Velocity correlations in a randomly stirred fluid: A variational principle for path-integral functionals

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A variational principle is applied to the generating functional of velocity correlations in an incompressible fluid described by the Navier-Stokes equations with random Gaussian stirring forces. The generating functional of velocity correlations gives, in a field-theoretic language, a complete statistical description of this driven, stationary Markov process in terms of path integrals and can be viewed as a generalized free energy. The statistical properties of velocity fluctuations generated by white-noise stirring forces whose energy-injection rate into the fluid is wave-number power-law distributed are investigated. A boundary dimension d^* depending upon the forcing spectrum is found above which long-wavelength velocity fluctuations are weakly coupled. In the strong-coupling regime $d < d^*$, static and dynamic exponents characterizing the wave-number dependence of long-wavelength fluctuations are evaluated. For $d > d^*$, corrections to the linear theory caused by the nonlinear mode-coupling terms are determined. The results are compared with renormalization-group calculations. Turbulent velocity fluctuations are discussed in the limit of vanishing viscosity. The relationship between forcing and energy spectrum in d dimensions is investigated and the results are compared with second-order closure approximations of statistical turbulence theories. A stirring-force spectrum $\sim k^{-d}$ leads to a Kolmogorov distribution of energy over wave number.

I. INTRODUCTION

A Lagrangian description of stochastic models simulating the dynamics of thermal systems near their critical point[1] has recently been demonstrated[2] to be an attractive alternative equivalent[3] to the operator formalism for classical dynamics of Martin, Siggia, and Rose.[4] The Lagrangian is derived from the path probability density[5] for nonlinear stochastic processes driven by Gaussian random forces. The Navier-Stokes equation (NSE) for incompressible fluids agitated by random stirring forces describes such a process.

The weight e^{-G} for different paths[2,5] is determined by a functional G with the physical meaning of a generalized Hamiltonian describing the dynamics and statics of the stochastic process. G is positive and contains for the NSE up to quartic field terms. In critical dynamics an integral representation for the exponential e^{-G} in terms of auxiliary response fields (equivalent to the conjugate fields of Martin *et al.*[4]) proved to be advantageous with respect to perturbative expansions.

For a variational approach, however, the original path probability density e^{-G} seems to be more appropriate[3] since the integral representation leads to nonreal expressions. The convexity of the weight e^{-G} suggests using the well-known inequality for such functions to apply a variational principle to the generating functional of correlations. In analogy with the free energy of Hamiltonian systems, the generating functional is given by the logarithm of an integral over all paths (configurations) with the weight e^{-G}.

The variational approach is a systematic method to determine fluctuation spectra of driven stochastic processes. Here we investigated the spectrum of velocity fluctuations generated by white-noise stirring forces whose energy injection rate into the fluid is for mathematical convenience power-law distributed over wave numbers. The variational correlation functions display all properties following from the symmetries and invariances of the fluid.

In Sec. III we apply the path-integral description for stochastic processes given in Sec. II to the forced NSE and we derive the generating functional e^{-F} for correlations. Section IV presents formulas for correlation functions which follow from the variational principle for F with a trial functional and variational parameters discussed in Sec. V. The minimization of F without auxiliary condition in Sec. VI leads to a Dyson equation for the velocity correlation function. The physical character of the variational approximation is explained. In Sec. VII we discuss static and dynamic behavior of velocity fluctuations in d dimensions for various stirring-force spectra. Long-wavelength properties for finite viscosities are compared with renormalization-group calculations.[6] Correlations for the inviscid case are compared with results from closure approximations[7] in turbulence theory. A quadratic random force spectrum leading to equilibrium dynamics and to statics determined by the linear theory is shown to require a variational principle with auxiliary conditions. That and the ensuing results are presented in Sec. VIII. Section IX summarizes our work.

II. PATH INTEGRALS FOR STOCHASTIC PROCESSES

Consider a Markovian stochastic process characterized by the nonlinear Langevin equations

$$\dot{x}_i(t)+K_i(X(t))=f_i(t) \tag{2.1}$$

for the set of random variables

$$X(t)=\{\ldots,x_i(t),\ldots\} \tag{2.2}$$

labeled by i. Let the set of random forces $F(t)$ driving the stochastic process be Gaussian distributed with zero mean and independent of $X(t)$. They enforce a statistically stationary state provided their correlations are time translational invariant. For the sake of simplicity the force correlations are usually assumed to have a white spectrum

$$\langle f_i(t)f_j(t')\rangle=D_{ij}\delta(t-t') \tag{2.3}$$

although general spectra do not pose a problem. According to Graham,[5] the probability density for paths between X_0 at t_0 and X_1 at t_1 for such a process (2.1) is given by the functional

$$W_{t_1t_0}[X]\sim e^{-G_{t_1t_0}[X]}J_{t_1t_0}[X] \tag{2.4a}$$

with

$$G_{t_1t_0}[X]=\frac{1}{2}\int_{t_0}^{t_1}dt f_i(t)[D^{-1}]_{ij}f_j(t)\,, \tag{2.4b}$$

where $f_i(t)$ stands for the left-hand side of Eq. (2.1). The term

$$J_{t_1t_0}[X]\sim\exp\left(-\frac{1}{2}\int_{t_0}^{t_1}dt\frac{\partial K_i}{\partial x_i}\right) \tag{2.5}$$

stems from the functional determinant[5] of the mapping (2.1) of the two sets X and F.

In order to express correlation functions in terms of path integrals, one also needs the conditional probability density

$$P_{\text{cond}}(X_2|X_1;t_2-t_1)=\int_{X_1}^{X_2}d[X]\,W_{t_2t_1}[X] \tag{2.6}$$

which is normalized by

$$\int dX_3P_{\text{cond}}(X_3|X_2;t_3-t_2)=1\,, \tag{2.7}$$

$$P_{\text{cond}}(X_1|X_0;0)=\delta(X_1-X_0)\,. \tag{2.8}$$

The integration in (2.6) is done over all paths between X_1 at t_1 and X_2 at t_2.[8] The stationary probability distribution is given by

$$P_{\text{stat}}(X)=P_{\text{cond}}(X|X_0;\infty)\,, \tag{2.9}$$

if the conditional probability density for finding X at any finite time t is independent of the initial value X_0 at $t_0=-\infty$. One commonly makes this assumption,[9] since as a result of the action of the random forces, the system should have "forgotten" at any finite time t its initial condition X_0 at $t_0=-\infty$. (For processes generated by Hamiltonian dynamics, e.g., time-dependent Ginzburg-Landau models,[1] one can in fact prove[10] that this condition is satisfied. It follows from the H theorem that the system approaches equilibrium after sufficient time and consequently that $P_{\text{stat}}[X]\sim e^{-\mathcal{H}[X]}$.) The correlation functions are then defined by

$$\langle x_j(t_2)x_i(t_1)\rangle=\int dX_2\,(x_j)_2P_{\text{cond}}(X_2|X_1;t_2-t_1)\times(x_i)_1P_{\text{stat}}(X_1)\,. \tag{2.10}$$

The above expression can be transformed into an integral over all paths starting at $t_0=-\infty$ and ending at $t_3=+\infty$ by using (2.9) for $P_{\text{stat}}(X_1)$, inserting the equality (2.7), and finally expressing the three conditional probabilities according to (2.6) as a product of three path integrals. Since the latter can be combined together with the integration over X_1 and X_2 to one extended path integral, one obtains

$$\langle x_i(t_1)x_j(t_2)\rangle=\int d[X]\,x_i(t_1)x_j(t_2)\,W_{\infty,-\infty}[X]\,. \tag{2.11}$$

When J defined in (2.5) is independent of X, as we shall assume throughout this paper, one finally deduces

$$\langle x_i(t_1)x_j(t_2)\rangle=\frac{\int d[X]\,x_i(t_1)x_j(t_2)e^{-G[X]}}{\int d[X]e^{-G[X]}} \tag{2.12}$$

with

$$G[X]=G_{\infty,-\infty}[X]\,. \tag{2.13}$$

The notation (2.12) stresses that G may be interpreted as the dynamic generalization of a Hamiltonian. Note that $G[X]$ describes the dynamics as well as the statics of the system.

III. NAVIER-STOKES EQUATIONS AND THE GENERATING FUNCTIONAL FOR CORRELATIONS

In this section we want to apply the concepts presented so far to the nonlinear NSE for the velocity field of an incompressible fluid stirred by random forces:

$$(\partial_t+\nu k^2)u_\alpha(\vec{k},t)+iP_{\alpha\beta}(\vec{k})k_\gamma\times\int\frac{d\vec{q}}{(2\pi)^d}\,u_\beta(\vec{q},t)\,u_\gamma(\vec{k}-\vec{q},t)=f_\alpha(\vec{k},t)\,. \tag{3.1}$$

Here ν is the viscosity and $P_{\alpha\beta}$ is the transverse projector

$$P_{\alpha\beta}(\vec{k})=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2\,. \tag{3.2}$$

It appears when the pressure gradient is eliminated using the incompressibility condition

$$k_\alpha u_\alpha(\vec{k}, t) = 0 = k_\alpha f_\alpha(\vec{k}, t) . \tag{3.3}$$

We also assumed the random stirring forces to be transverse. Before discussing applications of the NSE, let us introduce a more economical and condensed notation in terms of fields:

$$\phi(1) = \phi_{\alpha_1}(p_1) = \int d\vec{r} \int dt\, e^{i(\vec{k}_1\cdot\vec{r} - \omega_1 t)} u_{\alpha_1}(\vec{r}, t) , \tag{3.4}$$

$$\zeta(1) = \zeta_{\alpha_1}(p_1) = \int d\vec{r} \int dt\, e^{i(\vec{k}_1\cdot\vec{r} - \omega_1 t)} f_{\alpha_1}(\vec{r}, t) . \tag{3.5}$$

Throughout this paper we will use the abbreviations

$$p \equiv (\vec{k}, \omega), \quad \int_p = \int_{\vec{k}} \int_\omega = \int \frac{d\vec{k}}{(2\pi)^d} \int \frac{d\omega}{2\pi} . \tag{3.6}$$

Since $\phi(1)$, $\zeta(1)$ are Fourier transforms of real fields, one has

$$\phi(-1) = \phi_{\alpha_1}(-p_1) = \phi^*(1) \tag{3.7}$$

and a similar relation for $\zeta(1)$. NSE's for the fields thus read

$$\chi^{-1}_{\rm lin}(12)\phi(2) + W(123)\phi(2)\phi(3) = \zeta(1) , \tag{3.8}$$

where integration ($\int_p$) and summation (α) over repeated arguments is implied. Except for the projector (3.2) the inverse response function of the linear NSE is

$$\chi^{-1}_{\rm lin}(12) = \delta_{\alpha_1\alpha_2}\Delta(p_1 - p_2)[-i\omega_1 + \Omega_{\rm lin}(k_1)] \tag{3.9a}$$

with

$$\Omega_{\rm lin}(k) = \nu k^2 \tag{3.9b}$$

and

$$\Delta(p) = (2\pi)^{d+1}\delta(\vec{k})\delta(\omega) . \tag{3.10}$$

The symmetrized form of the nonlinear coupling function is

$$W(123) = P(11')P(22')P(33')w(1'2'3') , \tag{3.11a}$$

$$w(123) = -\tfrac{1}{2} i\Delta(p_1 - p_2 - p_3) w_{\alpha_1\alpha_2\alpha_3}(\vec{k}_1) , \tag{3.11b}$$

$$w_{\alpha\beta\gamma}(\vec{k}) = \delta_{\alpha\beta}k_\gamma + \delta_{\alpha\gamma}k_\beta , \tag{3.11c}$$

$$W(123) = W(132) = -W^*(123) = -W(-1-2-3) . \tag{3.12}$$

The projectors

$$P(12) = \Delta(p_1 - p_2)P_{\alpha_1\alpha_2}(\vec{k}_1) \tag{3.13}$$

explicitly display the transversality (3.3) of the fields

$$\phi(1) = P(12)\phi(2), \quad \zeta(1) = P(12)\zeta(2) . \tag{3.14}$$

We will consider random Gaussian correlated forces with zero mean. Their spectrum will be white in frequency, but may depend on the wave vector

$$\langle\zeta^*(1)\zeta(2)\rangle = P(11')D(1'2')P(2'2) , \tag{3.15}$$

$$D(12) = \delta_{\alpha_1\alpha_2}\Delta(p_1 - p_2)D(k_1) . \tag{3.16}$$

Rotational symmetry implies $D(k)$ to be a function of $|\vec{k}|$ only. Time and space translational invariance of the system is reflected by $D(12)$ being diagonal with respect to frequency and wave vector. The spectrum $D(k)$ is positive and measures the rate of energy input at wave vector $\vec{k}$ into the fluid by the stirring forces. The energy input is necessary to set up a stationary state. It balances the dissipation described by the viscosity term and the energy transfer caused by the nonlinear mode coupling term in (3.8).

The NSE's are the starting point for statistical theories of strongly developed homogeneous isotropic turbulence. Such turbulent states are supposed to be created by injecting enough energy into a fluid within a small band at low wave numbers through stirring. It is commonly believed that the ensuing inertial range energy cascade towards high wave vectors is independent of the detailed form of the injection spectrum $D(k)$ as long as the latter is restricted to long wavelengths. Experiments so far on man-made low-Reynolds-number turbulence created by grids in wind tunnels are unable to decide this question, although the small-scale motion in the tunnel center far enough downstream seems to be nearly isotropic and statistically independent of the details of the grid.[11] The relation between the energy spectrum of infinite-Reynolds-number turbulence and the spectrum of the random stirring forces will be investigated in Sec. VII.

Recently, Eqs. (3.1) and (3.8) together with an injection spectrum $D(k) \sim k^2$ have also been used for a description of hydrodynamic excitations from thermal equilibrium.[6,12] The idea of using a stirring force with spectrum $\sim k^2$ to simulate the generation of thermally agitated long-wavelength low-frequency excitations in fluids in thermal equilibrium seems to be a reasonable one even though one can question its usage with the not always physically valid assumption of incompressibility. It is supported by the fact that (only) a spectrum $D(k) \sim k^2$ fulfills the potential conditions[5,9,13] necessary for the random process (3.1), (3.8) to obey detailed balance. In that case the stationary probability distribution is according to class B of Deker and Haake[13] Gaussian in the fields $u_\alpha(\vec{k})$,

$$P_{\mathrm{stat}}[u] \sim \exp\left(-\frac{1}{2}\int_{\vec{k}} u_\alpha^*(\vec{k})\,\frac{2\nu k^2}{D(k)}\,u_\alpha(\vec{k})\right), \quad (3.17)$$

with

$$\langle u_{\alpha_1}^*(\vec{k}_1)u_{\alpha_2}(\vec{k}_2)\rangle = P_{\alpha_1\alpha_2}(\vec{k}_1)(2\pi)^d\delta(\vec{k}_1-\vec{k}_2)C(k_1), \quad (3.18)$$

so that the equal-time correlations are constant,

$$C(k)=D(k)/2\nu k^2, \quad (3.19)$$

and a fluctuation-dissipation-type relation[13] connects response and correlation functions. With the special choice $D_{\mathrm{th}}=2\nu k_B T/\rho$ for the proportionality constant in $D(k)$ the Einstein relation is satisfied. The resulting value of the equal-time velocity correlation $C(k)=(1/n)v_{\mathrm{th}}^2$ is independent of k and ν. It coincides with the equilibrium average value for a classical fluid of particle density n in a canonical or grand canonical ensemble.

The injection spectrum $D(k)\sim k^2$ discussed above is a special, albeit important, case: For general stirring spectra neither detailed balance nor the fluctuation dissipation relation will obtain and the stationary distribution will not be Gaussian nor independent of the nonlinear terms in the NSE. Note also that in the special case (3.19) energy balance between injection and dissipation is not only global but also local in $\vec{k}$ space: $2\nu k^2 C(k)\,d^dk$ is the kinetic energy taken out of the volume element d^dk around $\vec{k}$ by dissipation and $D(k)\,d^dk$ is the energy injected into it with $C(k)d^dk$ the kinetic energy contained in d^dk.[14-16] Since this balance works via the Einstein relation even for $\nu\to 0$, there is no way to set up an energy cascade characteristic for turbulence.

We now proceed to obtain the dynamical functional $G[\phi]$ for the NSE. Since J [Eq. (2.5)] is independent of the velocity field—contributions from the nonlinear term in (3.1) vanish—correlation functions (2.12) are determined by the functional G alone. Replacing the time integral in

$$G=\frac{1}{2}\int_{-\infty}^{\infty}dt\int\frac{d\vec{k}}{(2\pi)^d}\,f_\alpha^*(\vec{k},t)D^{-1}(k)f_\alpha(\vec{k},t) \quad (3.20a)$$

by an integral over all frequencies, one finally obtains

$$G[\phi]=\tfrac{1}{2}\zeta^*(1)D^{-1}(12)\zeta(2) \quad (3.20b)$$

where ζ abbreviates the left-hand side of Eq. (3.8).

The dynamical functional G is real and positive. It contains terms quadratic, cubic, and quartic in the fields:

$$\begin{aligned}G[\phi]&=\tfrac{1}{2}\phi^*(1)C_{\mathrm{lin}}^{-1}(12)\phi(2)\\&+\tfrac{1}{2}\phi^*(1)\phi^*(2)T(12,34)\phi(3)\phi(4)\\&+(\text{cubic terms}). \quad (3.21)\end{aligned}$$

Here $C_{\mathrm{lin}}(12)$ is, except for projectors (3.13), the Fourier transform of the two-point velocity-correlation function within the linear theory

$$C_{\mathrm{lin}}(12)=\chi_{\mathrm{lin}}(13)D(34)\chi_{\mathrm{lin}}^*(42) \quad (3.22a)$$

$$=\delta_{\alpha_1\alpha_2}\Delta(p_1-p_2)C_{\mathrm{lin}}(p_1) \quad (3.22b)$$

with

$$C_{\mathrm{lin}}(k,\omega)=C_{\mathrm{lin}}(k)\,\frac{2\Omega_{\mathrm{lin}}(k)}{\omega^2+\Omega_{\mathrm{lin}}^2(k)} \quad (3.22c)$$

and $C_{\mathrm{lin}}(k)$ denotes the equal-time correlation function of the linear theory

$$C_{\mathrm{lin}}(k)=C_{\mathrm{lin}}(k,t=0)=\frac{1}{2}\,\frac{D(k)}{\Omega_{\mathrm{lin}}(k)}=\frac{D(k)}{2\nu k^2}. \quad (3.22d)$$

Note that the choice $D(k)\sim k^2$ causes the equal-time correlations $C(k)$ [Eq. (3.19)] to be those of the linear theory $C(k)=C_{\mathrm{lin}}(k)$. For later use we have expressed $C_{\mathrm{lin}}(k,\omega)$ in terms of the static correlations $C_{\mathrm{lin}}(k)$ and the characteristic frequency $\Omega_{\mathrm{lin}}(k)$ of the linear theory.

The "coefficient" of the quartic term

$$T(12,34)=W^*(512)D^{-1}(56)W(634) \quad (3.23)$$

is quadratic in the coupling function, i.e., proportional to $k^2/D(k)$.

The generating functional for correlations is

$$e^{-F[h]}=\int d[\phi]\exp\{-G[\phi]+h^*(1)\phi(1)\}, \quad (3.24)$$

where an integration over all "paths" between $\omega=-\infty$ and $\omega=\infty$ must be carried out, i.e., $d[\phi]$ stands for $\Pi_n d\phi(n)$. Considering paths as functions of $\phi_\alpha(\vec{k},\omega)$ instead of $u_\alpha(\vec{k},t)$ implies a linear transformation whose Jacobian is irrelevant for calculating averages according to

$$\langle\phi(1)\cdots\phi(n)\rangle=e^{F[h]}\,\frac{\delta}{\delta h^*(1)}\cdots\frac{\delta}{\delta h^*(n)}\,e^{-F[h]}\Big|_{h=0}. \quad (3.25)$$

F can be thought of as being the dynamic generalization of a free energy.

IV. VARIATIONAL PRINCIPLE

The variational principle for $F[h]$ relies on the convexity of the exponential function[8,17] $\langle e^x\rangle \geq e^{\langle x\rangle}$ which requires x to be real and the weight factors implied in the average to be positive and normalized to unity. The term $h^*(1)\phi(1)$ in (3.24) is real as long as $h(1)$ is the Fourier transform of a real field which we assume. Let $G_t[\phi,\lambda]$ be a real, positive trial functional depending on a set of parameters λ. Then[17,18]

$$e^{-F[h]}\geq\exp\{-\langle G[\phi]-G_t[\phi,\lambda]\rangle_t^h\}\,e^{-F_t[h]}, \quad (4.1)$$

where $e^{-F_t[h]}$ is given by (3.24) with G_t replacing

G. Averages with a subscript t are defined by

$$\langle A[\phi]\rangle_t^h = \frac{\int d[\phi]A[\phi]\exp\{-G_t[\phi]+h^*(1)\phi(1)\}}{\int d[\phi]\exp\{-G_t[\phi]+h^*(1)\phi(1)\}}. \quad (4.2)$$

For the case $h=0$ the superscript h will be omitted.

It seems to be attractive to apply a variational principle to the more general functional[2,3] which generates correlation—as well as response—functions. However, the exponent

$$J[\phi,\tilde{\phi}] = \tfrac{1}{2}\tilde{\phi}^*(1)D(12)\tilde{\phi}(2) - i\tilde{\phi}^*(1)\zeta(1)$$

produced by the integral representation of the exponential function by

$$e^{-G[\phi]} = \exp[-\tfrac{1}{2}\zeta^*(1)D^{-1}(12)\zeta(2)]$$

$$\sim \int d[\tilde{\phi}]e^{-J[\phi,\tilde{\phi}]}$$

is complex. Our search for an inequality as powerful as the one in Eq. (4.1) has been fruitless.

The variational "free energy"

$$F_{\text{var}}[h;\lambda_0] = \min\{\langle G-G_t\rangle_t^h + F_t[h]\} \quad (4.3)$$

$$= \langle G-G_0\rangle_0^h + F_0[h] \quad (4.4)$$

is determined by the parameter set λ_0 which minimizes the curly bracket in (4.3) via

$$G_0 = G_t[\phi,\lambda_0]. \quad (4.5)$$

The extremal condition in (4.3) with respect to λ is

$$0 = -\left\langle \frac{\delta G_t}{\delta\lambda}(G-G_t)\right\rangle_t^h + \left\langle \frac{\delta G}{\delta\lambda^t}\right\rangle_t^h \langle G-G_t\rangle_t^h \quad (4.6)$$

and λ_0 is the solution thereof.

The correlation functions generated by the functional $F_{\text{var}}[h;\lambda_0]$ (4.4) are, according to Eq. (3.25),

$$\langle\phi(1)\rangle_{\text{var}} = \langle\phi(1)\rangle_0 - s(1), \quad (4.7)$$

$$\langle\phi^*(1)\phi(2)\rangle_{\text{var}} = \langle\phi^*(1)\phi(2)\rangle_0 - S(12) + s^*(1)s(2) \quad (4.8)$$

with

$$s(1) = \langle\phi(1)[G-G_0]\rangle_0 - \langle\phi(1)\rangle_0\langle G-G_0\rangle_0, \quad (4.9)$$

$$S(12) = \langle\phi^*(1)\phi(2)[G-G_0]\rangle_0 - \langle\phi^*(1)\phi(2)\rangle_0\langle G-G_0\rangle_0, \quad (4.10)$$

and the subscript 0 on the average brackets refers to G_0.

V. TRIAL FUNCTIONAL AND VARIATIONAL PARAMETERS

So far we have not said a word about the trial functional. In analogous problems in the literature it is usually chosen in accordance with two criterions: (i) physical intuition—which means in this case the ability to guess a trial functional containing the basic physics of G, and (ii) "calculability." The latter restricts us to functionals G_t quadratic in the field ϕ. Of course, one might argue that this choice can not satisfy the first criterion since a Gaussian dynamic weight for paths $\phi(\vec{k},\omega)$ will not reproduce all physical properties of the correlations in strongly developed turbulence. However, by optimizing the approximation to the nonlinear mode coupling terms—i.e., the cubic and quartic contributions to G—by relaxational quadratic terms one may hope to retain some of the characteristics of the nonlinear terms. Without calculations one expects an approximation to the nonlinear NSE by an optimized linear equation (leading to a quadratic functional G_t) to be reasonable for small field amplitudes and for small nonlinear couplings—i.e., for the long-wavelength limit $k\to 0$. Whether the variational principle allows one to treat also turbulence characterized by strong nonlinear mode coupling and large amplitudes can be decided only *a posteriori*. To that end we proceed towards the calculations.

The most general quadratic form for $G_t[\phi]$ which ensures that the mean velocity vanishes

$$\langle\phi(1)\rangle_{\text{var}} = 0 \quad (5.1)$$

is (see Appendix A)

$$G_t[\phi] = \tfrac{1}{2}\phi^*(1)C_t^{-1}(12)\phi(2) \quad (5.2)$$

with

$$\langle\phi^*(1)\phi(2)\rangle_t = P(11')C_t(1'2')P(2'2). \quad (5.3)$$

Throughout this paper we will employ this trial functional for which $s(1)$ [Eq. (4.9)] vanishes (see Appendix A). Homogeneity and stationarity requires

$$C_t(12) = \delta_{\alpha_1\alpha_2}\Delta(p_1-p_2)C_t(k_1,\omega_1) \quad (5.4)$$

to be diagonal in frequency and momentum. The spectrum $C_t(k,\omega)$ of velocity correlations evaluated with the weight $e^{-G_t[\phi]}$ has to be real, positive, even in ω, and a function only of $|\vec{k}|$.

In principle one should take the matrix $C_t(12)$ as a set of variational parameters and ensure symmetries of the variational correlation functions (4.10) by auxiliary conditions. We will use a trial and error approach with $\lambda(p) = C_t(k,\omega)$ as variational parameters and see whether the spectrum $C_0(k,\omega)$ and $C_{\text{var}}(k,\omega)$ shows the invariances discussed above. The result of this procedure (see Sec. VI) with no auxiliary conditions imposed is indeed that all correlation functions meet all symmetry requirements discussed so far. If there is more information on the spectrum of velocity correlations available than the mentioned symmetries one can incorporate it by auxiliary conditions (see Sec. VIII). The question of invariance of G_0 under the transformation $\phi(1)\to\phi(1)+v_{\alpha_1}\Delta(p_1)$ and under

Galilean transformations will be treated in a forthcoming publication.

VI. MINIMIZATION WITHOUT AUXILIARY CONDITIONS

In this section we will evaluate $\lambda_0(k,\omega)=C_0(k,\omega)$ and $C_{\text{var}}(k,\omega)$ without auxiliary conditions. To determine the minimal "free energy" by (4.6) we use the chain rule of differentiation to express $\delta G_t/\delta\lambda(p)$ as

$$\delta G_t/\delta\lambda(p)=\tfrac{1}{2}g_p(12)\phi^*(1)\phi(2)\,, \tag{6.1}$$

where

$$g_p(12)=\frac{\delta C_t^{-1}(12)}{\delta\lambda(p)}=-\delta_{\alpha_1\alpha_2}\Delta(p_1-p_2)\frac{\Delta(p_1-p)}{C_t^2(k,\omega)} \tag{6.2}$$

and

$$\tfrac{1}{2}\phi^*(1)\phi(2)=\delta G_t/\delta C_t^{-1}(12)\,. \tag{6.3}$$

Thus, the extremal condition (4.6) reads

$$0=-\tfrac{1}{2}g_p(12)S(12)\,, \tag{6.4}$$

or since $g_p(12)$ [Eq. (6.2)] is a diagonal, negative matrix in 1 and 2,

$$0=S(12)=\langle\phi^*(1)\phi(2)[G-G_0]\rangle_0 -\langle\phi^*(1)\phi(2)\rangle_0\langle G-G_0\rangle_0\,. \tag{6.5}$$

This implies, according to (4.10),

$$\langle\phi^*(1)\phi(2)\rangle_{\text{var}}=\langle\phi^*(1)\phi(2)\rangle_0 \tag{6.6}$$

and $C_0(k,\omega)$ satisfies Eq. (6.5). Since the cubic field terms of G do not contribute to $S(12)$ one finds

$$S(12)=S_{\text{II}}(12)+S_{\text{IV}}(12)\,, \tag{6.7}$$

where

$$S_{\text{II}}(12)=\langle\phi^*(1)\phi(3)\rangle_0[C_{1\text{in}}^{-1}(34)-C_0^{-1}(34)] \times\langle\phi^*(4)\phi(2)\rangle_0 \tag{6.8}$$

gives the contribution to (6.5) from the quadratic field term in $G-G_0$. The contribution from the quartic term reads with an obvious notation

$$S_{\text{IV}}(12)=\tfrac{1}{2}[\langle 1^*23^*4^*56\rangle_0 -\langle 1^*2\rangle_0\langle 3^*4^*56\rangle_0]T(34,56) \tag{6.9}$$

$$=\langle\phi^*(1)\phi(3)\rangle_0\bar{\Sigma}_0(34)\langle\phi^*(4)\phi(2)\rangle_0\,. \tag{6.10}$$

We have used the symmetry of W [Eqs. (3.11), (3.12)] as well as (5.3), (5.4) to find (6.10) with

$$\bar{\Sigma}_0(12)=4T(13,42)\langle\phi^*(3)\phi(4)\rangle_0\,. \tag{6.11}$$

(In evaluating Gaussian averages the recursion formula

$$\langle\phi(1)\cdots\phi(n)\rangle_0 =\sum_{\nu=2}^{n}\langle\phi(1)\phi(\nu)\rangle_0 \times\langle\phi(1)\cdots\phi(\nu-1)\phi(\nu+1)\cdots\phi(n)\rangle_0$$

is helpful.) Note that only 8 out of the possible 12 pairings of (6.9) contribute since $W(122)=0$.

The final result for $S(12)$

$$S=PC_0P[C_{1\text{in}}^{-1}-C_0^{-1}+\Sigma_0]PC_0P \tag{6.12}$$

shows that the extremal condition (6.5) leads to a Dyson equation for $C_0(12)$ with a self-energy $\Sigma_0(12)$ defined in analogy to (5.3) by

$$P(11')\Sigma_0(1'2')P(2'2)=\bar{\Sigma}_0(12)\,. \tag{6.13}$$

It turns out (see Appendix B) that its spectrum is white:

$$\Sigma_0(12)=\Delta(p_1-p_2)\Sigma^0_{\alpha_1\alpha_2}(\vec{k}_1)\,. \tag{6.14}$$

The absence of memory in the self-energy kernel is caused by the white force spectrum and the Markovian character of the NSE. This can be inferred most easily from the graphical representation of $\Sigma_0(12)$ in Fig. 1: Since $D^{-1}(12)$ as well as the nonlinear coupling functions $W(123)$ are frequency independent except for a δ function the frequency dependence of $C_0(k,\omega)$ is integrated out in the loop (3, 4, 5, 6) of Fig. 1. Note that the self-energy is quadratic in the coupling function W [Eq. (3.11)] and has a mode coupling structure except that one internal "leg" is replaced by D^{-1}. As a result of the unrestricted variational procedure the nonlinear, Markovian NSE is approximated by a linear Markovian process whose correlations are determined according to (6.12) by

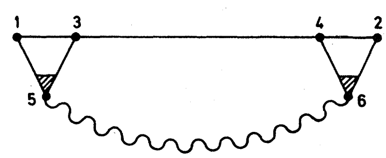
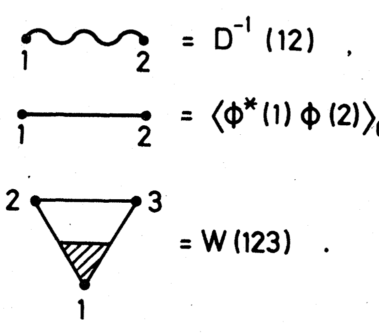


FIG. 1. Graphical representation of the self-energy contribution $T(13,42)\langle\phi^*(3)\phi(4)\rangle_0$. Here



$$0=P_{\alpha\nu}(\vec{k})\left(\frac{\delta_{\nu\mu}}{C_{1in}(k,\omega)}-\frac{\delta_{\nu\mu}}{C_0(k,\omega)}+\Sigma^0_{\nu\mu}(\vec{k})\right)P_{\mu\beta}(\vec{k}). \tag{6.15}$$

The optimal dynamical weight e^{-G_0} for paths is given according to (6.15), (6.11), and (6.13) by

$$G_0[\phi]=\tfrac{1}{2}\phi^*(1)[C^{-1}_{1in}(12)+4T(13,42)\times\langle\phi^*(3)\phi(4)\rangle_0]\phi(2). \tag{6.16}$$

The quartic field term of G [Eq. (3.21)] has been approximated by replacing a product of fields by its correlation (note however the factor of 4). The latter is determined self-consistently by the functional G_0 and thus the spirit of the variational solution is that of a self-consistent "mean-field-products" approximation. In models for critical statics the variational approach is equivalent[19] to a mean- (single) field approximation. Here the mean field (5.1) vanishes so that cubic terms of the functional G do not contribute to (5.2) and (6.16)—only even powers of the coupling function W [Eq. (3.11)] enter G_0.

VII. CORRELATIONS

The velocity correlation function $C_0(k,\omega)$ of the variational approach is determined by (6.15). Taking the trace one obtains

$$C_0(k,\omega)=C_0(k)\frac{2\Omega_0(k)}{\omega^2+\Omega_0^2(k)} \tag{7.1}$$

with

$$\Omega_0^2(k)=\Omega^2_{1in}(k)+D(k)\Sigma_0(k), \tag{7.2}$$

$$C_0(k)=C_{1in}(k)\frac{\Omega_{1in}(k)}{\Omega_0(k)}. \tag{7.3}$$

Here $(d-1)\Sigma_0(k)$ denotes the trace over the last term in (6.15) (see Appendix B). The "self-energy"

$$\Sigma_0(k)=\int_{\vec\kappa}\int_{\vec q}(2\pi)^d\delta(\vec\kappa-\vec q-\vec k)\times C_0(q)\frac{\kappa^2}{D(\kappa)}f(x,y,z) \tag{7.4}$$

is positive since $f(x,y,z)\geq 0$ for $d\geq 2$. This function is also bounded from above and dimensionless since it only depends on the three cosines x, y, z of the triangle formed by $\vec{k}$, $\vec{q}$, $\vec\kappa$ (see Appendix B).

The equal-time correlation function $C_0(k)$ of the variational approach is the solution of the integral equation defined by (7.2)–(7.4). The frequency spectrum (7.1) of velocity fluctuations reflects pure relaxational dynamics: The net effect of the transfer terms in the NSE [Eq. (3.8)] coupling different Fourier modes together via the triad interaction W [Eq. (3.11)] has been approximated here by a relaxation (or decay) mechanism produced by nonlinear interaction of fields. In turbulence theory this relaxation rate $[D(k)\Sigma_0(k)]^{1/2}$ which according to (7.2) enhances the viscous damping $\Omega_{1in}(k)=\nu k^2$ would be interpreted by an effective eddy damping

$$[D(k)\Sigma_0(k)]^{1/2}=\mu(k)k^2. \tag{7.5}$$

The eddy viscosity $\mu(k)$ describes damping of eddies (i.e., velocity modes) by decay owing to coupling to others. Note that $\mu(k)$ [Eqs. (7.5), (7.4)] retains the nonlocal triad coupling scheme of wavevectors characteristic of the NSE. Thus, our eddy viscosity does not show the deficiency of earlier theories[20] approximating energy-transfer terms by interactions between pairs of wave numbers.

In the description of long-wavelength, low-frequency fluctuations within the language of generalized hydrodynamics[21] the eddy damping would be hidden in the complex renormalized viscosity $\nu_R(k,\omega)$ introduced as a relaxation kernel[22] into the equations of motions for velocity fluctuations by

$$C(k,\omega)=-2\,\mathrm{Im}\frac{1}{\omega+ik^2\nu_R(k,\omega)}C(k). \tag{7.6}$$

In view of the Markovian character of the variational solution one has to identify

$$\frac{\Omega_0(k)}{k^2}=\nu_R^0(k,\omega=0)=[\nu^2+\mu^2(k)]^{1/2}. \tag{7.7}$$

Before discussing the results (7.1)–(7.7) further, one should note that the stirring force spectrum $D(k)\sim k^2$ (see Sec. III) requires a special treatment. Since $\Sigma_0(k)$ is positive the variational equal-time correlations $C_0(k)$ [Eq. (7.3)] are smaller than the ones stemming from linear theory. The equilibrium dynamics enforced by $D(k)\sim k^2$, however, requires that $C(k)=C_{1in}(k)$—static correlations are independent of mode-coupling effects since the velocity field amplitudes $u_\alpha(\vec{k})$ are Gaussian distributed according to (3.17) with a variance (3.22d) given by the linear theory. It is easy to see why the unrestricted variational approach gives wrong equal-time correlations in this case. The latter are determined by the stationary distribution $P_{stat}[u]$ of the field amplitudes $u_\alpha(\vec{k},t=0)$ which according to (2.9) and (2.6) is given by an integral over paths from $t=-\infty$ to $t=0$ with the weight e^{-G}. In thermal equilibrium, mode-coupling effects on the paths $u_\alpha(\vec{k},t)$ cancel out in (3.17) when the integration along the paths is performed with the proper weight. The variational dynamical weight e^{-G_0} for paths, which only approximates the effect of mode coupling on the time evolution, does not yield the delicate cancellation in path integrals. Consequently the distribution of final values $u_\alpha(\vec{k}, t=0)$ of the paths differs from the result in linear theory.

For general nonquadratic stirring spectra $D(k)$ (e.g., for one creating turbulence) the stationary distribution will not be determined by linear theory, i.e., mode-coupling effects will show up. In this case our result indicates that the variational treatment of nonlinear interactions is more successful. This is not at all trivial since the variational procedure effectively amounts to an approximation for paths, or better their weight e^{-G_0}, which might be a reasonable one for the dynamics. The static behavior of the system is determined by the stationary distribution which involves an integration over the approximated paths and an "accumulation" of small errors could take place.

The deviation from the linear result when $D(k) \sim k^2$ must be small in the limit of weak nonlinear coupling ($k\to 0$, $d>2$). Then also the eddy viscosity (7.5) is small and as we shall show later $\Omega_0(k) \to \Omega_{1\text{in}}(k)$, i.e., $C_0(k)\to C_{1\text{in}}(k)$. In the strong coupling regime ($d<2$) large nonlinear field interactions are approximated by a large eddy viscosity and the integral equation [(7.2)–(7.4)] gives a wrong power law for $\nu_R^0(k\to 0, \omega=0)$ (7.7). An *ad hoc* remedy would be to violate the self-consistency restriction (7.3) and replace $C_0(k)$ in Eqs. (7.1), (7.2), and (7.4) by the correct correlation $C_{1\text{in}}(k)$. Then $\nu_R^0(k\to 0, \omega=0)$, [Eq. (7.7)] shows for $d<2$ the expected divergence $\sim k^{(d-2)/2}$.[6] A systematic treatment when $D(k)\sim k^2$ is presented in Sec. VIII.

In the remainder of this section we will discuss the variational solution (7.1)–(7.7) for nonquadratic stirring spectra

$$D(k) = Dk^{\varphi}. \tag{7.8}$$

Investigating pure power laws [Eq. (7.8)] (with arbitrary exponent φ) is common practice so far and a matter of convenience: The connection between the behavior of static correlations $C_0(k)$ [Eq. (7.3)] and the dynamic frequency $\Omega_0(k)$ [Eq. (7.2)] on the one hand and the stirring spectrum on the other hand is most easily established for power laws. In order to extract the qualitative behavior of the solution to the integral equation [Eqs. (7.2)–(7.4)] for given exponent φ we will henceforth discuss power laws. Suppose

$$C_0(k) \approx k^{c} \tag{7.9}$$

solves Eqs. (7.2)–(7.4) for a broad range of wave numbers then also $\Sigma_0(k)$ [Eq. (7.4)] and $\mu(k)$ [Eq. (7.5)] display approximately power-law behavior:

$$\mu^2(k) \approx Kk^{d-2+c}. \tag{7.10}$$

To derive this approximation (7.10) for $\mu^2(k)$ we have evaluated $\Sigma_0(k)$ [Eq. (7.4)] with the approximate solution (7.9) by introducing dimensionless variables $\vec{\xi}=\vec{\kappa}/k$, $\vec{\eta}=\vec{q}/k$, $\vec{e}=\vec{k}/k$, so that

$$K=\int_{\vec{\xi}}\int_{\vec{\eta}}(2\pi)^d\delta(\vec{\xi}-\vec{\eta}-\vec{e})C_0(\eta)\xi^{2-\varphi}f(x,y,z). \tag{7.11}$$

There are three regions in ξ,η space which might give divergent contributions to K: (i) Long-wavelength contributions from $\eta\to 0$, $\xi\to 1$. For $C_0(k)$ showing power-law behavior even in the limit $k\to 0$ there is no divergence if $c>-d$ that means as long as the kinetic energy per unit mass $\int_{\vec{k}} C_0(k)$ displays no divergence from long-wavelength fluctuations. To ensure a finite kinetic energy for $c<-d$ one has to assume a rounding for very small k. Note, however, that the coupling function $f(x,y,z)$ is not defined at $\xi=1$, $\eta=0$. (ii) Long-wavelength contributions from $\xi\to 0$, $\eta\to 1$. They converge for $\varphi<2+d$ which will be the case in our investigations. (iii) uv contributions from $\eta\to\infty$ ($\xi\approx\eta$). They remain finite if $C_0(k)$ show the physically expected exponential screening of very large wave-number fluctuations. Convergence problems do not occur, however, with a locality enforcing cutoff procedure like, e.g., $\min(1,\eta,\xi) > \frac{1}{2}\max(1,\eta,\xi)$ which preserves scaling properties and thus can be used to discuss overall power-law behavior.

Let us first discuss the long-wavelength behavior of velocity fluctuations generated by the stirring forces. In the first limiting case of weak nonlinear coupling

$$\mu(k\to 0)/\nu \ll 1 \tag{7.12}$$

the nonlinear interaction of velocity modes yields only small corrections to the linear theory

$$\Omega_0(k)=\Omega_{1\text{in}}(k)\left[1+\left(\frac{\mu(k)}{\nu}\right)^2\right]^{1/2}, \tag{7.13a}$$

$$C_0(k)=C_{1\text{in}}(k)\left[1+\left(\frac{\mu(k)}{\nu}\right)^2\right]^{-1/2}. \tag{7.13b}$$

The condition $\mu(k\to 0)/\nu\ll 1$ imposes a consistency restriction since it requires that the exponent of $\mu^2(k)$ [Eq. (7.10)] be positive

$$d-2+c>0, \tag{7.14}$$

$$c=\varphi-2. \tag{7.15}$$

And $c=\varphi-2$ characterizes the power law of $C_0(k)$ in the weak-coupling limit [Eqs. (7.12) and (3.22d)]. The restriction (7.14) defines a boundary dimension

$$d^*=4-\varphi \tag{7.16}$$

depending on φ above which velocity fluctuations are weakly coupled: $[\mu(k)/\nu]^2 \sim k^{d-d^*}$. In this regime $d>d^*$ the long-wavelength form of (7.13) is

$$\Omega_0(k) = \nu k^2[1+(\text{const})k^{d-d^*}]^{1/2}, \tag{7.17a}$$

$$C_0(k) = (D/2\nu)k^{\varphi-2}[1+(\text{const})k^{d-d^*}]^{-1/2}. \tag{7.17b}$$

Since the variational solution is Markovian the effects of nonlinear mode coupling—which also cause long-time tail behavior of $\nu_R(k,\omega)$ (Ref. 23)—show up here only in k space, e.g., as corrections k^{d-d^*} in the characteristic frequency $\Omega_0(k)$.

For the other limiting case of strong nonlinear coupling

$$\nu/\mu(k\to 0) \ll 1 \tag{7.18}$$

viscous damping of velocity fluctuations is small compared with relaxation due to nonlinear interaction. Then

$$\Omega_0(k) = \mu(k)k^2\left[1+\left(\frac{\nu}{\mu(k)}\right)^2\right]^{1/2}, \tag{7.19a}$$

$$C_0(k) = \frac{D(k)}{2\mu(k)k^2}\left[1+\left(\frac{\nu}{\mu(k)}\right)^2\right]^{-1/2}. \tag{7.19b}$$

Again there is for finite ν a consistency restriction which, according to (7.10) reads

$$d-2+c<0. \tag{7.20}$$

The exponent c in the strong coupling regime is determined by (7.19b) and (7.10):

$$c=\varphi-2-\tfrac{1}{2}(d-2+c). \tag{7.21}$$

Combining the two last equations we find that long-wavelength fluctuations are strongly coupled for $d<d^*$ [Eq. (7.16)]:

$$[\nu/\mu(k)]^2 \sim k^{2(d^*-d)/3}. \tag{7.22}$$

We have used (7.21) and (7.16) to express c as

$$c=\varphi-2+\tfrac{1}{3}(d^*-d). \tag{7.23}$$

The characteristic frequency $\Omega_0(k)$ and the equal-time correlation $C_0(k)$ for long-wavelength fluctuations in the strong coupling limit $d<d^*$ then read

$$\Omega_0(k) \sim k^{2-2(d^*-d)/3}[1+(\text{const})\nu^2k^{2(d^*-d)/3}]^{1/2}, \tag{7.24a}$$

$$C_0(k) \sim k^{\varphi-2+(d^*-d)/3}[1+(\text{const})\nu^2k^{2(d^*-d)/3}]^{-1/2}. \tag{7.24b}$$

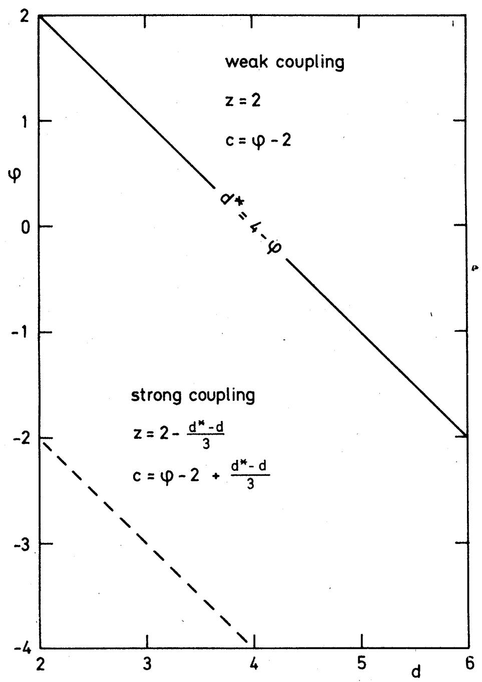


FIG. 2. Long-wavelength behavior of velocity fluctuations generated by stirring forces with a spectrum $D(k)=Dk^{\varphi}$ ($\varphi\neq 2$). The dynamical exponent z is defined by the characteristic frequency $\Omega_0(k)\sim k^z$, the static exponent c by $C_0(k)\sim k^c$. A Kolmogorov spectrum ($m=\frac{5}{3}$) for the energy $E(k)\sim k^{-m}$ is found along the dashed line $\varphi=-d$ for infinite Reynolds number ($\nu\to 0$).

The results obtained so far are summarized in Fig. 2. They coincide with renormalization-group calculations[6] done for $\varphi=0$ and $d \gtrless d^*=4$. At $d=d^*$ viscous and eddy relaxation are equally important and one expects logarithmic terms[6] in (7.24). Within our qualitative discussion of Eqs. (7.2)–(7.4) we can not argue about logarithmic behavior of the solutions to (7.2) and (7.3) at $d=d^*$.

Note that the dynamic exponent $z=2-\frac{1}{3}(d^*-d)$ [Eq. (7.24a)] of velocity fluctuations generated by nonquadratic stirring spectra differs in its dependence on d^*-d from $z=2-\frac{1}{2}(d^*-d)$ which describes the case $D(k)\sim k^2$ discussed in Sec. VIII (also the *ad hoc* treatment for $\varphi=2$ yields the correct exponent). We have not investigated the crossover.

In order to discuss the behavior of turbulent velocity fluctuations we shall investigate correlations in the limit of vanishing viscosity (infinite Reynolds number). To obtain a solution to (7.2)–(7.4) for finite viscosity, say, in the inertial range between the energy containing eddies and the dissipation cutoff requires more numerical effort.

In the limit of $\nu\to 0$ the dissipation range is pushed to higher and higher wave numbers and so the fluid balances in a stationary state the average rate of energy input $D(k)d^dk$ into d^dk by the stirring forces against the net average rate with which the nonlinear mode-coupling mechanism transfers energy out of the volume d^dk. Fournier and Frisch[7] have shown within the eddy-damped quasinormal Markovian (EDQNM) approximation[14,16] that a velocity spectrum $\sim k^{\varphi-2+(d^*-d)/3}$ leads to a convergent negative transfer integral as long as $d<d^*$. So we will restrict ourselves in the following to forcing spectra $\varphi<4-d$.

In turbulence theory the spectral distribution of energy (per unit mass) $E(k)$ over wave numbers is of considerable interest:

$$\tfrac{1}{2}\langle \vec{u}^2(\vec{r},t)\rangle = \int_0^\infty dk E(k) \,. \tag{7.25}$$

Since the average velocity square $\langle \vec{u}^2\rangle$ can be written

$$\mathrm{Tr}_{\alpha_1=\alpha_2}\int_{P_1}\int_{P_2}\langle\phi^*(1)\phi(2)\rangle=(d-1\ )\int_{\vec{k}}C(k), \tag{7.26}$$

one finds

$$E(k)=(\mathrm{const})k^{d-1}C(k). \tag{7.27}$$

For vanishing viscosity the variational approximation (7.2)–(7.4) is determined by (7.19):

$$\Omega_0(k)=\mu(k)k^2\sim k^{2-(d^*-d)/3}, \tag{7.28a}$$

$$C_0(k)=\frac{D(k)}{2\mu(k)k^2}\sim k^{\varphi-2+(d^*-d)/3}, \tag{7.28b}$$

and the characteristic exponents are

$$z=\tfrac{2}{3}+\tfrac{1}{3}(\varphi+d), \tag{7.29a}$$

$$m=\tfrac{5}{3}-\tfrac{2}{3}(\varphi+d). \tag{7.29b}$$

Here m is defined by

$$E(k)\sim k^{-m} \tag{7.30}$$

and z is the exponent of the frequency (7.28a). Thus stirring forces which inject energy into the volume $d\vec{k}$ around $\vec{k}$ at a rate $D(k)=Dk^{-d}$ cause the kinetic energy contained in the velocity field to be distributed according to Kolmogorov's law.[24] Note that in this case ($\varphi=-d$) the energy input rate into the wave-number band dk around k varies as k^{-1}.

VIII. SPECIAL CASE $D(k)=Dk^2$ MINIMIZATION WITH AUXILIARY CONDITIONS

A stirring spectrum $D(k)=Dk^2$ enforces a Gaussian distribution (3.17) for the velocities with wave-number-independent equal-time correlations C_{lin} determined by the linear theory. In our variational approach we will guarantee

$$C_{\mathrm{var}}(k)=C_{\mathrm{lin}} \tag{8.1}$$

via an auxiliary condition. Instead of (4.3) one now has to find the minimum of

$$K[\lambda,\mu]=F_t[\lambda]+\langle G[\phi]-G_t[\phi,\lambda]\rangle_t+H[\lambda,\mu], \tag{8.2}$$

where

$$H[\lambda,\mu]=-\Delta(0)\frac{d-1}{4}\int_{\vec{k}}\int_\omega \mu(\vec{k})[C_{\mathrm{var}}(k,\omega)-C_{\mathrm{lin}}(k,\omega)] \tag{8.3}$$

represents the auxiliary condition (8.1). The Lagrange multiplier $\mu(\vec{k})$ should not be confused with the eddy viscosity of Sec. VII. We have introduced $\Delta(0)$ in (8.3) to compensate for a term arising from functional derivatives. The trial functional $G_t[\phi,\lambda]$ and the parameter set $\lambda(p)=C_t(p)$ are those of Sec. V. Since the variational correlation function $C_{\mathrm{var}}(p)$ was defined by

$$C_{\mathrm{var}}(p_1)=\frac{1}{d-1}\mathrm{Tr}_{\alpha_1=\alpha_2}\int_{P_2}\langle\phi^*(1)\phi(2)\rangle_{\mathrm{var}}, \tag{8.4}$$

we obtain with (4.8) and (6.12)

$$C_{\mathrm{var}}(p)=C_0(p)-C_0(p)[C_{\mathrm{lin}}^{-1}(p)-C_0^{-1}(p)+\Sigma_0(k)]C_0(p). \tag{8.5}$$

$\Sigma_0(k)$ [Eq. (7.4)] is a linear functional of the optimal parameter set $\lambda_0(p)=C_0(p)$ determined by the minimum of (8.2).

Variation of the two first terms in (8.2) with respect to $\lambda(p)$ yields [cf. (4.6), (6.1)–(6.4), (6.12)]

$$-\tfrac{1}{2}g_p(12)S(12)=\Delta(0)\tfrac{1}{2}(d-1)\times[C_{\mathrm{lin}}^{-1}(p)-C_0^{-1}(p)+\Sigma_0(k)] \tag{8.6}$$

and the variation of the third term reads

$$\frac{\delta H[\lambda,\mu]}{\delta\lambda(p)}=\Delta(0)\frac{d-1}{2}\{\mu(\vec{k})C_0(p)[C_{\mathrm{lin}}^{-1}(p)-C_0^{-1}(p)+\Sigma_0(k)]+\Gamma_0(\vec{k})\}, \tag{8.7}$$

where

$$\Gamma_0(\vec{k})=\frac{1}{2}\int_{p'}\mu(\vec{k}')C_0^2(p')\frac{\delta\Sigma_0(k')}{\delta\lambda(p)}. \tag{8.8a}$$

A straightforward evaluation making use of the symmetry $f(x,y,z)=f(x,z,y)$ [Eq. (B8)] in $\Sigma_0(k)$ [Eq. (7.4)] shows that the functional dependence of

$$\Gamma_0(\vec{k})=\frac{1}{2D}\int_\omega\int_{\vec{\chi}}\int_{\vec{q}}(2\pi)^d\delta(\vec{\kappa}-\vec{q}-\vec{k})\mu(\vec{q})\times C_0^2(q,\omega)f(x,y,z) \tag{8.8b}$$

is quadratic in $C_0(p)$ and linear in $\mu(\vec{k})$. The first extremal condition of (8.2) is

$$\frac{\delta K[\lambda,\mu]}{\delta\lambda(p)}=0=\Gamma_0(k)+[1+\mu(\vec{k})C_0(p)]\times[C^{-1}_{1\mathrm{in}}(p)-C_0^{-1}(p)+\Sigma_0(k)]\,, \quad (8.9\mathrm{a})$$

the second reads

$$\frac{\delta K[\lambda,\mu]}{\delta\mu(\vec{k})}=0=\int_\omega \{C_0(p)-C_0(p)[C^{-1}_{1\mathrm{in}}(p)-C_0^{-1}(p)+\Sigma_0(k)]C_0(p)-C_{1\mathrm{in}}(p)\}. \quad (8.9\mathrm{b})$$

Equations (8.9) are coupled, nonlinear integral equations of $C_0(p)$ and $\mu(\vec{k})$. They can be rewritten as a system of linear, algebraic equations in $\mu(\vec{k})$ and $\Gamma_0(\vec{k})$:

$$C_0(p)[R(p)+\Sigma_0(k)C_0(p)]\mu(\vec{k})+C_0(p)\Gamma_0(\vec{k})=-C_0(p)\Sigma_0(k)-R(p)\,, \quad (8.10\mathrm{a})$$

$$[C_0(k)-C_{1\mathrm{in}}]\mu(\vec{k})+C_0(k)\Gamma_0(\vec{k})=-C_0(k)\Sigma_0(k)-\Omega_R(k)\,, \quad (8.10\mathrm{b})$$

where the coefficients are functionals of $C_0(p)$ and $C_{1\mathrm{in}}(p)$. Here we defined

$$R(p)=[C_0(p)-C_{1\mathrm{in}}(p)]/C_{1\mathrm{in}}(p)\,, \quad (8.11)$$

$$\Omega_R(k)=\int_\omega R(p)\,. \quad (8.12)$$

One obtains a single integral equation for $C_0(p)$ alone by inserting the solutions $\mu(\vec{k})$ and $\Gamma_0(\vec{k})$ of (8.10) into Eq. (8.8). We will not try to solve this equation for $C_0(p)$ but rather guess the zero-frequency long-wavelength behavior of $C_0(p)$ which is consistent with (8.10) and (8.8). But first let us rewrite $C_{\mathrm{var}}(p)$ [Eq. (8.5)] in the following form:

$$C_{\mathrm{var}}(p)=C_0(p)+[C_{1\mathrm{in}}-C_0(k)]\times\frac{C_0(p)[\Gamma_0(\vec{k})+\Sigma_0(k)]+R(p)}{C_0(k)[\Gamma_0(\vec{k})+\Sigma_0(k)]+\Omega_R(k)} \quad (8.13)$$

which obviously displays property (8.1) since the frequency integral over the quotient gives unity. Equation (8.13) was obtained by inserting $\mu(\vec{k})$ of (8.10b) into (8.9a).

To discuss the zero-frequency long-wavelength properties of the variational spectrum (8.13) we define frequencies $\Omega(k)$ characterizing low-frequency fluctuations according to (7.6) by zero-frequency values of generalized viscosities

$$\Omega(k)=k^2\nu(k,\omega=0)\,. \quad (8.14)$$

Then Eq. (8.13) determines $\Omega_{\mathrm{var}}(k)$ by

$$\frac{C_{1\mathrm{in}}}{\Omega_{\mathrm{var}}(k)}=\frac{C_0(k)}{\Omega_0(k)}+\frac{C_{1\mathrm{in}}-C_0(k)}{\Omega_0(k)}\times\frac{C_0(k)[\Gamma_0(\vec{k})+\Sigma_0(k)]+\frac{1}{2}\Omega_0(k)R(k,\omega=0)}{C_0(k)[\Gamma_0(\vec{k})+\Sigma_0(k)]+\Omega_R(k)} \quad (8.15)$$

and

$$R(k,\omega=0)=\frac{C_0(k)}{C_{1\mathrm{in}}}\frac{\Omega_{1\mathrm{in}}(k)}{\Omega_0(k)}-1\,.$$

The appearance of another frequency $\Omega_R(k)$ [Eq. (8.12)] appears at first sight to be a salient feature of (8.15) and of Eq. (8.10) at $\omega=0$ for $\mu(\vec{k})$ and $\Gamma_0(\vec{k})$. We will show however, that this spectral moment $\Omega_R(k)$ for which we assume long-wavelength power-law behavior

$$\Omega_R(k)\sim\begin{cases}k^{r_+} & \text{for } d>2\\ k^{r_-} & \text{for } d<2\end{cases} \quad (8.16)$$

does not influence the characteristic exponent of the low frequency $\Omega_{\mathrm{var}}(k)$. With the following ansatz for the solution of Eq. (8.8) and Eq. (8.10) at $\omega=0$

$$\frac{C_0(k)}{C_{1\mathrm{in}}}=\begin{cases}1+(\mathrm{const})k^{d-2} & \text{for } d>2\\ 1+(\mathrm{const})k^{2-d} & \text{for } d<2,\end{cases} \quad (8.17)$$

$$\frac{\Omega_0(k)}{\Omega_{1\mathrm{in}}(k)}=\begin{cases}1+(\mathrm{const})k^{d-2} & \text{for } d>2\\ (\mathrm{const})k^{-(2-d)/2} & \text{for } d<2,\end{cases} \quad (8.18)$$

one finds the solution $\mu(\vec{k})$ and $\Gamma_0(\vec{k})$ of the inhomogeneous linear system (8.10) at $\omega=0$ to behave like

$$\mu(\vec{k})\sim k^2\times\begin{cases}1+(\mathrm{const})k^{r_+-d} & \text{for } d>2\\ k^{-(2-d)/2}[1+(\mathrm{const})k^{r_--d}] & \text{for } d<2,\end{cases} \quad (8.19)$$

$$\Gamma_0(\vec{k})\sim k^d\times\begin{cases}1+(\mathrm{const})k^{r_+-d} & \text{for } d>2\\ 1+(\mathrm{const})k^{r_--d} & \text{for } d<2\end{cases} \quad (8.20)$$

for long wavelengths. To obtain (8.19) and (8.20) we used the approximation [cf. (7.9)–(7.11)] $\Sigma_0(k)\sim k^d$. Inserting (8.19) into (8.8b) one verifies that Eqs. (8.19) and (8.20) are consistent with (8.8b) evaluated in the scaling approximation (7.9)–(7.11) for all cases $r_\pm \gtreqless d$ within the additional approximation

$$\int_\omega [C_0(k,\omega)]^2\approx\Omega_0^{-1}(k)\left[\int_\omega C_0(k,\omega)\right]^2 \quad (8.21)$$

under the integral of (8.8b). Thus $C_0(k)$ [Eq. (8.17)] and $\Omega_0(k)$ [Eq. (8.18)] satisfy the extremal conditions (8.9) in the considered frequency and wavenumber regime. The results (8.16)–(8.18) and (8.20) cause the quotient in (8.15) which contains

$\Gamma_0(\vec{k})$ and $\Omega_R(k)$ to be constant for small k so that the characteristic exponent of $\Omega_{var}(k)$ is not influenced by $\Omega_R(k)$. The k^0 behavior of the quotient in (8.15) for $d \gtrless 2$ holds for $r_\pm \geq d$ as well as for $r_\pm < d$ since in the first case according to (8.20) $\Gamma_0(\vec{k}) \sim k^d$ and in the latter $\Gamma_0(\vec{k}) \sim k^{r_\pm}$. The final result for the characteristic frequency of long-wavelength fluctuations

$$\frac{\Omega_{var}(k)}{\Omega_{lin}(k)} = \begin{cases} 1 + (\text{const})k^{d-2} & \text{for } d > 2 \\ (\text{const})k^{-(2-d)/2} & \text{for } d < 2 \end{cases} \tag{8.22}$$

has been obtained previously by other methods.[6,23]

IX. SUMMARY

We have applied a variational principle to the generating functional of velocity correlations in randomly stirred fluids described by forced NSE. The generating functional is given by an integral over all paths with a weight $e^{-G[\phi]}$. The generalized Hamiltonian $G[\phi]$ contains quadratic, cubic, and quartic terms in the field ϕ and describes the statics as well as the dynamics of the Markov process defined by the forced NSE. The optimal quadratic trial functional $G_0[\phi]$ without subsidiary conditions amounts to a self-consistent mean-field-products approximation to $G[\phi]$ where two fields of the quartic nonlinearity are replaced by their correlation evaluated with weight $e^{-G_0[\phi]}$.

The resulting spectrum of velocity fluctuations is positive. It displays a pure relaxational dynamics since the effect of nonlinear terms coupling different modes together via triad interaction is approximated by a relaxation mechanism described by a wave-number-dependent eddy viscosity. The expression for the eddy viscosity explicitly displays the triad coupling scheme with a kinematic coefficient found also in second-order closure approximations.

We investigated the relationship between velocity fluctuation spectra and Gaussian white-noise stirring forces injecting energy at a rate $\sim k^\varphi$ into a volume element d^dk around $\vec{k}$. Power-law injection spectra were discussed for mathematical convenience only. There exists a boundary dimension $d^*(\varphi) = 4 - \varphi$ above (below) which long-wavelength velocity fluctuations are weakly (strongly) coupled. In the strong coupling regime static $[c = \varphi - 2 + \frac{1}{3}(d^* - d)]$ and dynamic $[z = 2 - \frac{1}{3}(d^* - d)]$ exponents characterizing the wave-number dependence of velocity fluctuations were evaluated. In the weak coupling limit we determined corrections ($\sim k^{d-d^*}$) to the results of the linear theory ($c = \varphi - 2$, $z = 2$) due to the nonlinear mode-coupling terms. Our results agree with renormalization-group calculations done so far for $\varphi = 0$ and $\varphi = 2$.

The latter case requires a variational principle with a subsidiary condition since a quadratic stirring spectrum enforces an equilibrium situation with a Gaussian distribution of velocity amplitudes whose variance is determined solely by the linear theory. One then obtains $z = 2 - \frac{1}{2}(d^* - d)$ for $d < d^* = 2$ and corrections $\sim k^{d-d^*}$ to the characteristic frequency of the linear theory for $d > d^*$. Random forces with quadratic stirring spectrum obeying the Einstein relation are irrelevant for turbulence questions: They do not allow energy accumulation at low or high wave numbers to set up a cascade since the energy input is balanced locally in $\vec{k}$ space against dissipation.

In nonequilibrium situations ($\varphi \neq 2$, no Einstein relation) vanishing viscosity, i.e., infinite Reynolds number, entails strong coupling of velocity modes which is characteristic for turbulence. In a stationary state the energy input is then balanced against transfer since the dissipation range is pushed to higher and higher wave numbers for $\nu \to 0$. The energy spectrum $E(k) \sim k^{-5/3 + 2(\varphi + d)/3}$ resulting from our variational principle for $\nu = 0$ was recently also derived with the Galilean invariant closure EDQNM.[7] The non-Galilean invariant DIA[25] gives according to Fournier and Frisch[7] the same power law as long as long-wavelength contributions of that power law to the total energy do not diverge, i.e., for $\varphi > 1 - d$. Our variational result bears some structural resemblance to the EDQNM but we have not investigated that point further.

Stirring forces injecting energy at a rate k^{-1} (i.e., $\varphi = -d$) into the wave-number band dk around k lead to a Kolmogorov $k^{-5/3}$ distribution of energy over wave numbers. A study of the commonly assumed universality of the Kolmogorov $\frac{5}{3}$ exponent with respect to details of injection spectra that are better limited to long wavelengths than power laws seems to be feasible within the presented framework and of interest too.

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APPENDIX A

Consider the trial functional

$$G_t = \tfrac{1}{2}\,[\phi(1) - \mu_t(1)\,]^* C_t^{-1}(12)\,[\phi(2) - \mu_t(2)\,] \qquad \text{(A1)}$$

with the variational parameters

$$\mu_t(1) = \langle \phi(1) \rangle_t \qquad \text{(A2)}$$

and with $C_t(12)$ the correlation function for fluctuations around μ_t. Since it has to be diagonal in frequency and momentum with a real, positive spectrum one finds

$$\delta G_t / \delta \mu_t^*(1) = -\,C_t^{-1}(12)\,[\phi(2) - \mu_t(2)\,]. \qquad \text{(A3)}$$

The extremal condition (4.6) with respect to the parameter set μ_t thus reads

$$0 = C_t^{-1}(12) s(2), \qquad \text{(A4)}$$

where $s(1)$ is defined by (4.9). Since $C_t(12)$ is diagonal in 1, 2, and positive (A4) implies

$$s(1) = 0, \text{ and } \langle \phi(1) \rangle_{\text{var}} = \mu_0(1)\,, \qquad \text{(A5)}$$

so that only $\mu_0 \equiv 0$ ensures $\langle \phi(1) \rangle_{\text{var}} = 0$. One can verify directly that $s(1)$ [Eq. (4.9)] vanishes for $\mu_t = 0$ caused by the coupling function $w_{\alpha\beta\gamma}(\vec{k})$ [Eq. (3.11c)] being odd in $\vec{k}$.

APPENDIX B

First we will evaluate the self-energy

$$\tilde{\Sigma}_0(12) = 4T(13,42)\langle \phi^*(3)\phi(4)\rangle_0 . \qquad \text{(B1)}$$

The projectors from the correlation function can be absorbed into the coupling functions so that

$$\tilde{\Sigma}_0(12) = 4W^*(513)D^{-1}(56)W(642)C_0(34). \qquad \text{(B2)}$$

Two integrations (summations), e.g., over 6 and 4 can easily be done to yield

$$\Sigma_0(12) = \Delta(p_1 - p_2) P_{\alpha_1 \alpha_1'}(\vec{k}_1) \times \Sigma^0_{\alpha_1' \alpha_2'}(\vec{k}_1) P_{\alpha_2' \alpha_2}(\vec{k}_1)\,, \qquad \text{(B3a)}$$

$$\Sigma^0_{\alpha\beta}(\vec{k}) = \int_{p_5}\int_{p_3} \Delta(p_5 - p_3 - p)\,\frac{C_0(p_3)}{D(k_5)} \times F_{\alpha\beta}(\vec{k}_3, \vec{k}_5) \qquad \text{(B3b)}$$

with

$$F_{\alpha\beta}(\vec{q}, \vec{\kappa}) = P_{\nu\mu}(\vec{q}) P_{\lambda\sigma}(\vec{\kappa}) w_{\lambda\nu\alpha}(\vec{\kappa}) w_{\sigma\mu\beta}(\vec{\kappa})\,. \qquad \text{(B4)}$$

The integrals over ω_5 and ω_3 can be performed too with the final result

$$\Sigma^0_{\alpha\beta}(\vec{k}) = \int_{\vec{\kappa}}\int_{\vec{q}} (2\pi)^d \delta(\vec{\kappa} - \vec{q} - \vec{k}) C_0(q, t=0) \times \frac{\kappa^2}{D(\kappa)} f_{\alpha\beta}(\vec{q}, \vec{\kappa})\,, \qquad \text{(B5)}$$

where we replaced $F_{\alpha\beta}(\vec{q}, \vec{\kappa})$ [Eq. (B4)] by $\kappa^2 f_{\alpha\beta}(\vec{q}, \vec{\kappa})$. The factor κ^2 comes from the square of the coupling functions and the matrix

$$f_{\alpha\beta}(\vec{q}, \vec{\kappa}) = \delta_{\alpha\beta}\left[1 - \left(\frac{\vec{q}\cdot\vec{\kappa}}{q\kappa}\right)^2\right] + \frac{\kappa_\alpha \kappa_\beta}{\kappa^2}\left[d - 3 + 4\left(\frac{\vec{q}\cdot\vec{\kappa}}{q\kappa}\right)^2\right] - \frac{q_\alpha \kappa_\beta + \kappa_\alpha q_\beta}{q\kappa}\left(\frac{\vec{q}\cdot\vec{\kappa}}{q\kappa}\right) \qquad \text{(B6)}$$

is dimensionless.

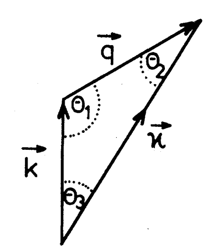
In the following the trace over Eq. (B3a) with respect to α_1, α_2 will be determined. It turns out that

$$P_{\alpha\beta}(\vec{k}) f_{\beta\gamma}(\vec{q}, \vec{\kappa}) P_{\gamma\alpha}(\vec{k}) = (d-1) f(x, y, z)\,, \qquad \text{(B7)}$$

$$(d-1) f(x,y,z) = 2(d-2) - 4y^2 z^2 - 2xyz + (3-d)(y^2 + z^2) \qquad \text{(B8)}$$

depends only on the three cosines

$$-\frac{\vec{k}\cdot\vec{q}}{kq} = x = \cos\theta_1\,, \quad \frac{\vec{q}\cdot\vec{\kappa}}{q\kappa} = y = \cos\theta_2\,, \quad \frac{\vec{k}\cdot\vec{\kappa}}{k\kappa} = z = \cos\theta_3\,, \qquad \text{(B9)}$$



of the triangle formed by $\vec{k}$, $\vec{q}$, $\vec{\kappa}$. One thus has

$$P_{\alpha\beta}(\vec{k}) \Sigma_{\beta\gamma}(\vec{k}) P_{\gamma\alpha}(\vec{k}) = (d-1)\int_{\vec{\kappa}}\int_{\vec{q}} (2\pi)^d \delta(\vec{\kappa} - \vec{q} - \vec{k}) \times C_0(q, t=0)\,\frac{\kappa^2}{D(\kappa)} f(x,y,z)\,. \qquad \text{(B10)}$$

Incidentally $a^{(d)}_{\kappa q k} = \frac{1}{4}(d-1) f(x,y,z)$ appears as kinematical coefficient[14,16] in the numerous and widely used second-order spectral equations approximating the dynamics of turbulence. Using trigonometric transformations and the boundedness of the cosines one can show[14,16]

$$0 \leq f(x, y, z) \quad \text{for } 2 \leq d. \tag{B11}$$

A crude upper bound for $f(x,y,z)$ is easily obtained to be $4/(d-1)$ for $2 \leq d \leq 3$ and 2 for $d \geq 3$.

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