4] we obtain an equation of motion for $p(n,t)$ in the form

$$\frac{\partial p(n)}{\partial t} = [-A(n+1)+B(n+1)^2]p(n) + [An - Bn^2]p(n-1) - Cnp(n) + C(n+1)p(n+1)$$
$$+ \tfrac{1}{2}D\{[2(n+1)^2 + (n+1)(n+2)]p(n) - [2n^2 + 4n(n+1)]p(n-1) + 3n(n-1)p(n-2)\} . \qquad (29)$$

It will be seen that the term in D, representing cooperative effects, contains contributions proportional to $p(n)$ and $p(n-1)$ that augment the B terms representing successive photon emissions and reabsorptions by a single atom. This is a reflection of the fact that there are more ways in which emission and absorption processes can take place among a group of cooperating atoms. On the other hand, the term proportional to $p(n-2)$ represents a two-photon emission process that cannot occur at all for singly excited atoms. It makes the equation of motion significantly more complicated than that of Scully and Lamb[4] in that it couples $p(n)$ not only to $p(n-1)$ and $p(n+1)$ but also to $p(n-2)$. Fortunately, we can introduce a simplification by writing

$$[p(n) - p(n-1)] - [p(n-1) - p(n-2)] \equiv p'' , \qquad (30)$$

where p'' is a close approximation to the second derivative of $p(n)$. If we take the Scully-Lamb[4] solution for $p(n)$ in the steady state, which can be cast into the approximate form

$$p_{\mathrm{SL}}(n) = (\mathrm{const}) \exp[-\tfrac{1}{2}(n/n_0 - a/\sqrt{2})^2] , \qquad (31)$$

as a zeroth-order approximation to $p(n)$, we can make an estimate of p''. Here n_0 is of the order of the mean number of photons present at the laser threshold, and a is a dimensionless laser pump parameter that is negative below threshold and positive above threshold. From Eq. (31) we find

$$p'' \approx [p(n)/n_0^2][(n/n_0 - a/\sqrt{2})^2 - 1] , \qquad (32)$$

so that, in the neighborhood of threshold, and for those values of n for which $p(n)$ is not negligible, p'' is of order $p(n)/n_0^2$. If we substitute for $p(n-2)$ in terms of p'' from Eq. (30) into Eq. (29), we find that the term in D can be rewritten

$$D[(5n+2)p(n) - 5np(n-1) + \tfrac{3}{2}n(n-1)p''] .$$

Since n and n_0 are typically of order several thousand, and p'' is of order $p(n)/n_0^2$, we see that the term $\tfrac{3}{2}n(n-1)p''$ is expected to be very small compared with the dominant terms $5np(n)$ and $5np(n-1)$, and may be neglected. With the help of the further approximation of replacing $(5n+2)p(n)$ by $(5n+5)p(n)$, Eq. (29) simplifies and can be written

$$\frac{\partial p(n)}{\partial t} = [-(A-5D)(n+1) + B(n+1)^2]p(n) + [(A-5D)n - Bn^2]p(n-1) - Cnp(n) + C(n+1)p(n+1)$$
$$\approx \frac{-(n+1)(A-5D)}{1+(n+1)B/(A-5D)}p(n) + \frac{n(A-5D)}{1+nB/(A-5D)}p(n-1) - nCp(n) + (n+1)Cp(n+1) , \qquad (33)$$

to the first order in $nB/(A-5D)$.

This equation has precisely the same structure as the Scully-Lamb equation of motion[4] for $p(n)$, except that the coefficients are different. The steady-state solution may therefore be written at once, in complete analogy with their calculation, and we obtain

$$p(n)=(\text{const})\prod_{r=0}^{n}\left(\frac{(A-5D)/C}{1+Br/(A-5D)}\right). \qquad (34)$$

The term in D, representing cooperative effects, now appears in association with the gain coefficient A rather than with the saturation coefficient B as in Eq. (29). The net effect of the cooperation is to reduce the laser gain by making reabsorption somewhat more probable, although, as we show below, the ratio of D/A is expected to be small in practice. We might point out that Dicke superradiance,[8] which is probably the most familiar form of cooperative atomic interaction, but is not adequately described by our single-mode treatment, would also have the effect of reducing the laser gain by depleting the excited population.

With the help of Stirling's expression for the factorial in the denominator in Eq. (34), and with the introduction of threshold and pump parameters n_0 and a defined by[1]

$$n_0^2\equiv\frac{A-5D}{B}=\left(\frac{3}{(fT_2)^2}\sum_{i=1}^{N_2}|u(\vec{r}_i)|^2-5\sum_{\substack{i=1\\ i\neq j}}^{N_2}\sum_{j=1}^{N_2}|u(\vec{r}_i)|^2|u(\vec{r}_j)|^2\right)\Big/\sum_{i=1}^{N_2}|u(\vec{r}_i)|^4 \qquad (35)$$

and

$$\exp\left(\frac{a}{\sqrt{2}\,n_0}\right)\equiv\frac{A-5D}{C}=\left(R_2(fT_2)^2\sum_{i=1}^{N_2}|u(\vec{r}_i)|^2-\frac{5R_2(fT_2)^4}{3}\sum_{\substack{i=1\\ i\neq j}}^{N_2}\sum_{j=1}^{N_2}|u(\vec{r}_i)|^2|u(\vec{r}_j)|^2\right)\Big/\frac{c\mathfrak{L}}{2\pi l}, \qquad (36)$$

Eq. (34) takes the familiar form

$$p(n)=(\text{const})\exp[-\tfrac{1}{2}(n/n_0-a/\sqrt{2})^2]. \qquad (37)$$

The same result is also obtained from semiclassical theories[9] dealing with the light intensity, if we identify intensity with the photon flux per unit area averaged over the cross section of the laser beam. However, the manner in which $p(n)$ depends on cavity length l through the parameters n_0 and a defined by Eqs. (35) and (36), will in general be different when compared with the familiar laser theories. The number n_0 is $(\frac{1}{2}\pi)^{1/2}$ times the average photon number at the laser threshold, and the pump parameter a is zero at threshold where the gain just equals the loss, and is generally a small positive or negative number in the neighborhood of threshold.

V. LENGTH DEPENDENCE OF THE LIGHT INTENSITY AND ITS FLUCTUATIONS

The light intensity $\hat{I}(\vec{r},t)$ may also be defined[10] as $\hat{\vec{E}}^{(-)}(\vec{r},t)\cdot\hat{\vec{E}}^{(+)}(\vec{r},t)$, where $\hat{\vec{E}}^{(-)}(\vec{r},t)$ and $\hat{\vec{E}}^{(+)}(\vec{r},t)$ are the positive and negative frequency parts of the real electric field $\hat{\vec{E}}(\vec{r},t)$. Hence from Eq. (5) we have, for a single-mode laser field at an internal point $\vec{r}$,

$$\hat{I}(\vec{r},t)=(\hbar\omega/2\epsilon_0)|u(\vec{r})|^2\hat{a}^\dagger(t)\hat{a}(t). \qquad (38)$$

With the help of Eq. (37) we then find that (cf. Ref. 9) in the steady state

$$\langle\hat{I}(\vec{r},t)\rangle=\frac{\hbar\omega}{2\epsilon_0}|u(\vec{r})|^2\sum_{n=0}^{\infty}np(n)$$
$$=\frac{n_0\hbar\omega}{2\sqrt{2}\,\epsilon_0}|u(\vec{r})|^2\left(a+\frac{2\exp(-\frac{1}{4}a^2)}{\sqrt{\pi}[1+\mathrm{erf}(\frac{1}{2}a)]}\right), \qquad (39)$$

where n_0 is given by Eq. (35), and again from Eq. (37)

$$\frac{\langle(\Delta\hat{I}(\vec{r},t)^2\rangle}{\langle\hat{I}(\vec{r},t)\rangle^2}=\frac{\langle(\Delta n)^2\rangle}{\langle n\rangle^2}$$
$$=2\left(1-\frac{a\exp(-\frac{1}{4}a^2)}{\sqrt{\pi}[1+\mathrm{erf}(\frac{1}{2}a)]}-\frac{2\exp(-\frac{1}{2}a^2)}{\pi[1+\mathrm{erf}(\frac{1}{2}a)]^2}\right)\Big/\left(a+\frac{2\exp(-\frac{1}{4}a^2)}{\sqrt{\pi}[1+\mathrm{erf}(\frac{1}{2}a)]}\right)^2. \qquad (40)$$

The dependence of the light intensity and its fluctuations on cavity length l are therefore completely determined by the manner in which n_0 and a given by Eqs. (35) and (36) vary with l. We therefore start by examining these parameters.

The summations in Eqs. (35) and (36) can be well approximated by integrals. Thus if the active medium is in the form of an axial cylinder of radius r_0 lying between the planes $z=z_1$ and $z=z_2$ as in Fig. 1, if the laser atoms are uniformly distributed with density η, and if p_2 is the probability that an atom is in the upper or excited state at any instant, then

$$\sum_{i=1}^{N_2}|u(\vec{r}_i)|^2\approx p_2\eta\int_{z_1}^{z_2}dz\int_0^{r_0}2\pi r\,dr|u(r,z)|^2, \tag{41}$$

$$\sum_{\substack{i=1\\i\neq j}}^{N_2}\sum_{j=1}^{N_2}|u(\vec{r}_i)|^2|u(\vec{r}_j)|^2 \approx\left(p_2\eta\int_{z_1}^{z_2}dz\int_0^{r_0}2\pi r\,dr|u(r,z)|^2\right)^2, \tag{42}$$

$$\sum_{i=1}^{N_2}|u(\vec{r}_i)|^4\approx p_2\eta\int_{z_1}^{z_2}dz\int_0^{r_0}2\pi r\,dr\,|u(r,z)|^4. \tag{43}$$

When the radius r_0 of the active medium is appreciably larger than the effective beam radius $w(z)$ [given by Eq. (2b)] of the Gaussian mode throughout the medium, as is frequently the case in practice, the integrals are easily evaluated. We may then replace the upper limit of the r integral by ∞ to a good approximation, and we find with the help of Eqs. (1) and (2b)

$$\sum_{i=1}^{N_2}|u(\vec{r}_i)|^2\approx p_2\eta\int_{z_1}^{z_2}dz\int_0^{\infty}2\pi r\,dr|u(r,z)|^2 = p_2\eta[(z_2-z_1)/l], \tag{44}$$

exactly as for a plane-wave laser mode,[10] and

$$\sum_{i=1}^{N_2}|u(\vec{r}_i)|^4 = p_2\eta\int_{z_1}^{z_2}dz\int_0^{\infty}2\pi r\,dr|u(r,z)|^4 = \frac{p_2\eta k}{2\pi l^2}\left(\tan^{-1}\frac{z_2}{[l(R-l)]^{1/2}} - \tan^{-1}\frac{z_1}{[l(R-l)]^{1/2}}\right). \tag{45}$$

The last expression may be further simplified if $z_1, z_2 \ll [l(R-l)]^{1/2}$, as is sometimes the case in practice, in which case the tangents are approximately equal to the angles, and we have

$$\sum_{i=1}^{N_2}|u(\vec{r}_i)|^4\approx\frac{p_2\eta k(z_2-z_1)}{2\pi l^2[l(R-l)]^{1/2}}. \tag{46}$$

However, even in this limiting case, the dependence on cavity length l is more complicated than for a plane-wave laser mode,[11] for which the same sum is proportional to $1/l^2$. From Eqs. (35), (36), (44), and (45) we then obtain

$$n_0^2\approx\frac{[3(z_2-z_1)/l(fT_2)^2][1-\frac{5}{3}p_2\eta(fT_2)^2(z_2-z_1)/l]}{(k/2\pi l^2)\{\tan^{-1}[z_2/l^{1/2}(R-l)^{1/2}]-\tan^{-1}[z_1/l^{1/2}(R-l)^{1/2}]\}} \tag{47}$$

and

$$\exp\left(\frac{a}{\sqrt{2}n_0}\right)\approx\frac{[R_2(fT_2)^2p_2\eta(z_2-z_1)/l][1-\frac{5}{3}p_2\eta(fT_2)^2(z_2-z_1)/l]}{c\mathcal{L}/2\pi l}. \tag{48}$$

If we neglect cooperative effects, represented by the term

$$5D/A=\tfrac{5}{3}p_2\eta(fT_2)^2(z_2-z_1)/l,$$

for the moment, then the ratio a/n_0 given by Eq. (48) is independent of cavity length l, exactly as was found for a plane-wave laser.[1,11] However, n_0, which is proportional to $\sqrt{l}$ for a plane-wave laser, has a much more complicated l dependence when the mode is Gaussian, and the same applies to the pump parameter a. The terms representing cooperative effects, when they are significant, further complicate the l dependence of both n_0 and a, and also make their ratio dependent on cavity length l to some extent.

We can obtain a rough estimate of the conditions under which cooperative effects play a role, by noting that this requires

$$p_2\eta(fT_2)^2(z_2-z_1)/l \not\ll 1.$$

If we make use of the definition (9) for f, and recall that $3\pi\epsilon_0\hbar c^3/\mu^2\omega^3$ is the natural lifetime T

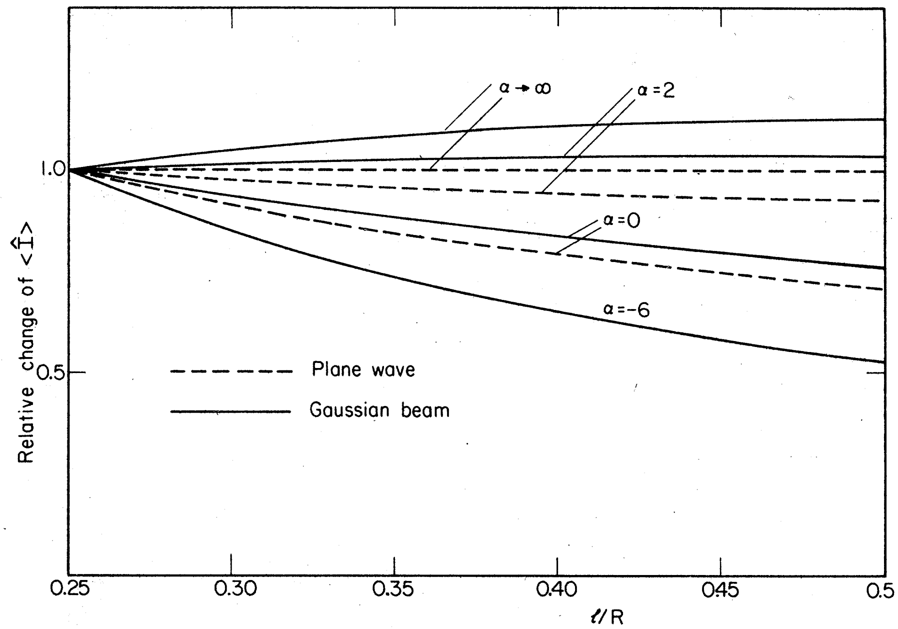

FIG. 2. Relative change of the light intensity $\langle \hat{I} \rangle$ with cavity length l for various initial values of the pump parameter a, for a laser with $z_2/R = 0.175$ and $z_1/R = 0.083$. The full curves apply to a Gaussian mode and the broken curves to a plane-wave mode.

of the upper- to lower-state transition, we obtain

$$5D/A \approx \tfrac{5}{3} p_2 \eta (f T_2)^2 (z_2 - z_1)/l$$
$$= (5/8\pi) p_2 \eta (T_2/T) \lambda^2 (z_2 - z_1)(cT_2/l), \quad (49)$$

which depends on T_2, T, l, and on the number of excited laser atoms within a cylinder of length $z_2 - z_1$ and radius λ. With some typical parameters for a small inhomogeneously broadened He:Ne laser,[12,13] $\lambda \approx 6328$ Å, $T_2 \approx 10$ nsec, $T \approx 700$ nsec, $l \approx 30$ cm, $z_2 - z_1 \approx 5$ cm, $\eta \approx 2 \times 10^{16}$ atoms/cm^3, $p_2 \approx 10^{-9}$, we find

$$5D/A = \tfrac{5}{3} (f T_2)^2 p_2 \eta (z_2 - z_1)/l \sim 10^{-2},$$

so that cooperative effects would be very small. However, they may become larger under other circumstances or in other lasers.

In order to illustrate the behavior and to allow a comparison with the results presented in Ref. 1, we show in Figs. 2 and 3 how the light intensity $\langle \hat{I}(\vec{r}) \rangle$ given by Eq. (39) and the relative intensity fluctuations given by Eq. (40) change with cavity

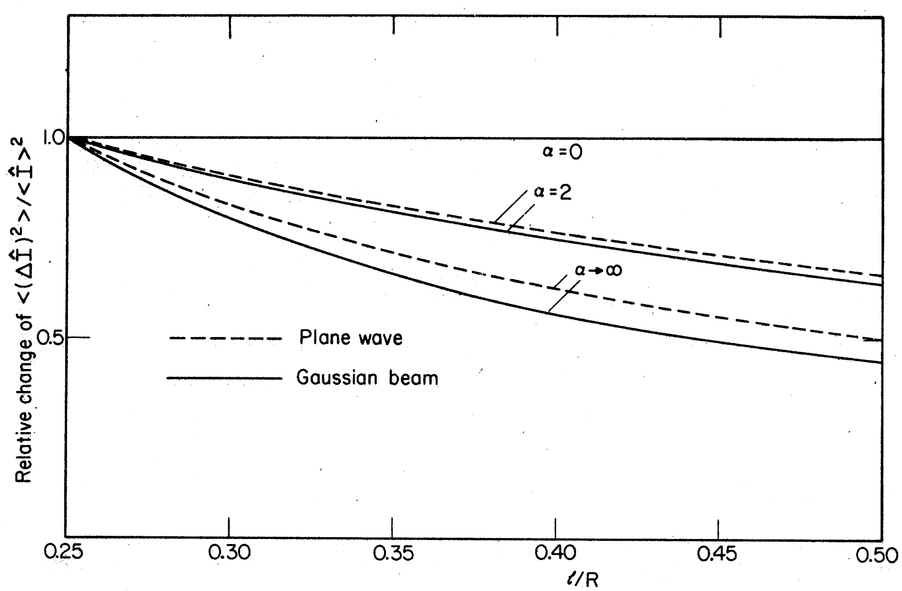

FIG. 3. Relative change of the relative intensity fluctuations $\langle (\Delta \hat{I})^2 \rangle / \langle \hat{I} \rangle^2$ with cavity length l for various initial values of the pump parameter a, for a laser with $z_2/R = 0.175$ and $z_1/R = 0.083$. The full curves apply to a Gaussian mode and the broken curves to a plane-wave mode.

length l for a particular laser. We assume that the initial working point of the laser is characterized by some pump parameter a that changes with l as the length is increased. Cooperative effects have been neglected. For comparison, the broken curves show the corresponding solution for a plane-wave laser. It will be seen that whereas in the latter case the light intensity always falls with increasing cavity length l, it rises slightly for the Gaussian-mode laser, once its working point is at least modestly above threshold. As regards the relative intensity fluctuations, these fall with increasing cavity length above threshold in both cases, but somewhat more rapidly for the Gaussian-mode laser. In both cases the light becomes increasingly coherent as the length increases, because stimulated emission, which depends on cavity length through the photon-occupation number of the laser mode, dominates increasingly over spontaneous emission.

Although these conclusions should lend themselves to experimental test, the curves in Figs. 2 and 3 are probably not easy to confirm directly, because of the difficulty of maintaining the laser-oscillation frequency and the pump-parameter constant as the cavity length is changed. If the active medium is inhomogeneously broadened, changes of frequency are reflected in changes in the density $p_2\eta$ of excited atoms taking part in laser action. However, we note from Eq. (40) that the relative intensity fluctuation $\langle(\Delta\hat{I})^2\rangle/\langle\hat{I}\rangle^2$ is completely determined by the pump parameter a, so that a measurement of this ratio can be used to determine a.

Although the light intensity inside the laser cavity is not easily measured, the intensity just outside the cavity is proportional to the intensity just inside, and this can be determined from the counting rate of a photodetector exposed to the laser beam. Moreover, as is now well known,[10] the relative intensity fluctuation is proportional to the normalized second factorial moment minus one of the number of photoelectric counts registered by the detector in a short time. $\langle(\Delta\hat{I})^2\rangle/\langle\hat{I}\rangle^2$ and the pump parameter a are therefore easy to determine experimentally. Once a has been found, we observe that from Eqs. (39) and (47),

$$\frac{\langle\hat{I}(\vec{r})\rangle}{a+[2\exp(-\frac{1}{4}a^2)]/\sqrt{\pi}[1+\mathrm{erf}(\frac{1}{2}a)]}=\frac{\hbar\omega|u(\vec{r})|^2}{2\sqrt{2}\,\epsilon_0}\left(\frac{[6\pi l(z_2-z_1)/k(fT_2)^2][1-\frac{5}{3}p_2\eta(fT_2)^2(z_2-z_1)/l]}{\tan^{-1}z_2/[l(R-l)]^{1/2}-\tan^{-1}z_1/[l(R-l)]^{1/2}}\right)^{1/2}, \tag{50}$$

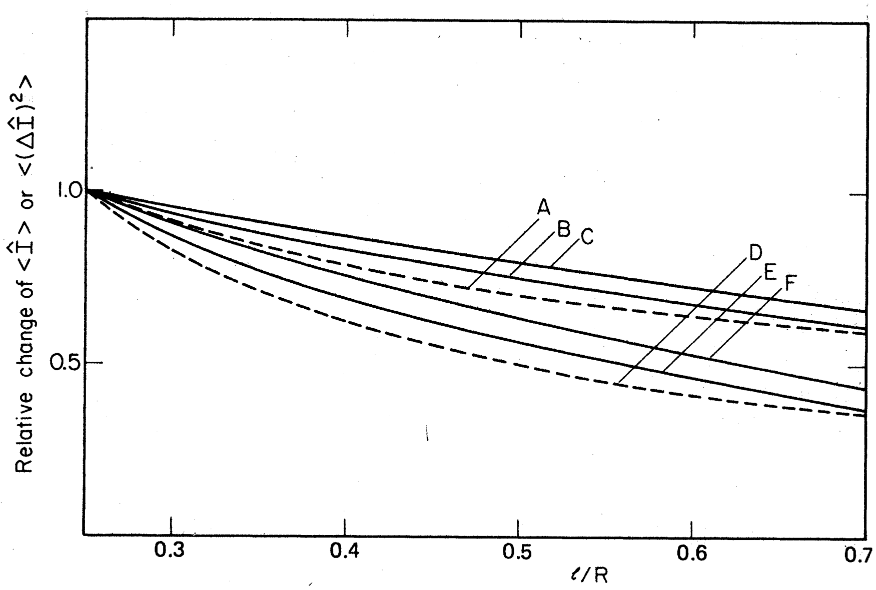

FIG. 4. Relative change of both $\langle\hat{I}\rangle$ and $\langle(\Delta\hat{I})^2\rangle$ with cavity length l at constant pump parameter, for a laser with $z_2/R=0.175$ and $z_1/R=0.083$. Curve A: $\langle\hat{I}\rangle$ for a plane-wave mode; curve B: $\langle\hat{I}\rangle$ for a Gaussian mode; curve C: $\langle\hat{I}\rangle$ for a Gaussian mode with cooperative effects $(p_2\eta(fT_2)^2(z_2-z_1)/R=0.03)$; curve D: $\langle(\Delta\hat{I})^2\rangle$ for a plane-wave mode; curve E: $\langle(\Delta\hat{I})^2\rangle$ for a Gaussian mode; curve F: $\langle(\Delta\hat{I})^2\rangle$ for a Gaussian mode with cooperative effects $(p_2\eta(fT_2)^2(z_2-z_1)/R=0.03)$.

and we may look on this equation as furnishing a convenient test of the length dependence of the laser field intensity. In a conventional plane-wave-laser theory the right-hand side would simply be proportional to $1/\sqrt{l}$. Moreover, at constant pump parameter, $\langle(\Delta\hat{I})^2\rangle$ is proportional to the square of $\langle\hat{I}\rangle$.

Figure 4 shows a plot of the variation of both $\langle\hat{I}\rangle$ and $\langle(\Delta\hat{I})^2\rangle$ at constant a with the ratio l/R over a certain range, for a laser for which $z_2/R = 0.175$ and $z_1/R = 0.083$, both without and with the inclusion of cooperative effects. In the latter case we have taken the factor $p_2\eta(fT_2)^2(z_2 - z_1)/R$ to be 0.03, which is about 10 times larger than it would normally be in a small He:Ne laser, but it might reach this value in other higher-gain lasers. For comparison we also show the corresponding behavior of a plane-wave laser. The plane-wave curves A and D and the Gaussian-wave curves B and E actually cross over when $l/R \approx 0.75$. The differences between the various curves, while they are not very large, should be clearly distinguishable by experiment. The departure of the mode from a plane wave should therefore be manifest in the photoelectric counting statistics.

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