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# Improved calculation of the muonic-helium Lamb shift

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Several small, previously uncalculated contributions to the $2s_{1/2}$-$2p_{3/2}$ splitting $S_1$ in muonic $^4$He are calculated, and previously calculated contributions are verified. The improved theoretical result is $1817.5 \pm 0.6 + 0.4\langle r^2\rangle^{1/2} - 106.2\langle r^2\rangle + 1.4\langle r^2\rangle^{3/2} \simeq 1527.3 - 102.6(\langle r^2\rangle - 2.80)$ meV, where $\langle r^2\rangle$ is the mean-square charge radius of $^4$He in fm$^2$. Using the most recent value of the rms charge radius we get $S_{1\text{th}} = 1527.1 \pm 4.2$ meV, in excellent agreement with the experimental result $S_{1\exp} = 1527.5 \pm 0.3$ meV.

Recent precision measurements of the $2s_{1/2}$-$2p_{3/2}$ splitting in muonic helium[1,2] have inspired a number of theoretical improvements[3-7] over the original calculation of Campani,[8] both with regard to the purely quantum-electrodynamical (QED) effects (of which the most important is vacuum polarization), and with regard to the nuclear polarization. The purpose of the present work is to examine some previously neglected contributions and to reevaluate more accurately those which have been discussed previously. We have attempted to evaluate all corrections (except for nuclear polarization) to an accuracy of 0.01 meV. We do not claim the final result to be this accurate, but believe that the uncertainties in the purely QED corrections are not significantly larger. At this level of accuracy, the dependence of some of the lower-order radiative corrections on the nuclear radius must also be taken into account. Among the previously neglected or only estimated higher-order effects are higher-order binding corrections [of relative order $(\alpha Z)^2 \ln^2(\alpha Z)$] to the vertex graph,[9] two-photon recoil corrections,[10-12] the virtual Delbrück effect,[13] fourth-order Lamb shift,[14] $\alpha(\alpha Z)^{n\geq 3}$ vacuum polarization,[15] and hadronic vacuum polarization.[16] With the exception of the two-photon recoil correction, all of these effects contribute less than 0.1 meV. All numerical values given here use the most recent values of the fundamental constants.[17]

We find an improved value for the $2s_{1/2}$-$2p_{3/2}$ splitting in muonic $^4$He of $1817.5 \pm 0.6 + 0.4\langle r^2\rangle + 1.4\langle r^2\rangle^{3/2}$ meV. For rms radii in the vicinity of the electron-scattering value, the theoretical result can also be parametrized with no loss of numerical accuracy in the somewhat simpler form $1527.3 - 102.6(\langle r^2\rangle - 2.80)$ meV. The best value of $\langle r^2\rangle^{1/2}$ from electron scattering[18] is $1.674 \pm 0.012$ fm (differing slightly from the value $1.65 \pm 0.025$ fm used in previous theoretical work[3,4]), giving a theoretical splitting of $1527.1 \pm 4.2$ meV, to be compared with the experimental value of $1527.5 \pm 0.3$ meV.

The various improvements to the calculation will be presented. The following discussion is motivated largely by analytic analyses of the corrections based upon point-nucleus solutions. For the numerical results in Table I and in Eqs. (3) and (5), however, we have relied upon purely numerical calculations for the contrbutions which depend significantly upon the nuclear radius. Here we parametrize the results in the lowest-order form which would be expected from analytic calculation and adjust the coefficients to produce a best fit to the numerical results.

## I. ORDER-$\alpha$ CORRECTIONS

New results for the electronic-vacuum-polarization correction of order $\alpha(Z\alpha)$ are given in Table I. As in previous work[3,4] the Dirac equation was solved numerically in the field of a Gaussian charge distribution for values of the radial parameter in the neighborhood of the electron-scattering value. The relevant numerical parameters in the calculation were varied to ensure stability of the eigenvalues and expectation values at a level of 0.001 meV, so that one may have some confidence that all digits quoted in Table I are significant. Variation in the higher iterations of the $\alpha(Z\alpha)$ potential owing to changes in the nuclear radius appeared

 

TABLE I. Contributions to the $2s$-$2p$ splittings in the $(\mu^4\mathrm{He})^+$ system, in meV. $r$ is the rms charge radius of $^4$He in fm$^2$. All formulas are evaluated using the value $r^2 = 2.802$ fm$^2$.

| Contributions | | $S_1 = 2p_{3/2}-2s_{1/2}$ | $S_2 = 2p_{1/2}-2s_{1/2}$ |
|---|---|---|---|
| Electronic VP | | | |
| Uehling $\alpha Z\alpha$ first iteration | | $1666.57 - 0.76r^2 = 1664.44$ | $1666.30 - 0.76r^2 = 1664.17$ |
| | higher iterations | 1.70 | 1.70 |
| Källen-Sabry | $\alpha^2 Z\alpha$ | 11.55 | 11.55 |
| | $\alpha(Z\alpha)^n$, $n \geq 3$ | −0.02 | −0.02 |
| | $\alpha^2(Z\alpha)^2$ | 0.02 | 0.02 |
| muon VP | $\alpha Z\alpha$ | 0.33 | 0.33 |
| $\mu - e$ VP | $\alpha^2 Z\alpha$ | 0.02 | 0.02 |
| hadron VP | | 0.05 | 0.05 |
| Total | | $1680.22 - 0.76r^2 = 1678.09$ | $1679.95 - 0.76r^2 = 1677.82$ |
| Vertex corrections | | | |
| $\alpha Z\alpha$ | | $-10.90 + 0.23r = -10.52$ | $-11.23 + 0.23r = -10.85$ |
| $\alpha(Z\alpha)^n$, $n \geq 2$ | | −0.16 | −0.16 |
| $\alpha^2 Z\alpha$ | | −0.03 | −0.03 |
| Total | | $-11.09 + 0.23r = -10.71$ | $-11.42 + 0.23r = -11.04$ |
| Recoil | Breit | $+0.17r = 0.28$ | $0.17r = 0.28$ |
| | two-photon | −0.44 | −0.44 |
| Total | | $-0.44 + 0.17r = -0.16$ | $-0.44 + 0.17r = -0.16$ |
| Point Coulomb (fine structure) | | 145.70 | 0.00 |
| Total *QED* | | $1814.39 + 0.40r - 0.76r^2 = 1812.92$ | $1668.09 + 0.40r - 0.76r^2 = 1666.62$ |
| Nuclear polarization | | $3.1 \pm 0.6$ | $3.1 \pm 0.6$ |
| Finite size | | $-105.46r^2 + 1.40r^3 = -288.9 \pm 4.1$ | $-105.46r^2 + 1.40r^3 = -288.9 \pm 4.1$ |
| Total | | $1817.5 + 0.4r - 106.2r^2 + 1.4r^3 = 1527.1 \pm 4.2$ | $1671.2 + 0.4r - 106.2r^2 + 1.4r^3 = 1380.8 \pm 4.2$ |

at a level of $10^{-6}$ meV and were thus ignored.

At this level of accuracy, there is an additional modification of the $\alpha(Z\alpha)$ vertex correction not included previously. This is owing to the effect of the $\alpha(Z\alpha)$ vacuum polarization upon the muon wave functions used to compute expectation values of $\nabla^2 V$ and $r^{-1}\, dV/dr$. This modification increases the $\alpha(Z\alpha)$ vertex correction to the $2p$-$2s$ intervals by approximately 0.05 meV. Including this effect and variations owing to the finite nuclear size yields the formulas in Table I.

## II. BINDING CORRECTIONS

A new analysis of higher-order binding corrections to the $2s$-$2p$ splitting in normal atoms has been given by Mohr.[9] Both previous analyses of the muonic Lamb shift[3,4] included the contribution of relative order $\alpha Z$ [denoted in Ref. 4 as the vertex correction of order $\alpha(Z\alpha)^2$] and neglected the higher orders. Including the higher orders [with a value[9] $G_{\mathrm{SE}}(2\alpha) = -23.0 \pm 1.2$] gives a total higher-order correction of 0.16 meV in the point approximation.

The remaining correction (of order $\alpha Z$) is as given in Refs. 3 and 4. Inclusion of finite nuclear size results for the $2s_{1/2}$ state in a reduction by $\simeq 2.5\%$ (for a uniform nucleus of radius $R_0$ the reduction factor is easily found to be $1 - \frac{3}{2} m_r \alpha Z R_0 \simeq 0.975$ for $^4$He).

## III. ORDER-$\alpha^2$ CORRECTIONS

The fourth-order muon Lamb shift (order $\alpha^2 Z\alpha$) has been discussed in Ref. 13. An improved value for the fourth-order contribution to the muon radius has been given by Barbieri *et al.*[19] The contribution owing to the fourth-order vertex and muon vacuum polarization amounts to $-0.03$ meV, while the mixed $\mu$-$e$ vacuum polarization contributes $+0.02$ meV to the energy difference.

The virtual Delbrück contribution was calculated using the effective potential developed by Borie.[13] The result is 0.02 meV. In addition, we have confirmed the previous value[4] of $-0.02$ meV for the contribution of order $\alpha(\alpha Z)^{n\geq 3}$. The contribution of order $\alpha^3 Z\alpha$ may be comparable, but we have not calculated it. We also give a more accurate result for the correction of order $\alpha^2 Z\alpha$.

## IV. RECOIL CORRECTIONS

The recoil correction to muonic $^4$He given by Rinker[4] is based on the Breit equation analysis due to Friar and Negele.[20] For a uniform nucleus, the additional contribution to the $s$-state shift owing to finite nuclear size is given by

$$\Delta B_{2s} = \frac{9}{70}\,\frac{(Z\alpha)^5}{m_N}\,R_0 m_r^3\{1+\mathcal{O}[\alpha Z m_\mu R_0 \times \ln(\alpha Z m_\mu R_0)]\} \simeq 0.26 \text{ meV}. \quad (1)$$

A numerical calculation using a more realistic charge distribution gives a correction 0.28 meV, which is the value given in the table. In addition there is a further two-photon contribution[10-12] which is given in the point approximation by

$$\Delta S_{1,2} = -\frac{(Z\alpha)^5}{6\pi}\,\frac{m_\mu^2}{m_n}\left[\frac{97}{12}+2\ln\frac{K_0(2,1)}{K_0(2,0)} - \frac{1}{2}\ln(Z\alpha)\right] \simeq -0.44 \text{ meV}. \quad (2)$$

The effect of finite nuclear size may reduce this contribution by 2%, as is the case for the self-energy correction. This contribution was estimated by Borie[3] but was not included in previous compilations.

## V. HADRONIC VACUUM POLARIZATION

The effect of hadronic vacuum polarization was calculated using the formulas given by Sundaresan and Watson.[16] Masses and widths of the vector mesons were taken from the Particle Data Group,[17] and the pion form factor was parametrized according to the Gounaris-Sakurai fit of Ref. 21. The contribution is 0.05 meV.

## VI. FINITE NUCLEAR SIZE

The finite-size correction to the point Dirac energy has been previously parametrized in slightly different ways. We have reexamined this contribution. Numerical values of the energy shift of the $2s_{1/2}$ state were found by solving the Dirac equation in the field of a Gaussian charge distribution for several values of the radial parameter $r_0 = (\frac{2}{3}\langle r\rangle)^{1/2}$ which are close to the electron scattering value, and fitting the resulting energy shifts to a polynomial in $\langle r^2\rangle^{1/2}$. Other models (modified Gaussian, Fermi) for the charge distribution gave the same results. The energy shifts were carefully checked for numerical accuracy, using three different computer programs. An excellent fit to these "data" was obtained with the parametrization

$$\Delta B_{2s} = -105.46\langle r^2\rangle + 1.40\langle r^2\rangle^{3/2}, \quad (3)$$

where $\Delta B_{2s}$ is in meV and $\langle r^2\rangle^{1/2}$ is in fm. We observe that this parametrization is very close to that given by perturbation theory. The $2p_{1/2}$ state is also shifted by $-0.01$ meV for $\langle r^2\rangle = 2.80$ fm$^2$, while the shift of the $2p_{3/2}$ state is completely negligible. For radii close to the electron scattering value of $1.674 \pm 0.012$ fm, the numerical value of the energy shift given above does not differ significantly from previous parametrizations.[3,4]

The table summarizes our best values for all known contributions to the splittings $S_1$ ( of the $2s_{1/2}$-$2p_{3/2}$ levels) and $S_2$ ($2s_{1/2}$-$2p_{1/2}$) in the muonic ion $(\mu\,^4\text{He})^+$. For the nuclear polarization correction the value $3.1 \pm 0.6$ meV was adopted, as recommended in Refs. 4, 5, and 7. The results can be summarized in the form

$$S_1 = 1817.4 \pm 0.6 + \Delta B_{FS},$$

$$S_2 = 1671.1 \pm 0.6 + \Delta B_{FS},$$

where $\Delta B_{FS}$ is the shift of the $2s$ level owing to the effect of the finite size of the helium nucleus, including finite-size effects in the radiative corrections. The theoretical uncertainty quoted here is due entirely to the nuclear polarizability, since we believe that all other contributions have been calculated to a total accuracy better than 0.05 meV. Our best parametrization for $\Delta B_{FS}$ is

$$\Delta B_{FS} = 0.4\langle r^2\rangle^{1/2} - 106.2\langle r^2\rangle + 1.4\langle r^2\rangle^{3/2}, \quad (5)$$

where the effect of nuclear finite size on the vacuum polarization (here only the Uehling term is significantly affected) and vertex correction is included, as well as the correction to the point Dirac result.

The precision experiment[2] may be used to determine a more precise value of the charge radius of $^4$He. We find $\langle r^2\rangle^{1/2} = 1.673 \pm 0.001$ fm, where most of the error is owing to the uncertainty in the nuclear polarization correction.

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