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Fuda's off-shell Jost function for Coulomb, Hulthén, and Eckart potentials and limiting relations

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We study the off-shell Jost function $f(k,q)$, introduced by Fuda, for the Coulomb, the Hulthén, and two modified Eckart potentials. A simple closed expression for the $l = 0$ Coulomb off-shell Jost function has been obtained. This function is discontinuous at $q = k$. Its on-shell limiting behavior is given by the singular factor $(q-k)^{-i\gamma}$, where γ is the Sommerfeld parameter. We also discuss the off-shell Jost solution $f(k,q,r)$, which is an off-shell generalization of the Jost solution $f(k,r)$. We consider the Hulthén potential as a screened Coulomb potential, let the screening parameter a go to infinity, and derive the limiting behavior of the Jost solution, the Jost function, the off-shell Jost function, and the half-shell T matrix for the Hulthén potential as $a \to \infty$. We obtain discontinuities given by the singular factor $a^{i\gamma}$. For comparison, we introduce two modifications of an Eckart potential which can be considered to be a screened r^{-2} potential and derive a number of limiting relations in analogy to those for the Hulthén-Coulomb pair of potentials.

I. INTRODUCTION

The concepts of Jost[1] function and Jost solution are very well known in the theory of nonrelativistic two-body scattering by a spherically symmetric potential.[2-4] Their usefulness in the study of the Schrödinger equation may be recognized from some of their properties: The phase of the Jost function is the negative of the phase shift for the physical wave function. For a local potential, the Jost function is identical to the Fredholm determinant and its zeros determine the bound-state energies of the two-body system.

For many-particle systems one needs off-shell quantities, in particular the off-shell T matrix. The on-shell restriction of the T matrix is proportional to the two-particle scattering amplitude. Fuda and Whiting[5] have introduced and studied a generalization of the Jost function which they call the off-shell Jost function $f_l(k,q)$. It is a function of the wave number k and an off-shell momentum q. These authors have discussed its usefulness in off-shell scattering. In particular, they have proved a simple relation connecting the half-shell T matrix $T_l(k,q;k^2)$ and the off-shell Jost function, see Eq. (16). Further, they have discussed an off-shell extension of the Jost solution. This is a solution $f_l(k,q,r)$ with prescribed asymptotic behavior e^{iqr}, $r\to\infty$, of an inhomogeneous "Schrödinger equation." We shall call this function $f_l(k,q,r)$ the off-shell Jost solution.

Recently Fuda[6] has developed a momentum-space formulation of the off-shell Jost function and derived two integral representations for it. Finally we note that very recently Pasquier and Pasquier[7,8] have studied an off-shell generalization of the Jost formalism in a more general context.

For some particular potentials the Jost function and the Jost solution are known in closed form, see Newton,[2] Chap. 14. In the case of s waves ($l=0$) the off-shell T matrix $T(p,q;k^2)$ has been obtained for a small number of potentials only. Most of these explicit expressions are in terms of generalized hypergeometric functions[9,10] ${}_mF_n$.

Fuda[11] has derived $T_{l=0}$ for the exponential potential in terms of ${}_1F_2$ and ${}_2F_3$, while Bahethi and Fuda[12] have given an expression for the case of the Hulthén potential in terms of ${}_3F_2$ and ${}_4F_3$. Finally Fuda and Whiting[5] have simplified these expressions. For the Coulomb potential, van Haeringen and van Wageningen[13] have derived the $l=0$ T matrix in terms of ${}_2F_1$.

In the case $l>0$, no closed expression for T_l is known for the exponential and Hulthén potentials. Van Haeringen[14] has obtained the $l=1$ Coulomb T matrix, also in terms of the Gaussian hypergeometric function ${}_2F_1$. This expression is much more complicated than the Coulomb $T_{l=0}$ expression.

The s-wave off-shell Jost function and solution for the exponential potential and for the Hulthén potential have been given by Fuda and Whiting.[5] Fuda and Girard[15] have derived integral representations for the s wave, off-shell Jost function, and half-off-shell T matrix for a superposition of Yukawa potentials. Some work has also been done on other potentials, cf. the references quoted by Fuda and Girard.[15]

In this paper we extend these investigations on off-shell Jost functions and solutions, in particular to the Coulomb case. Here we meet special difficulties, which are due to the long range of the

Coulomb potential.

In Sec. II we study the off-shell Jost solution and function for the Coulomb potential for arbitrary values of l. For $l=0$ we obtain a very simple closed expression for the Coulomb off-shell Jost function $f_C(k,q)$, see Eq. (24). In connection with the derivation of the $l=0$ off-shell Jost function for the Hulthén potential, we pay attention to a statement concerning the hypergeometric function ${}_3F_2$ which is of general interest. It is formulated as follows: any Saalschützian ${}_3F_2$ of argument 1, with one of its three first parameters equal to 1, can be summed in terms of Γ functions.

The Hulthén potential goes over into the Coulomb potential when the so-called screening parameter a goes to infinity. We investigate whether or not the limits for $a\to\infty$ of the Hulthén Jost functions and solutions are equal to the Coulomb Jost function and solution, respectively. We establish the type of singularity for those cases for which this limit is nonexistent. It is given by $a^{i\gamma}$, where γ is the Sommerfeld parameter.[16]

Further, we investigate the continuity of the off-shell Jost solutions and functions with respect to the off-shell variable q at $q=k$. According to Fuda and Whiting,[5] the off-shell Jost function and solution are continuous at $q=k$ with limit equal to the ordinary Jost function and solution, if the potential has a short range. We prove that the Coulomb off-shell Jost function is not continuous but singular at $q=k$, and we also show that the source of the singularity lies in the factor $(q-k)^{-i\gamma}$.

Finally, we take a third and a fourth limit into consideration. Besides $a\to\infty$ and $q\to k$ we also consider $r\to\infty$ and $r\to 0$. A survey of our main results for these limiting relations may be found in Eqs. (40)–(41), and in Fig. 1.

In the final part of Sec. III we prove an interesting limiting relation for the half-shell Hulthén T matrix, see Eq. (48). In Sec. IV we consider a potential from the Eckart[17] class and two modifications, and we derive some limiting relations. In Sec. V we give a summary and a short discussion.

We shall mainly use the notation of Newton[2] and of Fuda and Whiting.[5]

II. COULOMB AND HULTHÉN FUNCTIONS

The Jost[1] solution $f_l(k,r)$ is that solution of the radial Schrödinger equation,

$$\left(k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right)f_l(k,r)=0\,, \tag{1}$$

which satisfies the asymptotic condition

$$\lim_{r\to\infty} f_l(k,r)e^{-ikr}=1\,. \tag{2}$$

This function $f_l(k,r)$ is well defined if the potential $V(r)$ is not singular and satisfies

$$V(r)=O(r^{-\alpha})\,,\quad \alpha>1\,,\quad r\to\infty\,.$$

In the Coulomb case ($\alpha=1$), a different asymptotic behavior has to be prescribed, because there is no solution of Eq. (1) satisfying Eq. (2) in this case. Then one may define $f_{C,l}$ such that

$$\lim_{r\to\infty} f_{C,l}(k,r)e^{-ikr+i\gamma\ln(2kr)}=1\,, \tag{3}$$

where γ is Sommerfeld's parameter. The factor $(2k)^{i\gamma}$ in Eq. (3) is usually included for convenience. The Coulomb potential is given by

$$V_C(r)=2k\gamma/r\,.$$

The Coulomb Jost solution can be given in several equivalent closed forms, e.g.,

$$f_{C,l}(k,r)=e^{ikr+\pi\gamma/2}(-2ikr)^{l+1}\times U(l+1+i\gamma,2l+2,-2ikr) \tag{4}$$

$$=e^{ikr-i\gamma\ln(2kr)}\,{}_2F_0(l+1+i\gamma,i\gamma-l;(2ikr)^{-1})\,. \tag{5}$$

Here U is an irregular solution of the confluent hypergeometric differential equation,[18] and ${}_2F_0$ is a generalized hypergeometric function. Note that Eq. (3) is easily read off from Eq. (5) since

$$\lim_{z\to 0}{}_2F_0(a,b;z)=1\,.$$

The Jost *function* is defined by[19] [Newton,[2] Eqs. (12.140) and (12.142)]

$$f_l(k)\equiv\lim_{r\to 0} f_l(k,r)(-2ikr)^l l!/(2l)!\,. \tag{6}$$

This definition is also valid in the Coulomb case. By using (Ref. 18, p.288)

$$\lim_{z\to 0} z^{c-1}U(a,c,z)=\Gamma(c-1)/\Gamma(a)\,,\quad \mathrm{Re}\,c>1\,,$$

one obtains from Eq. (4) the well-known expression

$$f_{C,l}(k)=e^{\pi\gamma/2}\Gamma(l+1)/\Gamma(l+1+i\gamma) \tag{7}$$

for the Coulomb Jost function.

Now we turn to the *off-shell* Jost solution and the *off-shell* Jost function, introduced by Fuda and Whiting.[5] The off-shell Jost solution $f_l(k,q,r)$ is that solution of the so-called inhomogeneous Schrödinger equation

$$\left(k^2+\frac{d^2}{dr^2}-\frac{l(l+1)}{r^2}-V(r)\right)f_l(k,q,r)=(k^2-q^2)\,i^l\,qrh_l^{(+)}(qr)\,, \tag{8}$$

which satisfies the asymptotic condition

$$\lim_{r\to\infty} f_l(k,q,r)e^{-iqr}=1\,. \tag{9}$$

Here $h_l^{(+)}$ is the spherical Hankel function,[16] for which the following useful equality holds:

$$i^l z h_l^{(+)}(z) = e^{iz}\,{}_2F_0(l+1, -l; (2iz)^{-1}) . \tag{10}$$

We have found that one can also define for the Coulomb potential an off-shell Jost solution which satisfies Eqs. (8) and (9). We shall denote this solution by $f_{C,l}(k,q,r)$. It is remarkable that its asymptotic behavior is the same as for a short-range potential. Recall from Eq. (3) that the ordinary Coulomb Jost solution $f_{C,l}(k,r)$ has a more complicated asymptotic behavior. In particular, one may expect that

$$\lim_{q\to k} f_{C,l}(k,q,r) \neq f_{C,l}(k,r) , \tag{11}$$

in contrast to the short-range case, where the equality holds, see Eq. (13) below.

The off-shell Jost function is defined by Fuda and Whiting[5,19] with

$$f_l(k,q) \equiv \lim_{r\to 0} f_l(k,q,r)(-2iqr)^l l!/(2l)! . \tag{12}$$

Note that this definition is completely analogous to the definition of $f_l(k)$ in Eq. (6). It also holds for the Coulomb case, just as Eq. (6) did.

Fuda and Whiting have studied the off-shell Jost solution, the off-shell Jost function, and some related functions for general short-range potentials. Some of their results, relevant for this paper, are

$$\lim_{q\to k} f_l(k,q,r) = f_l(k,r) , \tag{13}$$

$$\lim_{q\to k} f_l(k,q) = f_l(k) , \tag{14}$$

$$f_l(k,-q) = f_l^*(k,q) \quad (k \text{ and } q \text{ real}) , \tag{15}$$

and

$$T_l(k,q;k^2) = \left(\frac{k}{q}\right)^l \frac{f_l(k,q) - f_l(k,-q)}{i\pi q f_l(k)} . \tag{16}$$

These authors have also derived $f_l(k,q,r)$ and $f_l(k,q)$ in closed form for the square-well potential and, for $l=0$ only, for the exponential potential and the Hulthên potential.

We shall discuss now some functions for the Hulthên potential V_H and compare these with the corresponding functions for the Coulomb potential V_C. We use the subscripts H and C, respectively, and we restrict ourselves to the $l=0$ case and omit l.

The Hulthên potential is given by

$$V_H(r) = V_0 e^{-r/a}/(1-e^{-r/a}) .$$

V_H can be considered as a screened Coulomb potential with screening parameter a. For $a\to\infty$, V_H goes over into V_C. More precisely,

$$\lim_{\substack{a\to\infty \\ V_0\to 0 \\ aV_0\to 2k\gamma}} V_0 \frac{e^{-r/a}}{1-e^{-r/a}} = \frac{2k\gamma}{r} = V_C(r) . \tag{17}$$

The Jost solution and the Jost function for V_H are known in closed form (for $l=0$ only), see Newton[2] Chap. 14.4,

$$f_H(k,r) = e^{ikr}\,{}_2F_1(A,B;C;e^{-r/a}) , \tag{18}$$

$$f_H(k) = {}_2F_1(A,B;C;1) = \Gamma(C)/[\Gamma(1+A)\Gamma(1+B)] , \tag{19}$$

with

$$A = -iak + ia(k^2+V_0)^{1/2} ,$$
$$B = -iak - ia(k^2+V_0)^{1/2} ,$$
$$C = 1 - 2iak = 1 + A + B .$$

The expressions of Fuda and Whiting for the off-shell Jost solution and the off-shell Jost function are as follows:

$$f_H(k,q,r) = e^{iqr}\left(1 + \frac{ABe^{-r/a}}{(1+\sigma)(C+\sigma)}\,{}_3F_2(1, 1+A+\sigma, 1+B+\sigma; 2+\sigma, 1+C+\sigma; e^{-r/a})\right) , \tag{20}$$

$$f_H(k,q) = \frac{\Gamma(1+\sigma)\Gamma(C+\sigma)}{\Gamma(1+A+\sigma)\Gamma(1+B+\sigma)} , \tag{21}$$

with

$$\sigma = iak - iaq .$$

The derivation of Eq. (21) from Eq. (20) has been given by Bahethi and Fuda.[12] We now give a slightly different but essentially equivalent derivation because it is of general interest.

The ${}_3F_2$ of Eq. (20) is of the Saalschützian type.[9,10] Every Saalschützian ${}_3F_2$ of argument 1 with one of its three first parameters equal to 1 can be summed in terms of Γ functions. In order to find these Γ functions explicitly one can proceed as follows: According to a well-known theorem by Dixon,[9] any well-poised ${}_3F_2$ of unit argument can be summed in terms of Γ functions. A generalization of Dixon's theorem is (Slater,[9] p. 52)

$${}_3F_2(a,b,c;e,f;1) = \frac{\Gamma(e)\Gamma(f)\Gamma(s)}{\Gamma(a)\Gamma(b+s)\Gamma(c+s)} \times {}_3F_2(e-a, f-a, s; b+s, c+s; 1) , \quad \mathrm{Re}\,s>0, \quad \mathrm{Re}\,a>0 \tag{22}$$

with

$$s \equiv e+f-a-b-c .$$

For a Saalschützian ${}_3F_2$ we have $s=1$. $a=s=1$ if and only if both ${}_3F_2$'s of Eq. (22) are Saalschützian. In this case, they can be summed in terms of Γ functions, which is easily seen if one uses the general formula

$$z\frac{\alpha\beta\cdots}{\gamma\delta\cdots}\,{}_mF_n(1,\alpha+1,\beta+1,\ldots;\gamma+1,\delta+1,\ldots;z)$$
$$=-1+{}_mF_n(1,\alpha,\beta,\ldots;\gamma,\delta,\ldots;z). \quad (23)$$

By applying Eqs. (22) and (23) to the ${}_3F_2$ of Eq. (20) (with $c=1$), Eq. (21) is readily obtained.

In the first place, we are interested in the behavior of $f_H(k)$, $f_H(k,r)$, $f_H(k,q)$, and $f_H(k,q,r)$ as the screening parameter a goes to infinity. Do these four functions approach their Coulomb analogs, as one might hope in view of Eq. (17)? The Coulomb analogs of the first two functions, $f_C(k)$ and $f_C(k,r)$, are given by Eqs. (4) and (7). We have been able to derive a closed expression for $f_C(k,q)$ which is extremely simple,

$$f_C(k,q)=\left(\frac{q+k}{q-k}\right)^{i\gamma}. \quad (24)$$

Here k is real positive and q is complex with Imq >0. One obtains $f_C(k,q)$ for real positive k and q by taking the limit Im$q \to 0+$ which yields

$$f_C(k,q)=e^{\pi\gamma}\left|\frac{q+k}{q-k}\right|^{i\gamma} \quad \text{if } 0<q<k,$$
$$=\left|\frac{q+k}{q-k}\right|^{i\gamma} \quad \text{if } 0<k<q.$$

In contrast to $f_C(k,q)$, $f_C(k,q,r)$ is quite complicated. The function $f_C(k,q,r)$ can be expressed by an indefinite integral involving the Whittaker function W. We omit this expression since it does not seem to be very useful.

III. LIMITING RELATIONS

In this section, we shall consider the limits of various functions which have been discussed in Sec. II, for $a\to\infty$, for $q\to k$, for $r\to\infty$, and for $r\to 0$, respectively.

In the first place, we consider the limit of the functions f_H for the Hulthén potential, for $a\to\infty$ and $V_0\to 0$ in such a way that their product remains constant, $a\,V_0\to 2k\gamma$ [cf. Eq. (17)]. In this connection we rewrite the Hulthén potential as

$$V_H(r)=\frac{2k\gamma}{a}\frac{e^{-r/a}}{1-e^{-r/a}}=\frac{2k\gamma/a}{e^{r/a}-1}. \quad (25)$$

The four parameters A, B, C, and σ are functions of a. We have, for $a\to\infty$,

$$1+A+\sigma \sim -ia(q-k)+1+i\gamma,$$
$$1+B+\sigma \sim -ia(q+k)+1-i\gamma,$$
$$1+C+\sigma = -ia(q+k)+2,$$
$$2+\sigma = -ia(q-k)+2.$$

We use the following property of the Γ function:

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)}=z^{\alpha-\beta}[1+O(z^{-1})],$$
$$z\to\infty, \quad |\arg(z)|<\pi. \quad (26)$$

Then we obtain from Eq. (19)

$$\lim_{a\to\infty}(2ak)^{-i\gamma}f_H(k)=i^{-i\gamma}/\Gamma(1+i\gamma)=f_C(k), \quad (27)$$

and from Eq. (21),

$$\lim_{a\to\infty}f_H(k,q)=f_C(k,q). \quad (28)$$

From Eq. (27), we see that the usual (on-shell) Hulthén Jost function has no limit for $a\to\infty$, since it is proportional to $a^{i\gamma}$ for $a\to\infty$. Remarkably enough, Eq. (28) shows that the *off-shell* Hulthén Jost function *does* have as its limit the off-shell Coulomb Jost function.

Further, we consider $f_H(k,r)$ for $a\to\infty$ [see Eq. (18)]. It is known that[18]

$$\lim_{c\to\infty}z^{-\lambda}\,{}_2F_1(\lambda,b;c;1-c/z)=U(\lambda,\lambda-b+1,z). \quad (29)$$

By applying the Euler transformation

$${}_2F_1(\lambda,b;c;\zeta)$$
$$=(1-\zeta)^{-\lambda}\,{}_2F_1[\lambda,c-b;c;\zeta/(\zeta-1)], \quad (30)$$

we get

$$\lim_{c\to\infty}c^{-\lambda}\,{}_2F_1(\lambda,b+c;c;1-z/c)=U(\lambda,\lambda+b+1,z). \quad (31)$$

From this equality, we derive

$$\lim_{a\to\infty}(-2iak)^{-i\gamma}\,{}_2F_1(i\gamma,C-1-i\gamma;C;e^{-r/a})$$
$$=U(i\gamma,0,-2ikr).$$

Finally, it is known that

$$U(\lambda,c,z)=z^{1-c}U(\lambda+1-c,2-c,z), \quad (32)$$

and so we obtain from Eq. (31)

$$\lim_{a\to\infty}(2ak)^{-i\gamma}e^{ikr}\,{}_2F_1(i\gamma,-i\gamma-2iak;1-2iak;e^{-r/a})$$
$$=-2ikri^{-i\gamma}e^{ikr}U(1+i\gamma,2,-2ikr), \quad (33)$$

which is equivalent to

$$\lim_{a\to\infty}(2ak)^{-i\gamma}f_H(k,r)=f_C(k,r), \quad (34)$$

according to Eqs. (4) and (18). So we have proved that the Hulthén Jost solution, just as the Hulthén Jost function, has no limit for $a\to\infty$.

We have seen in Eq. (27) that the limit for $a\to\infty$ of the *off-shell* Hulthén Jost *function* is equal to

$f_C(k,q)$. Therefore, we conjecture that the analogous relation for the off-shell Jost *solutions*, namely,

$$\lim_{a\to\infty} f_H(k,q,r) = f_C(k,q,r), \quad q\neq k, \tag{35}$$

will turn out to be true.

Let us now consider the limit for $q\to k$ of the off-shell Jost solutions and functions. In the short-range case, in particular for the Hulthén potential, it is known that

$$\lim_{q\to k} f(k,q) = f(k),$$

$$\lim_{q\to k} f(k,q,r) = f(k,r).$$

In the Coulomb case we have proved that $f_C(k,q)$ is not continuous at $q=k$. We have from Eq. (24),

$$\lim_{q\to k}\left(\frac{q-k}{q+k}\right)^{i\gamma}\frac{e^{\pi\gamma/2}}{\Gamma(1+i\gamma)} f_C(k,q) = f_C(k). \tag{36}$$

Therefore, we conjecture that the following equality will turn out to hold for the off-shell Coulomb Jost solution,

$$\lim_{q\to k}\left(\frac{q-k}{q+k}\right)^{i\gamma}\frac{e^{\pi\gamma/2}}{\Gamma(1+i\gamma)} f_C(k,q,r) = f_C(k,r). \tag{37}$$

We summarize the results obtained so far. For completeness we also give well-known relations and our conjectures. We use the abbreviations,

$$\alpha \equiv (2ak)^{-i\gamma}, \tag{38}$$

$$\omega \equiv \left(\frac{q-k}{q+k}\right)^{i\gamma}\frac{e^{\pi\gamma/2}}{\Gamma(1+i\gamma)} = \frac{f_C(k)}{f_C(k,q)}, \quad \mathrm{Im}\,q>0. \tag{39}$$

It is known that

$$\lim_{r\to 0} f_H(k,r) = f_H(k), \tag{40a}$$

$$\lim_{r\to 0} f_H(k,q,r) = f_H(k,q), \tag{40b}$$

$$\lim_{r\to 0} f_C(k,r) = f_C(k), \tag{40c}$$

$$\lim_{q\to k} f_H(k,q) = f_H(k), \tag{40d}$$

$$\lim_{q\to k} f_H(k,q,r) = f_H(k,r). \tag{40e}$$

We have proved

$$\lim_{r\to 0} f_C(k,q,r) = f_C(k,q), \tag{40f}$$

$$\lim_{q\to k} \omega f_C(k,q) = f_C(k), \tag{40g}$$

$$\lim_{a\to\infty} \alpha f_H(k) = f_C(k), \tag{40h}$$

$$\lim_{a\to\infty} f_H(k,q) = f_C(k,q), \quad q\neq k, \tag{40i}$$

$$\lim_{a\to\infty} \alpha f_H(k,r) = f_C(k,r). \tag{40j}$$

Our conjectures are

$$\lim_{a\to\infty} f_H(k,q,r) = f_C(k,q,r), \quad q\neq k, \tag{40k}$$

$$\lim_{q\to k} \omega f_C(k,q,r) = f_C(k,r). \tag{40l}$$

We found it convenient to arrange these twelve limiting relations of Eq. (40) schematically, see Fig. 1 [note that $f_H(k)$, $a\to\infty$ occurs twice].

So we see that difficulties occur in the two limits $a\to\infty$ and $q\to k$. The third limit $r\to 0$ is, in general, well defined. We note in particular that it may be interchanged with $\lim a\to\infty$ and with $\lim q\to k$, e.g.,

$$\lim_{r\to 0}\lim_{a\to\infty} f_H(k,q,r) = \lim_{a\to\infty}\lim_{r\to 0} f_H(k,q,r) = f_C(k,q),$$

and

$$\lim_{r\to 0}\lim_{q\to k} f_H(k,q,r) = \lim_{q\to k}\lim_{r\to 0} f_H(k,q,r) = f_H(k).$$

We can take a fourth limit into our considerations, namely, $\lim r\to\infty$. Of course, the asymptotic behavior of the Jost solutions is well known, because it is part of their definition. We shall give one interesting example, which involves the interchange of $\lim r\to\infty$ and $\lim a\to\infty$. From Eq. (40j), we have

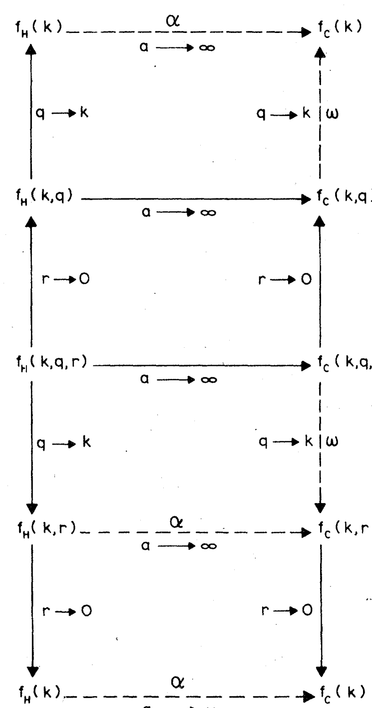

FIG. 1. Limiting relations for various functions. A full arrow indicates that the corresponding limit exists. For a dashed arrow, the limit does not exist, but a factor α or ω, respectively, is involved, see text.

$$\lim_{a\to\infty}(2ak)^{-i\gamma} f_H(k,r) = f_C(k,r)\,,$$

and from Eq. (3),

$$\lim_{r\to\infty}(2kr)^{i\gamma} e^{-ikr} f_C(k,r) = 1\,.$$

It follows from these two equations and Eq. (2) that

$$\lim_{r\to\infty}\lim_{a\to\infty}(r/a)^{i\gamma} e^{-ikr} f_H(k,r) = 1\,. \tag{41}$$

These two limits cannot be interchanged because

$$\lim_{a\to\infty}\lim_{r\to\infty}(r/a)^{i\gamma} e^{-ikr} f_H(k,r)$$

is not defined. Instead we have

$$\lim_{a\to\infty}\lim_{r\to\infty} e^{-ikr} f_H(k,r) = 1\,.$$

The off-shell Jost function has an analytic continuation into the complex q plane. From our closed form, Eq. (24), we have for the Coulomb off-shell Jost function, for real positive k and complex q,

$$f_C^*(k,q^*) = f_C(k,-q)\,. \tag{42}$$

This is a generalization of Fuda's relation for real q, see Eq. (15).

One should compare Eq. (42) with similar equations for the ordinary Jost function, see Eqs. (12.30) and (12.32a) of Newton.[2] It appears natural to define a "minus" Jost function f_- by

$$f_-(k,q) = f(k,-q)\,. \tag{43}$$

For a short-range potential, the half-off-shell T matrix can be expressed in terms of the off-shell Jost function, as Fuda and Whiting have shown, see our Eq. (16). We rewrite this equation as follows,

$$i\pi q f(k)\,\langle q|T|k\rangle = f(k,q) - f^*(k,q)\,, \tag{44}$$

where the T operator has energy variable k^2.

In virtue of Eq. (40.9) we have for $q\neq k$

$$\lim_{a\to\infty} f_H(k,q) = f_C(k,q)\,,$$

so

$$\lim_{a\to\infty} f_H^*(k,q) = f_C^*(k,q)\,. \tag{45}$$

Therefore, the limit for $a\to\infty$ of $f_H(k)\langle q|T_H|k\rangle$ exists for $q\neq k$. This implies that $\langle q|T_H|k\rangle$ has no limit for $a\to\infty$. Now, it is well known that the following equality holds,

$$\langle q|T_H|k\rangle = \langle q|V_H|k+\rangle_H\,, \tag{46}$$

where $|k+\rangle_H$ is the (outgoing) Hulthén $l=0$ scattering state. The Coulomb analog of Eq. (46) is not valid, since the Coulomb half-shell T matrix is not defined. However, $\langle q|V_C|k+\rangle_C$ is a well-defined quantity, for which we have been able to find the following closed form:

$$\langle q|V_C|k+\rangle_C = \frac{1}{i\pi q f_C(k)} \lim_{\eta\to 0+}\left[\left(\frac{q+i\eta+k}{q+i\eta-k}\right)^{i\gamma} - \text{c.c.}\right],\quad k>0\,,$$
$$= \lim_{\eta\to 0+}\frac{f_C(k,q+i\eta) - f_C^*(k,q+i\eta)}{i\pi q f_C(k)}\,,\quad k>0\,. \tag{47}$$

From Eqs. (44)–(47), we derive the following interesting equality:

$$f_C(k)\langle q|V_C|k+\rangle_C = \lim_{a\to\infty} f_H(k)\langle q|V_H|k+\rangle_H = \lim_{a\to\infty} f_H(k)\langle q|T_H|k\rangle\,. \tag{48}$$

IV. MODIFIED ECKART POTENTIALS AND LIMITING RELATIONS

It is very likely that all troubles with nonexistent limits we have encountered in Sec. III merely arise from the Coulomb tail. In order to investigate this point, one may study a screened $r^{-\alpha}$ type potential with $\alpha>1$. We feel that all limits of our scheme will be valid if V_C is replaced by a (nonsingular) potential of the form

$$V_\alpha(r) = O(r^{-\alpha})\,,\quad \alpha>1\,,\quad r\to\infty\,,$$

and V_H by an exponentially screened V_α potential. A convenient candidate for such a potential is

$$V_E(r) = \frac{2}{a^2}\,\frac{e^{-r/a}}{(1-e^{-r/a})^2}\,. \tag{49}$$

This is a member of the Eckart[17] class. Obviously we have

$$\lim_{a\to\infty} V_E(r) = 2/r^2\,. \tag{50}$$

The $l=0$ Jost solution for $V_E(r)$ is well known,[1]

$$f_E(k,r) = e^{ikr}\left(1+\frac{2}{(1-2iak)(e^{r/a}-1)}\right). \tag{51}$$

However, we have an annoying complication which should be avoided in this investigation. Since $V_E(r)$ is singular at the origin,

$$V_E(r) \simeq 2/r^2\,,\quad r\to 0\,,$$

the usual definition of the Jost function is meaningless. This problem may be disposed of in either of two ways, by not considering V_E but one of the potentials $V_E^{(1)}$, $V_E^{(2)}$ defined below.

In the first approach we define

$$V_E^{(1)}(r) \equiv V_E(r) - 2/r^2\,. \tag{52}$$

Then the ($l=0$) Jost solution $f_E(k,r)$ is just the $l=1$ Jost solution for $V_E^{(1)}$: $f_E(k,r) = f_{E,1}^{(1)}(k,r)$, since the term $2/r^2$ equals the centrifugal-barrier term for $l=1$. So we can find the $l=1$ Jost function simply by applying Eq. (6),

$$f_{E,1}^{(1)}(k)=\lim_{r\to 0}-ikrf_E(k,r)=\frac{2iak}{2iak-1}\ . \tag{53}$$

Remarkably enough, for this nonsingular potential $V_E^{(1)}$, we did *not* succeed in finding the $l=0$ Jost solution, but the $l=1$ Jost solution (and $l=1$ Jost function).

In the second approach, we define

$$V_E^{(2)}(r)\equiv V_E(r+d)=\frac{2}{a^2}\frac{\exp[-(r+d)/a]}{\{1-\exp[-(r+d)/a]\}^2}\ , \tag{54}$$

with $d>0$. Obviously, $f_E(k,r+d)$ is a solution of the $l=0$ radial Schrödinger equation. Therefore, the Jost solution for $V_E^{(2)}$ is

$$f_E^{(2)}(k,r)=e^{-ikd}f_E(k,r+d)$$
$$=e^{ikr}\left(1+\frac{2(1-2iak)^{-1}}{\exp[(r+d)/a]-1}\right). \tag{55}$$

The $l=0$ Jost function for $V_E^{(2)}$ is then given by

$$f_E^{(2)}(k)=f_E^{(2)}(k,0)=1+\frac{2(1-2iak)^{-1}}{\exp(d/a)-1}\ . \tag{56}$$

Now we consider the limits of the above quantities for $a\to\infty$. We have

$$\lim_{a\to\infty}V_E^{(1)}=0\ ,$$

$$\lim_{a\to\infty}V_E^{(2)}=2(r+d)^{-2}\ , \tag{57}$$

$$\lim_{a\to\infty}f_{E,1}^{(1)}(k,r)=ikrh_1^{(+)}(kr)=e^{ikr}[1-(ikr)^{-1}]\ ,$$

$$\lim_{a\to\infty}f_{E,1}^{(1)}(k)=1\ ,$$

$$\lim_{a\to\infty}f_E^{(2)}(k,r)=e^{ikr}\{1-[ik(r+d)]^{-1}\}, \tag{58}$$

$$\lim_{a\to\infty}f_E^{(2)}(k)=1-(ikd)^{-1}. \tag{59}$$

It follows from these expressions that the limits of $f^{(1)}$ and $f^{(2)}$ do indeed correspond to the limits of $V_E^{(1)}$ and $V_E^{(2)}$, respectively.

Fuda has obtained an expression for the off-shell Jost function for a short-range potential [Ref. 6, Eq. (25)]. In the notation of Ref. 20, it reads

$$f_l(k,q)=1+\tfrac{1}{2}\pi q(q/k)^l\ {}_0\langle ql\!\downarrow|V_l|kl+\rangle f_l(k)\ , \tag{60}$$

where

$${}_0\langle ql\!\downarrow|r\rangle\equiv(2/\pi)^{1/2}i^{-l}h_l^{(+)}(qr)\ .$$

We find from Eq. (60) that the off-shell Jost functions $f_E^{(1)}$ and $f_E^{(2)}$ are continuous at $q=k$. Moreover, the limits $a\to\infty$ and $q\to k$ may be interchanged. We conjecture that the same holds for the off-shell Jost solutions. If this is true, a diagram can be given similar to Fig. 1 where now, however, all of the limits are valid.

So we have indeed succeeded in proving (except for the off-shell Jost solutions) that, for the above screened r^{-2}-type potentials, the screening can be turned off without any discontinuity problem, in contrast to the situation for the (Hulthén) screened r^{-1} potential.

V. SUMMARY AND DISCUSSION

We have studied in Sec. II the off-shell Jost solution and function for the Coulomb potential for arbitrary values of l. For $l=0$, we have obtained a very simple closed expression for the off-shell Coulomb Jost function $f_C(k,q)$, see Eq. (24).

The Hulthén potential goes over into the Coulomb potential when the screening parameter a goes to infinity. In Sec. III we have investigated whether or not the limits for $a\to\infty$ of the Hulthén Jost functions and solutions are equal to the Coulomb Jost functions and solutions, respectively. The limits of the ordinary (on-shell) Jost function and solution do not exist. We have proved that in both cases the singularity is due to the factor $a^{i\gamma}$. Further, we have proved that for the Coulomb case the off-shell Jost function is not continuous at $q=k$. Here, the singularity is given by the factor $(q-k)^{-i\gamma}$. We also have derived some relations for the limits $r\to\infty$ and $r\to 0$. The main results have been summarized in Eqs. (40)–(41) and in Fig. 1. In the final part of Sec. III we have given an interesting limiting relation, for $a\to\infty$, of the half-shell Hulthén T matrix, see Eq. (48).

The different kinds of singularities which we have found in Sec. III should be attributed to the long range of the Coulomb potential. In earlier studies of the Coulomb T matrix we have seen singularities of a similar type.[13,21] It is very likely that, with a screened $r^{-\alpha}$ potential with $\alpha>1$, no singularity will turn out to exist. With the aim of giving an illustrative and interesting example, we have studied in Sec. IV a potential of the Eckart[17] class, which may be considered as a screened r^{-2} potential. Its Jost solution is well known and has a simple form. However, this Eckart potential is singular at the origin, $V(r)\simeq 2r^{-2}$, $r\to 0$, which makes the usual definition of the Jost function meaningless.

One way of avoiding this complication consists in subtraction of the singular term, which can in this particular case be interpreted as a centrifugal-barrier term for $l=1$.

A second method is generally useful for an arbitrary potential. The central idea here is that the Jost solution for a shifted potential function follows in an easy way from the ordinary Jost solution. To be specific, let $f(k,r)$ be the $l=0$ Jost solution for any potential $V(r)$. Then $e^{-ikd}f(k,r+d)$ is the Jost solution for the shifted potential $V(r+d)$ (where d is some real parameter). This follows easily from the defining differential equation. By the same reasoning, $e^{-iqd}f(k,q,r+d)$ is the off-shell Jost solution for the shifted potential $V(r+d)$. Obviously, the above statements hold only for the $l=0$ case.

By applying this method to any potential $V(r)$ which is (too) singular at $r=0$, we obtain a potential which is regular at $r=0$, if we choose d to be positive. Therefore, it has a Jost solution which is sufficiently regular at $r=0$ that the corresponding Jost *function* is well defined [namely, by the limit of $f(k,r)$ for $r\to 0$].

For the two modified Eckart potentials, obtained in the above-described way, we have made an investigation similar to the one of Sec. III. We have shown that the limits for $a\to\infty$ of the Jost functions corresponding to the screened potentials exist and do indeed correspond to the Jost functions of the unscreened potentials.

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