Connectance of dynamical systems with increasing number of degrees of freedom

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A set of n coupled two-dimensional area-preserving mappings in taken as a model problem for the study of the stochasticity of dynamical systems with n degrees of freedom. The connectance of the system is defined as the percentage of two-dimensional mappings that are directly coupled. This work suggests that large complex dynamical systems may be expected to be stable below some critical level of connectance, but that as the connectance increases above that level they would become unstable.

I. INTRODUCTION

The behavior of dynamical systems is deeply influenced by the existence of isolating integrals and by the size of the domain of the phase space where such integrals exist.[1] Numerical experiments on a one-dimensional self-gravitating system have shown that the proportion of the measure of the ergodic domain with respect to the whole volume of the phase space increases very rapidly as the number of degrees of freedom increases.[2] This result, however was obtained with a system that is fully connected, i.e., where every variable has a direct effect on every other variable.

For another dynamical system, Casartelli *et al.*[3] using a Lennard-Jones potential between neighboring particles, found that, even for a large number of degrees of freedom, non-negligible integrable zones remained. Here the amount of connectedness ("connectance") is very small. Only neighboring particles interact. In many dynamical systems the connectance is far below 100%. Not every person in a slum has an immediate effect on every other person and not every cell in the brain affects every other cell directly[4]; in a planetary system each planet interacts with all the others though there is a dominant attraction to the massive star. The ratios of the masses of the planets to the mass of the star measure the connectance. We have studied, using a discrete nonlinear dynamical system as a model, the effect of the connectance on the existence and size of the domain of phase space where the system behaves as though it were integrable. The model is a $2n$-dimensional symplectic mapping built from n two-dimensional area-preserving mappings; such mappings have been studied extensively in the last few years and display the well-known features of conservative dynamical systems with two degrees of freedom.

II. THE MAPPING AS A MODEL

Let us consider $N=2n$ variables $x_1,\ldots,x_n$, $y_1,\ldots,y_n$, and $S(y_1,\ldots,y_n)$, an arbitrary function. The mapping

$$A\begin{cases} x'=x+f(y)\\ y'=y\end{cases}$$

with $f(y)=-\partial S/\partial y$ is canonical and one to one. It is also a symplectic mapping, i.e., if M is the Jacobian matrix of A we have ${}^tM\,JM=J$, where ${}^tM$ is the transpose matrix of M and

$$J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}.$$

In our model we have taken a mapping $T=A_1\circ A_2$ from the two symplectic mappings A_1 and A_2 generated by

$$S_1=\sum_{i=1}^{n}a_i\cos y_i+b\sum_{i=1}^{n}\cos\Big(\sum_{j=1}^{n}\alpha_{ij}y_j\Big),$$

$$A_1\begin{cases} x_i'=x_i-\dfrac{\partial S}{\partial y_i},\\ y_i'=y_i-\dfrac{\partial S_2}{\partial x_i} \quad i=1,\ldots,n.\end{cases}$$

$$S_2=-\sum_{i=1}^{n}\tfrac{1}{2}x_i^2,$$

$$A_2\begin{cases} x_i'=x_i,\\ y_i'=y_i-\dfrac{\partial S_2}{\partial x_i},\quad i=1,\ldots,n.\end{cases}$$

(The roles of x and y are exchanged in the definition of A_2.) Finally we obtain the mapping T:

$$T\begin{cases} x_i'=x_i+a_i\sin(x_i+y_i)\\ \qquad +b\displaystyle\sum_{j=1}^{n}\alpha_{ij}\sin\Big(\sum_{k=1}^{n}\alpha_{jk}(x_k+y_k)\Big),\\ y_i'=x_i+y_i;\quad i=1,\ldots,n.\end{cases}$$

We have taken $\alpha_{ij}=\alpha_{ji}$ and

$$\alpha_{ij}=1 \quad \text{for } j-i<N_c$$
$$=0 \quad \text{for } j-i\geqslant N_c$$

with $j\geqslant i$.

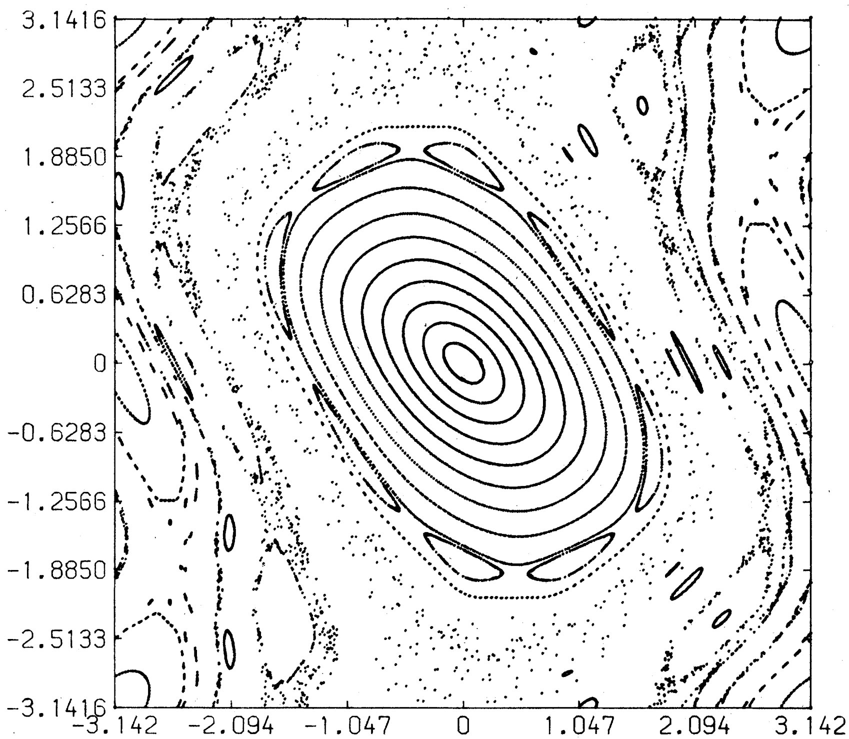

FIG. 1. The mapping τ_1 for $a_1=-1$.

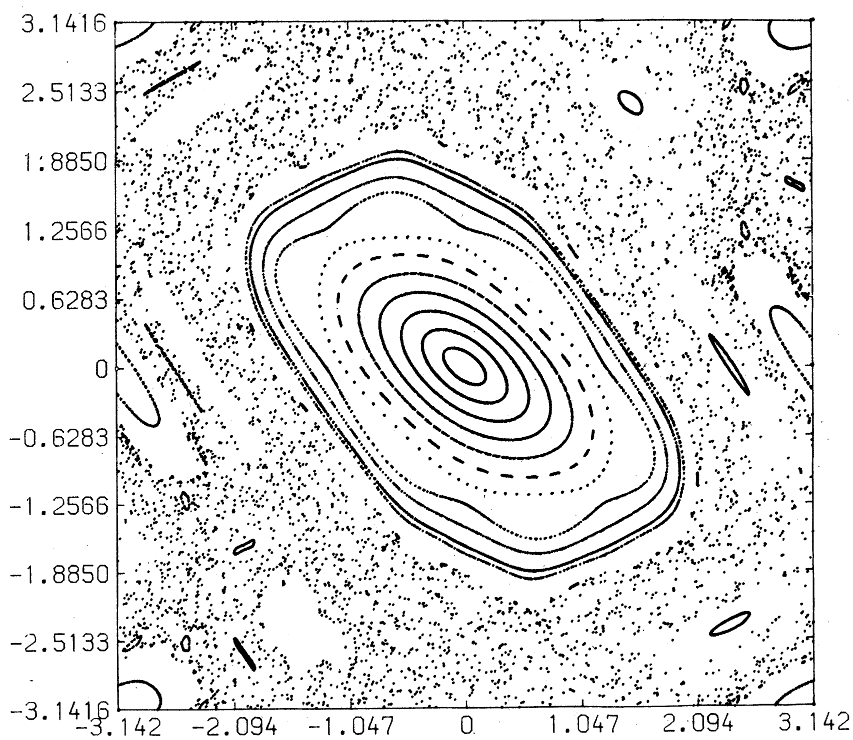

FIG. 2. The mapping τ_1 for $a_1=-1.3$.

TABLE I. Data for Figs. 1 and 2. (x_0, y_0) are the initial conditions and M the total number of iterations for each orbit.

Fig.	α_1	x_0	y_0	M
1	−1	0.1571	0	500
		0.3142	0	500
		0.4712	0	500
		0.6283	0	500
		0.7854	0	500
		0.9425	0	500
		1.0996	0	500
		1.2566	0	500
		1.4137	0	500
		1.5708	0	500
		1.7279	0	500
		1.8850	0	500
		2.0402	0	500
		2.1991	0	500
		2.5133	0	500
		2.6704	0	500
		2.8274	0	500
		2.9845	0	500
		3.1416	0	500
		0.6300	−1.8850	500
		0.6400	−1.8850	500
		0.6280	−1.8850	500
		−1.7500	−1.7000	500
		−2.0900	−2.5100	500
		−2.2000	0	500
		−3.1400	1.8800	500
		−2.5100	−1.9000	500
		2.5000	2.5100	500
2	−1.3	0.1571	0	500
		0.3142	0	500
		0.4712	0	500
		0.6283	0	500
		0.7854	0	500
		0.9425	0	500
		1.0996	0	500
		1.2566	0	500
		1.4137	0	500
		1.5708	0	500
		1.7279	0	500
		1.8850	0	500
		2.0420	0	500
		2.1991	0	500
		2.3562	0	500
		2.5133	0	500
		2.6704	0	500
		2.8274	0	500
		2.9845	0	500
		3.1416	0	500
		0.6300	−1.8850	500
		0.6400	−1.8850	500
		0.6280	−1.8850	500
		−1.7500	−1.7000	500
		−2.0900	−2.5100	500
		−2.2000	0	500
		−3.1400	1.8800	500
		−2.5000	1.9000	500
		2.5000	2.5100	500

We call N_c the connectance number. When $N_c = 1$, the mapping T is the product of n two-dimensional mappings τ_i,

$$\tau_i \begin{cases} x_i' = x_i + (a_i + b)\sin(x_i + y_i) \\ y_i' = x_i + y_i . \end{cases}$$

When $N_c = n - 1$ the connectance between the mappings τ_i is maximum; i.e., each mapping τ_i interacts directly with the others with the same strength. Already, for $N_c = \frac{1}{2}(n+2)$ if n is even and $N_c = \frac{1}{2}(n+1)$ if n is odd, each mapping interacts directly with the others but not with the same strength.

The mapping T has been studied extensively for $N = 4$ by Froeschlé[5] and Froeschlé and Scheidecker.[6] In particular, the shape of the set of points (i.e., the sections of the invariant manifolds and their disappearance) can be clearly illustrated with visual methods. We take at random the a_i between 1 and 1.3. Figures 1 and 2 display typical sets of points for the mapping τ_i with initial conditions given in Table I. Figures 1 and 2 exhibit all the characteristics and well-known features of problems with two degrees of freedom, such as invariant curves and islands, which correspond to the existence of isolating integrals, and also wild zones, sometimes called "ergodic," where the points seem to fill a broad region in the plane and which correspond to the nonexistence of isolating integrals. When a_i varies from −1.3 to −1, there is a continuous deformation of the invariant-curves zone around the invariant point (0, 0) and a shrinking of the ergodic zone. With n ranging from 10 to 50 it is too costly to explore the whole $2n$-dimensional phase space. Therefore, we have looked only at the dimension of the zone of stability (or integrable zone) around the point $x_i = 0,\ y_i = 0,\ i = 0,\ i = 1\text{–}n$.

Among the various possible tests of the stochasticity of a dynamical system, the divergence of two initially close orbits has been selected,[7] this being the cheapest and quickest test. Indeed, in the wild zones, the divergence of such orbits is roughly exponential, in opposition to the linear divergence in the integrable zone. For a given mapping T (n and N_c given), we have taken the initial condition of the first orbit along the line Δ: $x_i = y_i = \delta$, $i = 1, \ldots, n$ with a step equal to $\frac{1}{80}\pi$. In the second orbit, the initial value of the coordinate x_i is increased by a quantity $\epsilon = 10^{-10}$. We define the distance between these two orbits in the phase space by

$$d_K = \left[\frac{1}{n} \left(\sum_{j=1}^{n} [x_j(K) - x_j'(K)]^2 + \sum_{j=1}^{n} [y_j(K) - y_j'(K)]^2 \right) \right]^{1/2} ,$$

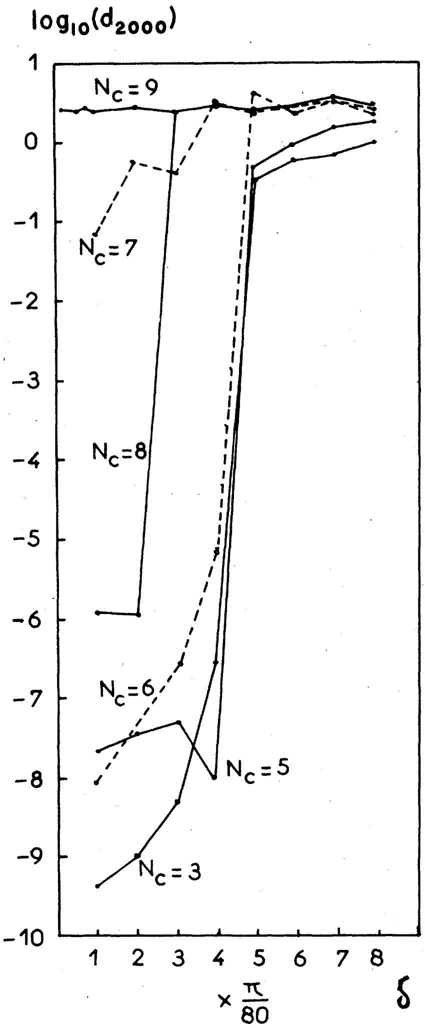

FIG. 3. Variation of d_{2000} as a measure of the stochasticity plotted against δ for $N = 30$, $b = 0.01$ and different values of the connectance number N_c.

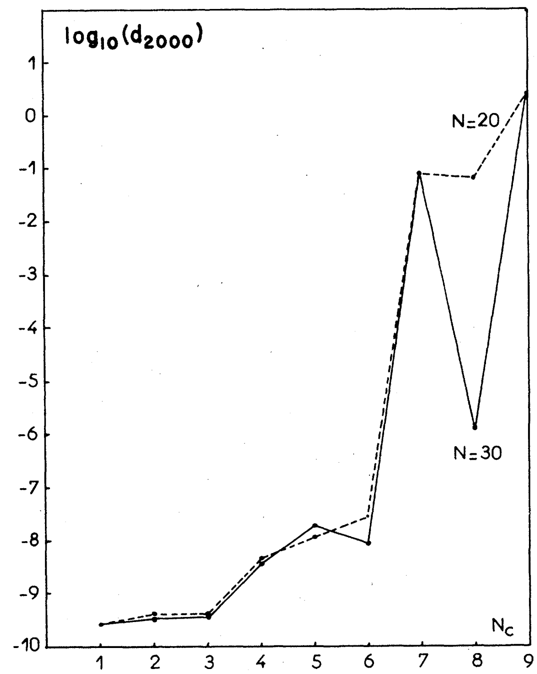

FIG. 4. Variation of d_{2000} as a measure of the stochasticity plotted against the connectance number N_c for $N = 20$ and 30 with the same initial conditions $x_i = y_i = \frac{1}{80}\pi$.

where $x_j x_j$ and $x_j' y_j'$ refer to the first and the second orbits, respectively. We take $K = 2000$. Figure 3 shows the variations of $\log_{10} d_{2000}$ when the initial conditions vary along the line Δ for different values of N_c, N being taken equal to 30. For $N_c = 3$ and 6 we see a drastic increase of d_{2000} for $x_i = \frac{1}{20}\pi$, which suggests the existence of an integrable zone around the origin and a sharp transition zone to the stochastic zone. This was well known for $N = 2$ and $N = 4$. On the other hand for $N_c = 9$ the integrable zone has disappeared. Figure 4 shows the variations of $\log_{10} d_{2000}$ for given initial conditions when N_c increases. As suggested already by Fig. 3 the transition is rather sharp. We note an interesting phenomenon for $N_c = 8$ and $x_i = \frac{1}{80}\pi$: we still have an integrable behavior in opposition to the wild behavior for $N_c = 7$. By analogy to the well-known features of dynamical systems with two degrees of freedom we will call this an island phenomenon. Figure 5 summarizes the results obtained for $N = 10, 20, 30, 40, 50$. We show for different values of the connectance number N_c the dimension of the main continent. For N small (even for 10), there is a continuous change. But already for $N = 20$ there is a region of steep decline, and above a certain critical value, $N_c = 8$, the integrable zone has a very small measure. Thus, even for $N = 20$, questions of stability can be answered simply by asking whether the connectance number N_c is above

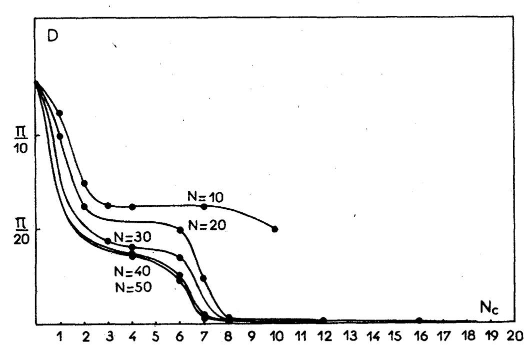

FIG. 5. Dimension of the main continent as a function of the connectance number N_c.

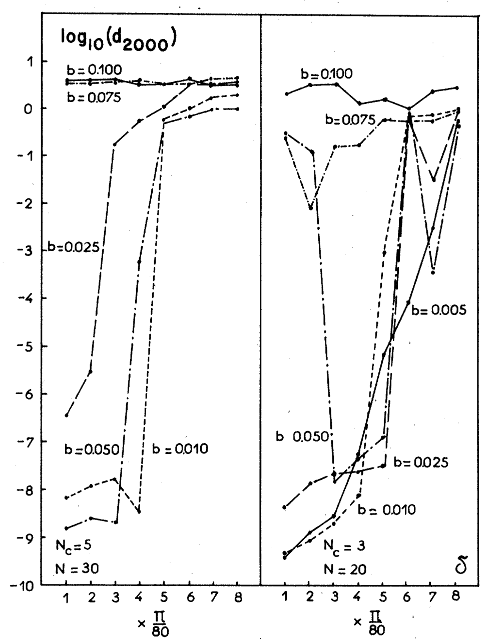

FIG. 6. Variation of d_{2000} as a measure of the stochasticity plotted against δ for $N=30$, $N_c=5$ and $N=20$, $N_c=3$ and different values of the coupling number b.

or below the critical value.

This feature repeats as N increases. The curves are shifted to the left but for $N=40$ and $N=50$ they are almost the same and for $N_c \geqslant N_c'$ ($N_c'=7$ is called the critical value of the connectance number N_c), the integrable zone has disappeared. If we consider the values of D for $N_c=6$, the separations are approximately $(12, 7, 3)\frac{1}{80}\pi$ which looks like a geometric progression. This is a further evidence that N_c has a nonzero limit when N tends to infinity. However, if we consider the connectance $c=N_c/N$ instead of N_c the critical value c' of c decreases with N. Gardner and Ashby[4] have shown for linear systems where the connectance is defined as the percentage of nonzero values in the distribution of the matrix elements (i.e., a kind of diffuse connectedness) that above $N=10$ questions of stability can be answered only by asking whether the connectance c (c is defined as the percentage of nonzero values in the distribution of the matrix elements) is below or above a fixed value $c'=0.13$. In the present model we find a similar kind of answer, but for N_c, not c, below or above 7. The reason for this is that in the mapping T there is a local connectedness: i.e., the mappings τ_i interact only with their neighbors.

Figure 6 shows the variations of $\log_{10}(d_{2000})$ with distance to the origin when N and N_c are given ($N=30$, $N_c=5$ and $N=20$, $N_c=3$) but for different values of the coupling number b. As expected, we see that the ergodicity of the system increases with b with however an exception for $N=20$, $N_c=3$, $b=0.05$, $\frac{3}{80}\pi<\delta<\frac{5}{80}\pi$ for which we have again an island phenomenon.

III. CONCLUSION

It appears that for this particular dynamical system the dimension of the zone of stability around the origin depends not only on the number of degrees of freedom but also on the connectance. Moreover, if N is quite large ($N \geqslant 40$) it seems to depend only on the connectance number. Above a certain critical value of this number the system is unstable and below it there is quite a large domain of stability. Of course there are only numerical results for a particular model: Casati and Ford[8] have shown numerically the total integrability of the Calogero Hamiltonian where each particle interacts with all other particles, i.e., maximum connectance. But integrable systems are particular cases of measure zero in the class of dynamical systems. On the other hand some preliminary results that we have obtained with Schneidecker using an isolated one-dimensional self-gravitating system consisting of N plane parallel sheets with a predominant mass seem to show the same results as those of the present paper.

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