Ray propagation and self-focusing in nonlinear absorbing media

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Two previously given theories are unified, facilitating the discussion of ray tracing in nonlinear and absorbing media. The media may be fairly general: dispersive, absorbing, anisotropic, inhomogeneous, and time varying. Self-focusing is demonstrated on a simple example, showing that absorption will slow down the buildup of high intensities expected from the convergence of the rays.

I. INTRODUCTION

The problem of ray propagation in weakly nonlinear systems has been discussed recently[1] for a general class of media. The theory has then been developed in a way that takes into account the structure of a ray bundle, facilitating the discussion of self-focusing effects.[2] The geometrical-optics approach towards self-focusing problems seems to be of great potential, and there are many specific problems waiting to be solved by machine computations.

It must be realized that the present model[1] is deficient in that it does not take into account the effects of absorption. To some extent, these are invariably present in all media. This is not a trivial problem since even for linear media the effect of losses on the ray equations is not yet fully explored. Recently[3] a method has been proposed for real ray tracing in absorbing media. The method facilitates the computation of real paths of wave packets in linear, dispersive, inhomogeneous, time-varying, and anisotropic media. It has been shown[4] that the same results can be derived by postulating a variational principle, which is an extension of the well-known Fermat principle. Since the formalism for ray tracing in nonlinear media can also be derived from an extended Fermat principle,[2] the two problems can be combined in a straightforward fashion.

The equations for rays in real space-time propagating in nonlinear absorbing media are presented. A simple example[2] is considered for the present case. The analysis shows that absorption and self-focusing are competitive phenomena. While power density along the beam is increased owing to self-focusing, power is also lost owing to dissipation in the medium. This qualitative conclusion could be anticipated, no doubt, from a simple intuitive discussion. However, with the present model it is feasible to compute the ray paths and the amplitudes along them.

II. GENERAL THEORY

Most of the preliminary material pertinent to the present discussion is given in Censor.[1,2,4] In the interest of space saving, formulas cited from Refs. 2 and 4 will be indicated by (... [a]) and (... [b]), respectively.

We start with Maxwell's equations (1[a]) and assume, instead of (2[a]), a solution of the form

$$E_i = \sum_{m=-\infty}^{\infty} E_{m,i} \exp(-|m| \operatorname{Im}\theta) \exp(im \operatorname{Re}\theta), \quad (1)$$

where the phase θ (3[a]) is now in general complex, since in absorbing media complex $\vec{K}$ must be allowed. Consequently (1) is no more a periodic solution. However, there is no point in considering wave packets and their evolution in space-time unless the attenuation over a distance of a wavelength and a time of a period are very small. Otherwise the concept of a ray path becomes meaningless. We shall therefore assume that $\operatorname{Im}\theta/\operatorname{Re}\theta \ll 1$ such that the wave (1) can be considered, to a good approximation as a series of harmonics. This leads again to the algebraic forms (6[a]) and (7[a]) for the fundamental harmonic. See also Ref. 1. Hence we end up with a system of homogeneous nonlinear equations.

$$G_r(\vec{K}, \vec{a}; \vec{X}) = 0, \quad r = 1, \ldots, 6 \quad (2)$$

where $\vec{a} = (\vec{E}, \vec{H})$ is a six-component vector of the electric and magnetic fields (8[a]). The determinant of the system of equations (2) must vanish, leading to the dispersion equation (10[a]),

$$F(\vec{K}, \vec{a}:\vec{X}) = 0, \quad (3)$$

and (2) and (3) with complex $\vec{K}$ provides the basis for deriving the relevant ray equations.

The extended Fermat principle[5] is now stipulated

$$0 = \delta \int \vec{K} \cdot d\vec{X}, \quad (4)$$

with fixed end points in space-time. Since $\vec{K}$ and $\vec{X}$ are dependent through (2) and (3), these constraints must be included in the integral by adequately using Lagrange multiplier functions. For lossless nonlinear media this can be written in the form (11[a]). For absorbing media and wave packets propagating in real space-time, we have to add the appropriate constraint as in (33[b]). Hence for the present case of nonlinear absorbing media the extended Fermat principle is written in the form

$$0=\delta\int\left(\vec{K}\cdot\frac{d\vec{X}}{d\tau}+\lambda(F+\lambda_r G_r+\Lambda_i X_i)\right)d\tau\,, \tag{5}$$

where λ is the Lagrange multiplier associated with $F=0$ [Eq. (3)], and is taken as a common multiplier outside the parentheses for convenience, $\lambda\lambda_r$ stand for the six Lagrange multipliers associated with $G_r=0$, and $\lambda\Lambda_i$, $i=1,\ldots,4$ takes care of the four conditions which constrain $\vec{X}$ to be in real space-time. For details see Ref. 4. The end points of the integral (5) are chosen in real space-time, i.e., $I\vec{X}=0$, where the symbol I, as well as R are defined in (9[b]).

The Euler equations associated with (5) are obtained by varying $\vec{K}$, $\vec{a}$, $\vec{X}$:

$$\begin{aligned}
\frac{d\vec{X}}{d\tau}&=-\lambda\left(\frac{\partial F}{\partial\vec{K}}+\lambda_r\frac{\partial G_r}{\partial\vec{K}}\right),\\
\frac{d\vec{K}}{d\tau}&=\lambda\left(\frac{\partial F}{\partial\vec{X}}+\lambda_r\frac{\partial G_r}{\partial\vec{X}}+\vec{\Lambda}\right),\\
\frac{\partial F}{\partial a_s}&+\lambda_r\frac{\partial G_r}{\partial a_s}=0\,.
\end{aligned}\tag{6}$$

Elimination of λ_r yields

$$\frac{d\vec{X}}{d\tau}=-\lambda F_{\vec{K}}\,,\quad \frac{d\vec{K}}{d\tau}=\lambda(F_{\vec{X}}+\vec{\Lambda})\,, \tag{7}$$

where the operator, defined by the subscript, is given by

$$F_l=\frac{\partial F}{\partial l}-\frac{\partial F}{\partial a_s}\left(\frac{\partial G_r}{\partial a_s}\right)^{-1}\frac{\partial G_r}{\partial l}\,, \tag{8}$$

denoting any component of $F_{\vec{K}}$ and $F_{\vec{X}}$ standing for the vectors with the components F_{K_i} and F_{X_i}, $i=1,\ldots,4$.

The special case of lossless media is obtained by setting $\vec{\Lambda}\equiv 0$, then (7) and (8) are essentially (13[a]), the latter already written in three-dimensional form with t as the parameter. On the other hand, for linear media $\partial F/\partial a_s\equiv 0$, since in linear media the dispersion equation does not contain amplitudes. Hence (7) and (34[b]) become identical (note that the definition of λ is slightly different for the two formulas). Without imposing the above special cases, (7) and (8) constitute the desired generalization which combines the effects of nonlinearity and dissipation of the medium. Continuing as in (36[b]) we now have

$$\frac{d\vec{X}}{d\tau}=-\lambda RF_{\vec{K}}\,,\quad IF_{\vec{K}}=0\,, \tag{9}$$

which in three-dimensional notation, with t as the parameter, becomes

$$\vec{V}=\frac{d\vec{x}}{dt}=-\mathrm{Re}\frac{F_{\vec{k}}}{F_\omega}\,,\quad \mathrm{Im}\frac{F_{\vec{k}}}{F_\omega}=0\,. \tag{10}$$

Similarly to (38[b]) and with $R\vec{\Lambda}\equiv 0$, we now have

$$I\vec{\Lambda}=(RF_{\vec{K}\vec{K}})^{-1}I(F_{\vec{X}}\cdot F_{\vec{K}\vec{K}}-F_{\vec{K}}\cdot F_{\vec{X}\vec{K}})\,,$$

providing the definition of $\vec{\Lambda}$ in (7). In three-dimensional notation, with t as the parameter, the analog of (41[b]) becomes (10) and

$$\begin{aligned}
\frac{d\vec{k}}{dt}&=\frac{F_{\vec{x}}}{F_\omega}+i\vec{\beta}\,,\\
\frac{d\omega}{dt}&=-\frac{F_t}{F_\omega}+i\vec{V}\cdot\vec{\beta}\,,\\
\vec{\beta}&=-\mathrm{Re}(\vec{V}_{\vec{k}}+\vec{V}_\omega\vec{V})^{-1}\,\mathrm{Im}\left(\vec{V}_{\vec{k}}\cdot\frac{F_{\vec{x}}}{F_\omega}-\vec{V}_\omega\frac{F_t}{F_\omega}+\vec{V}_t+\vec{V}_{\vec{x}}\cdot\vec{V}\right).
\end{aligned}\tag{11}$$

To complete the ray equations, the evolution of $\vec{a}$ along the ray path must be given. This is obtained from the fact that G_s must be satisfied along the path, i.e.,

$$\frac{dG_s}{d\tau}=0=\frac{\partial G_s}{\partial\vec{K}}\cdot\frac{d\vec{K}}{d\tau}+\frac{\partial G_s}{\partial\vec{X}}\cdot\frac{d\vec{X}}{d\tau}+\frac{\partial G_s}{\partial a_r}\cdot\frac{da_r}{d\tau}\,, \tag{12}$$

yielding,

$$\frac{da_r}{d\tau}=-\left(\frac{\partial G_s}{\partial a_r}\right)^{-1}\left(\frac{\partial G_s}{\partial\vec{K}}\cdot\frac{d\vec{K}}{d\tau}+\frac{\partial G_s}{\partial\vec{X}}\cdot\frac{d\vec{X}}{d\tau}\right). \tag{13}$$

In three-dimensional notation, with t as the parameter, we get

$$\frac{da_r}{d\tau}=-\left(\frac{\partial G_s}{\partial a_r}\right)^{-1}\left(\frac{\partial G_s}{\partial\vec{k}}\cdot\frac{d\vec{k}}{dt}+\frac{\partial G_s}{\partial\omega}\frac{d\omega}{dt}+\frac{\partial G_s}{\partial\vec{x}}\cdot\frac{d\vec{x}}{dt}+\frac{\partial G_s}{\partial t}\right). \tag{14}$$

Thus we have the complete information for ray tracing, by which individual noninteracting rays may be described.

For completeness, it must also be shown that $F=0$ [Eq. (3)] is satisfied along the ray path in the same manner. Thus

$$\frac{dF}{d\tau}=0=\frac{\partial F}{\partial\vec{K}}\cdot\frac{d\vec{K}}{\partial\tau}+\frac{\partial F}{\partial\vec{X}}\cdot\frac{d\vec{X}}{dt}+\frac{\partial F}{\partial a_r}\cdot\frac{da_r}{d\tau} \tag{15}$$

must be compatible with (13). Indeed, if we multiply (13) by $\partial F/\partial a_r$ and use (6) and $\vec{\Lambda}\cdot(d\vec{X}/d\tau)=0$, we get (15). In the linear case (15) is satisfied identically, both for lossless and lossy media. In the linear case we also have $F=0=\det(\partial G_s/\partial a_r)$, hence (13) is not applicable. This is consistent with the well-known fact that ray tracing in linear media cannot yield the amplitude and additional assumptions are necessary.

The above developed theory deals with tracing of individual rays. In order to bring out typical nonlinear effects, such as self-focusing, the characteristics of a bundle of rays, or otherwise termed, a beam, are essential. Since the advent of intensive research of nonlinear electromagmetic phenomena,[6] it is well known that the gradients of the fields play a dominant role (even the title of Askar'yan's paper indicates this). For the present formalism this means that instead of (12), we have to ensure that $G_s=0$ is satisfied throughout the beam. Therefore (15[a) is stipulated,

$$\frac{\partial a_r}{x_i} = -\left(\frac{\partial G_s}{\partial a_r}\right)^{-1}\left(\frac{\partial G_s}{\partial k_j}\frac{\partial k_j}{\partial x_i}+\frac{\partial G_s}{\partial \omega}\frac{\partial \omega}{\partial x_i}+\frac{\partial G_s}{\partial x_i}\right),$$
$$\frac{\partial a_r}{\partial t} = -\left(\frac{\partial G_s}{\partial a_r}\right)^{-1}\left(\frac{\partial G_s}{\partial k_j}\frac{\partial k_j}{\partial t}+\frac{\partial G_s}{\partial \omega}\frac{\partial \omega}{\partial t}+\frac{\partial G_s}{\partial t}\right). \tag{16}$$

Clearly this is stronger than (14), but contains it. To evaluate now (10), (11), and (16), a bundle of rays must be followed and gradients of fields within the bundle play a part. This is the reason for the emergence of the self-focusing effect in this geometrical-optics analysis.

III. SIMPLE EXAMPLE

Analytic investigation of special examples, using the above given theory, is hopelessly complicated. But, as done previously (Ref. 2), we are still able to consider the simplest case of a homogeneous, isotropic, and time-independent medium. The nonlinearity will be restricted to the dielectric properties of the medium, and only $\epsilon^{(2)}$, the first nonlinear effect will be retained. The losses will be included by replacing $\epsilon^{(1)}$ (16[a]) by $\epsilon+i\sigma/\omega$, where σ is the conductance, taken here as a constant. It follows (16[a]) that (2) and (3) have now the form

$$G=FE=[-k^2+\omega^2\mu(\epsilon+i\sigma/\omega+E\epsilon^{(2)})]E=0\,. \tag{17}$$

We have to satisfy the boundary condition $IF_{\vec{k}}=0$ [Eq. (9)] or $\mathrm{Im}(F_{\vec{k}}/F_\omega)=0$ [Eq. (10)]. It seems inconsistent that the first condition (9) constitutes four scalar equations while the second one (10) constitutes only three. The reason for that is that the representation of F is not unique, in this context, and has to be taken as $fF=0$, where f is resolved by the fourth condition $IfF_{\vec{k}}=0$. This problem does not arise in (10) since in $\mathrm{Im}(F_{\vec{k}}/F_\omega)=0$, f is eliminated.

Let us assume a beam of parallel rays entering the nonlinear absorbing medium perpendicularly to a plane interface. At the boundary the tangential components of $\vec{k}$ are continuous, owing to (4[a]), i.e., the Sommerfeld-Runge law of refraction $\nabla\times\vec{k}=0$. This is identically satisfied here. Also the time dependence of the fields is conserved at the boundary, in order that the tangential components of the fields be continuous. What happens to a wave packet as it enters the absorbing medium is explained by Censor (Ref. 4), using the Gaussian pulse as an example. The wave excites the absorbing medium at the boundary (42[b]) and propagates into the medium with complex $\vec{k}$, ω and a deformed envelope (46[b]). This trade-off between envelope and carrier allows for the appearance of the complex frequency. Following the envelope we have to take the carrier at $\vec{x}, t$ prescribed by the motion of the wave packet, accordingly the complex $\vec{k}$, ω of the carrier describe the attenuation of the wave in the absorbing medium.

In computing $F_{\vec{k}}/F_\omega$ we again note (Ref. 2) that since F and G [Eq. (17)] are proportional this expression is indeterminate. Taking $F=F(\vec{k},\omega)$, $G=G(\vec{k}',\omega')$ and finding the limit as $\vec{k}'\to k$, $\omega'\to\omega$ by l'Hospital's rule, we get

$$\frac{F_{\vec{k}}}{F_\omega}=\frac{\partial F/\partial\vec{k}}{\partial F/\partial\omega} \tag{18}$$

for this case. Now it is easy to verify that $\mathrm{Im}(F_{\vec{k}}/F_\omega)=0$ and $F=0$ are compatible if k is real and $\omega=\omega_0+i\eta$,

$$\eta=-\sigma/2(\epsilon+E\epsilon^{(2)})\,, \tag{19}$$

where ω_0 is the frequency on the boundary and η is the imaginary part of the frequency in the absorbing medium. Since from (11) $\vec{\beta}=0$, $F_t\equiv 0$, $d\omega/dt$ vanishes. In view of

$$\frac{d\omega}{dt}=\frac{\partial\omega}{\partial t}+\frac{\partial\omega}{\partial\vec{x}}\cdot\frac{d\vec{x}}{dt},$$
$$\frac{\partial\omega}{\partial\vec{x}}+\frac{\partial\vec{k}}{\partial t}=0, \tag{20}$$

and $\partial\omega/\partial t=0$, $\partial\vec{k}/\partial t=0$, ω is a constant; the second line (20) is given by the consistency relations (4[a]). The waves have therefore a factor of $\exp(\eta t)$ describing the attenuation along the ray.

According to (17) and (19), ω_0, $\vec{k}$ are related by

$$\mathrm{Re}F=-k^2+(\omega_0^2-\eta^2)\mu(\epsilon+E\epsilon^{(2)})-\mu\eta\sigma=0\,, \tag{21}$$

while $\mathrm{Im}F=0$ yields again (19).

Similarly to (20), (11) prescribes

$$\frac{dk_i}{dt}=\frac{\partial k_i}{\partial t}+\frac{\partial k_i}{\partial x_j}\frac{dx_j}{dt}=0\,, \tag{22}$$

but $\partial k_i/\partial x_j = 0$ is not implied and must be investigated. Finally (16) prescribes for the present problem

$$\frac{\partial a_r}{\partial x_i} = -\left(\frac{\partial G_s}{\partial a_r}\right)^{-1} \frac{\partial G_s}{\partial k_j}\frac{\partial k_j}{\partial x_i}, \qquad (23)$$

connecting the changes of k with the field gradients. The group velocity, according to (10) is given by

$$\frac{dx_i}{dt} = \frac{k_i}{\omega_0 \mu(\epsilon + E\epsilon^{(2)})}, \qquad (24)$$

which in lossless media becomes (18[a]). From (23) we get

$$\frac{\partial E}{\partial x_i} = \frac{2k_j}{\omega^2 \mu \epsilon^{(2)}}\frac{\partial k_j}{\partial x_i}, \qquad (25)$$

which is the analog of (19[a]), but here ω is complex. It follows that if $\vec{k}$ is taken real, as above, then the boundary condition $\partial E/\partial x_i$ must allow for complex E. This point is not further pursued here.

We are now ready to discuss the self-focusing effect. In view of (22) and k perpendicular to the boundary, say, in direction x_1, we have

$$\frac{\partial k_i}{\partial x_1}\frac{dx_1}{dt} = 0, \quad i = 1, 2 \qquad (26)$$

i.e., $\partial k_1/\partial x_1 = 0$, $\partial k_2/\partial x_1 = 0$, and in view of $\nabla \times \vec{k} = 0$, also $\partial k_1/\partial x_2 = 0$. Therefore the gradient of E must be in direction x_2, and from (25) we have

$$\frac{\partial E}{\partial x_2} = \frac{2k_2}{\omega^2 \mu \epsilon^{(2)}}\frac{\partial k_2}{\partial x_2} = \frac{1}{\omega^2 \mu \epsilon^{(2)}}\frac{\partial k_2^2}{\partial x_2}. \qquad (27)$$

This shows that owing to the nonlinear effects a component k_2 is created. Since the intensity falls off from the center of the beam outwards, we have $\partial E/\partial x_2 < 0$, hence for positive $\epsilon^{(2)}$ also $\partial k_2^2/\partial x_2 < 0$. The present result (27) is consistent with k_2 being negative and $\partial k_2/\partial x_2$ being positive, i.e., the rays converge towards the axis and the effect is larger near the axis, where the field intensity is large. This analytical treatment considers only the trend of the rays near the boundary. A detailed numerical scheme is yet to be found which will trace the beam inside the medium.

IV. CONCLUSIONS

The unification of two theories, both derived from a variational principle, i.e., the extended Fermat principle, facilitates the discussion of ray tracing in general absorbing and weakly nonlinear media. The separate theories, i.e., ray tracing in nonlinear media (Refs. 1 and 2) and ray tracing in absorbing media (Refs. 3 and 4) follow as limiting cases.

The end products are Eqs. (10) and (11), where the new operator is defined in (8). The gradients of the fields are incorporated into the formalism by means of (16), producing collective effects such as self-focusing.

A simple example has been analyzed, showing the existence of self-focusing and the attenuation caused by the absorbing medium. A deeper insight is necessary, but the complexity of the problem calls for a numerical computation project. This will be carried out in the future. However, the intuitively expected results are obtained. It is clear that the absorption slows down the buildup of high intensities produced by self-focusing.

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