Semiclassical and quantum theories of bistability in lasers containing saturable absorbers*

L. A. Lugiato† and P. Mandel‡
Université Libre de Bruxelles, CP 231, Bruxelles 1050, Belgium

S. T. Dembinski and A. Kossakowski
Institute of Physics, Nicholas Copernicus University, Toruń, Poland

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We study the properties of the laser field in the case where the cavity contains two cells, one which amplifies and one which absorbs the radiation. We adopt the simplest possible model, in which there is only one running mode in the cavity and all atoms are fixed and homogeneously distributed in space. We first set up a semiclassical description. In the frame of the stationary semiclassical (SC) theory the most striking feature is the occurrence of a domain in which three solutions coexist. A linear-stability analysis shows that two of these solutions are stable. This gives rise to an hysteresis cycle. Most of the results in this section are analytic. In the frame of a purely quantum description, we derive a generalized Fokker-Planck equation which is exactly solved in the stationary case. This yields a stationary quasiprobability distribution function for the field. In the domain where the SC theory predicts three solutions, the field-distribution function has two maxima and one minimum whose positions are precisely given by the SC intensities. With this distribution function we study the intensity and the intensity fluctuations: the intensity shows a rather abrupt increase when crossing the narrow transition region; in this region the intensity fluctuations are extremely high. By means of an approximate method to solve the time-dependent Fokker-Planck equation we study the linewidth and show that it drastically decreases above threshold. Finally we consider the linearized and quasilinearized Fokker-Planck equations which can be solved analytically. We show that the domain in which neither of these equations fits the results of the generalized Fokker-Planck equation is extremely small. The properties of the metastable states are analyzed in terms of the generalized and asymptotic Fokker-Planck equations.

I. INTRODUCTION

It has been theoretically shown that optical systems containing saturable absorbers can exhibit bistability. Examples are (i) the so-called optical bistability or optical transition[1] which concerns the light transmitted by an optical cavity filled with absorbing atomic material, (ii) the laser with saturable absorber,[2] and (iii) the dye laser.[3] From the experimental viewpoint there exists a very remarkable experiment on optical bistability performed by Gibbs *et al.*[4] which shows the hysteresis cycle (of the transmitted light versus the incident light) arising in the bistable situation. For the laser with saturable absorber we are not aware of any experiment as complete as that of Ref. 4; however, the existence of hysteresis effects has been verified experimentally by Lee *et al.*[5] and by Lisitsyn *et al.*[6]

In this paper we are concerned with the laser with saturable abosrber. Salomaa and Stenholm[2(c)] have given a complete numerical analysis of this problem on the semiclassical level for a gas laser. It turns out that for suitable values of the relevant parameters the semiclassical equations have two different steady-state solutions which are stable in the sense of the linear-stability analysis. In one of these states the laser does not oscillate, whereas in the other state the field has a finite intensity. Hence, one immediately obtains a hysteresis cycle for the laser intensity versus the pump parameter of the active atoms. Clearly, the existence of a bistable domain strongly suggests that a laser with saturable absorber is an example of a system exhibiting a kind of first-order transition far from thermal equilibrium, exactly as the usual laser (without absorber) is a prototype of second-order phase transition out of equilibrium.[7] In this spirit Scott *et al.*[2(b)] have extended the laser phase-transition analogy of Ref. 7(a) to the case of a laser with saturable absorber by constructing a free-energy analog for the laser. The discussion of the minima of such a free energy in the bistable case allows them to give a kind of Maxwell rule which fixes the value of the pump parameter at which the laser begins to oscillate. At this threshold the system should suddenly jump from the zero-intensity steady state to the finite-intensity steady state (first-order phase transition). A similar analysis for a four-level laser with saturable absorber has been given by two of us.[8] Although the approach which is based on the semiclassical equations and on a thermodynamical analogy is quite rewarding, it is limited insofar as it contains an admittedly *ad hoc* element, namely, the "Maxwell rule." Therefore, such an approach assumes the existence of a first-order transition rather than proving it.

The aim of the present paper is to show the existence of a first-order transition in the laser with saturable absorber on a firmer basis; namely, we show that the physical quantities actually undergo a rapid change in a narrow transition region. This behavior really resembles what one finds in first-order phase transitions occurring in systems at thermal equilibrium. Of course the transition is not infinitely sharp, because the laser is a finite open system. To fulfill our program we use a fully quantum-mechanical approach. The reason for introducing a quantum treatment is not only motivated by the need of a correct description of the fluctuations which are known to have a critical influence in the transition region. The main reason is that in our problem even the calculation of the mean values of the observables requires a fully quantum-mechanical theory. In fact in the bistable region the Glauber quasiprobability distribution function associated with the field in the stationary state has *two* maxima. This means that the average values depend on all the features of the distribution function, e.g., the relative heights of the two peaks and their widths.

We use a model which is exactly solvable on a semiclassical level. It describes the interaction of a single running mode of the laser cavity with two-level atoms which are fixed and homogeneously distributed in space. The atoms are of two different species: the first one (active atoms) being pumped to a positive inversion, the second one constituting the absorber. On the basis of this model we first make a complete semiclassical and linear-stability analysis. This study is parallel to that performed in[2(c)] for the gas laser, but is practically completely analytical. As a second point we calculate the steady-state Glauber distribution function of the field. It is important to stress in this connection that in the bistable case the laser with saturable absorber is an intrinsically high-intensity problem. Saturation effects are crucial for the discussion of this problem, and they must be fully taken into account. To deal with these saturation effects, we use the techniques developed by two of us to treat the problem of high intensity lasers.[9,10] This procedure allows us to obtain the distribution function of the field in a straightforward way. By means of this function we calculate the intensity, the intensity fluctuation, the linewidth, and the phase shift. The two latter quantities are calculated following a procedure devised by one of us[11] and which has been successfully used in the case of the laser with active atoms only. Similar problems for a simplified model were considered by two of us.[11(c)]

A point which is interesting to discuss in this context is metastability. In fact the quantum treatment allows us to see which one of the two steady-state semiclassical solutions (which are stable according to the linear-stability analysis) is absolutely stable and which one is only metastable. It is meaningful to ask questions concerning metastable states such as the fluctuations and the linewidth in these states. Strictly speaking, one should solve the quantum-mechanical equation for the whole time evolution to answer these questions since the metastable states are not stationary states. However, we show that one can study the properties of the metastable states by linearizing the quantum-mechanical equation around the metastable state itself. With this trick we associate to the metastable state a formal distribution function through which we can study the usual properties using standard methods. Of course this distribution describes the state of the system only as long as it remains trapped in the metastable state.

In Sec. II we state the von Neumann equation which is the starting point of our treatment. Section III is devoted to the semiclassical (SC) theory which is derived from the von Neumann equation in Sec. III A. In Sec. III B we study the stationary solutions for the intensity. This study is completed in Sec. III C by a linear-stability analysis. Section III D is devoted to the derivation of the frequency shift. In Sec. IV we make a fully quantum study of our model. In Sec. IV A we derive the generalized Fokker-Planck equation which governs the time evolution of the Glauber distribution function associated with the field. In Sec. IV B we study the stationary solution of this equation as well as the mean intensity and the intensity fluctuations. Section IV C is devoted to the analysis of the transient properties of the Fokker-Planck equation; this leads to a study of the linewidth and the frequency shift. Section V is devoted to a study of asymptotic equations and is divided into two subsections: In Sec. V A we consider the linearized Fokker-Planck equation whereas in Sec. V B we consider the quasilinearized Fokker-Planck equation. In each case we compare the results of the asymptotic and the exact theories. The last paragraph of the section is devoted to the discussion of the metastable states. Section VI contains some concluding remarks.

II. THE VON NEUMANN EQUATION

The physical situation which we describe in this paper is a solid-state laser containing two cells in its resonant cavity. Both cells contain an inert matrix doped with atoms which may act as active material under suitable conditions. The possibility that the two cells contain the same inert matrix and the same active atoms is not ruled out. The

basic difference between the two cells is that in one of them (the amplifying cell) the active atoms are submitted to an intense pumping process so that a state of population inversion between two levels is created. On the contrary in the second cell (the absorbing cell) the pumping is weak and no population inversion is induced.

In the model we shall use we describe the active and the passive atoms (i.e., the doping ions in the amplifying and absorbing cells, respectively) by two-level atoms. Both levels of each type of atoms have a finite lifetime. Furthermore we assume that the cavity can sustain only one running mode. Consequently, in the diagonal Glauber representation the von Neumann equation which we associate to our system is

$$i\hbar\partial_t W(t) = (L_A + L_F + L_{AF} + i\Lambda_A + i\Lambda_F)W(t) \ , \tag{2.1}$$

where

$$L_A X = H_A X - X H_A, \quad H_A = \sum_{p=1}^{N} \hbar\omega(p) a^\dagger(p) a(p) + \sum_{q=1}^{\bar{N}} \hbar\bar{\omega}(q) A^\dagger(q) A(q) \ ,$$

$$L_F = -\hbar\nu\left(\frac{\partial}{\partial\beta}\beta - \frac{\partial}{\partial\beta^*}\beta^*\right) \ ,$$

$$L_{AF}X = \hbar\sum_{p=1}^{N}\{g(p)a^\dagger(p)\beta X + g^*(p)a(p)[\beta^* - \partial/\partial\beta]X - g^*(p)\beta^* X a(p) - g(p)[\beta - \partial/\partial\beta^*]Xa^\dagger(p)\}$$
$$+\hbar\sum_{q=1}^{\bar{N}}\{\bar{g}(q)A^\dagger(q)\beta X + \bar{g}^*(q)A(q)[\beta^* - \partial/\partial\beta]X - \bar{g}^*(q)\beta^* XA(q) - g(q)[\beta - \partial/\partial\beta^*]XA^\dagger(q)\} \ ,$$

$$\Lambda_F = \hbar\kappa\left(\frac{\partial}{\partial\beta}\beta + \frac{\partial}{\partial\beta^*}\beta^*\right) \ ,$$

$$\Lambda_A = \frac{\hbar}{2}\sum_{p=1}^{N}\Lambda(p) + \frac{\hbar}{2}\sum_{q=1}^{\bar{N}}\bar{\Lambda}(q) \ ,$$

$$\Lambda(p)X = \gamma_\uparrow(p)\{[a^\dagger(p), Xa(p)]_- + [a^\dagger(p)X, a(p)]_-\} + \gamma_\downarrow(p)\{[a(p), Xa^\dagger(p)]_- + [a(p)X, a^\dagger(p)]_-\}$$
$$-\eta(p)\{a(p)a^\dagger(p)Xa^\dagger(p)a(p) + a^\dagger(p)a(p)Xa(p)a^\dagger(p)\} \ ,$$

$$\bar{\Lambda}(q)X = \bar{\gamma}_\uparrow(q)\{[A^\dagger(q), XA(q)]_- + [A^\dagger(q)X, A(q)]_-\} + \bar{\gamma}_\downarrow(q)\{[A(q), XA^\dagger(q)]_- + [A(q)X, A^\dagger(q)]_-\}$$
$$+\eta(q)\{A(q)A^\dagger(q)XA^\dagger(q)A(q) + A^\dagger(q)A(q)XA(q)A^\dagger(q)\} \ .$$

In these definitions X stands for an arbitrary operator, ν is the frequency of the field created in the cavity, κ is the damping constant of the field in the empty cavity, β is the Glauber variable associated to the field. For the active atoms the relevant parameters are the total number of atoms (N), the energy difference between the two levels [$\hbar\omega(p)$], the effective atomic transition rates between the two states ($\gamma_\uparrow$ and $\gamma_\downarrow$) and the influence of phase-destroying processes in the atomic damping (η). They are connected to the longitudinal ($\gamma_\parallel$) and transverse ($\gamma_\perp$) relaxation constants and to the unsaturated inversion (σ) through

$$\gamma_\perp = \tfrac{1}{2}(\gamma_\uparrow + \gamma_\downarrow + \eta), \quad \gamma_\parallel = \gamma_\uparrow + \gamma_\downarrow \ ,$$
$$\sigma = (\gamma_\uparrow - \gamma_\downarrow)/(\gamma_\uparrow + \gamma_\downarrow) \ . \tag{2.2}$$

For the passive atoms we use the same notation except that the symbols have a bar on them. The atomic operators $a^\dagger(p)$, $a(p)$ and $A^\dagger(q)$, $A(q)$ describe the transitions between the two levels and obey the anticommutation relations

$$[a^\dagger(p), a(p')]_+ = \delta(p,p') \ ,$$
$$[A^\dagger(q), A(q')]_+ = \delta(q,q') \ .$$

Finally the coupling parameter is expressed as

$$g(p) = -i[\hbar\nu v]^{-1/2}\omega(p)e^{i\kappa\cdot r}\vec{e}\cdot\vec{\mathcal{P}} \ ,$$
$$\vec{\mathcal{P}} = e\int_v d\vec{r}\ \varphi^*(0;\vec{r})\vec{r}\varphi(1;\vec{r}) \ ,$$

where v is the volume of the cavity, κ is the field wave vector ($\kappa^2c^2 = \nu^2$), $\vec{r}(p)$ is the position of the atom characterized by the variable(s) p, $\vec{e}$ is the unit polarization vector of the field mode, e the electronic charge, and $\varphi(i;\vec{r})$ the atomic wave function of state i. The variable p (and q of course) is a short-hand notation for all the variables needed to specify the atomic properties.

Typically it can represent the position and the frequency, in which case the sum over p is to be understood as

$$\sum_{p=1}^{N} f(p) = \int_v d\vec{r}\,\rho(\vec{r}) \int d\omega\, g(\omega) f(\vec{r},\omega) ,$$

where $\rho(\vec{r})$ is the atomic density and $g(\omega)$ is the distribution of atomic frequencies in the crystal.

III. SEMICLASSICAL DESCRIPTION

A. Derivation of the semiclassical equations

In order to derive the semiclassical (SC) equations from the von Neumann equation (2.1), we apply the following procedure.[12(a)-12(c)] We derive from Eq. (2.1) the equations for the mean values of β, $a(p)$, $A(q)$, $a^\dagger(p)a(p)$, and $A^\dagger(q)A(q)$ under the factorization assumption

$$\langle XYZ\rangle = \langle X\rangle\langle Y\rangle\langle Z\rangle , \tag{3.1}$$

where $X(Y, Z)$ is an arbitrary operator which is a function of the field (active-atom, passive-atom) variables. In other terms we neglect all atom-field and passive-active atom correlations. The mean value is defined by

$$\langle O\rangle = \mathrm{Tr}\int d^2\beta\, O W(t) , \tag{3.2}$$

where O is an arbitrary operator, $W(t)$ is the solution of Eq. (2.1), and Tr stands for the trace over all atomic variables. A straightforward application of the rule (3.1) leads to

$$\left[i\left(\frac{d}{dt}+\kappa\right)-\nu\right]\langle\beta\rangle = \sum_{p=1}^{N} g^*(p)\langle\langle a(p)\rangle\rangle + \sum_{q=1}^{\bar{N}} \bar{g}^*(q)\langle\langle A(q)\rangle\rangle , \tag{3.3}$$

$$\left[i\left(\frac{d}{dt}+\gamma_\perp(p)\right)-\omega(p)\right]\langle\langle a(p)\rangle\rangle = -D(p;t)g(p)\langle\beta\rangle , \tag{3.4}$$

$$i\left(\frac{d}{dt}+\gamma_\parallel(p)\right)D(p;t) = i\gamma_\parallel(p)\sigma(p) + 2g(p)\langle\langle a^\dagger(p)\rangle\rangle\langle\beta\rangle + 2g^*(p)\langle\langle a(p)\rangle\rangle\langle\beta^*\rangle , \tag{3.5}$$

$$\left[i\left(\frac{d}{dt}+\bar{\gamma}_\perp(q)\right)-\bar{\omega}(q)\right]\langle\langle A(q)\rangle\rangle = -\bar{D}(q;t)\bar{g}(q)\langle\beta\rangle , \tag{3.6}$$

$$\left[i\left(\frac{d}{dt}+\bar{\gamma}_\parallel(q)\right)\bar{D}(q;t)\right] = i\bar{\gamma}_\parallel(q)\bar{\sigma}(q) + 2\bar{g}(q)\langle\langle A^\dagger(q)\rangle\rangle\langle\beta\rangle - 2\bar{g}^*(q)\langle\langle A(q)\rangle\rangle\langle\beta^*\rangle , \tag{3.7}$$

with $D(p,t) = \langle\langle a^\dagger(p)a(p)\rangle\rangle - \langle\langle a(p)a^\dagger(p)\rangle\rangle$ and $\bar{D}(q,t) = \langle\langle A^\dagger(q)A(q)\rangle\rangle - \langle\langle A(q)A^\dagger(q)\rangle\rangle$ which are the perturbed atomic inversions. Because $\langle\langle a^\dagger(p)\rangle\rangle = \langle\langle a(p)\rangle\rangle^*$ and $\langle\langle A^\dagger(q)\rangle\rangle = \langle\langle A(q)\rangle\rangle^*$, this set of five equations is closed.

It is more convenient and physical to introduce the decomposition

$$\langle\beta\rangle = E(t)e^{-i\Omega t} ,$$

$$g^*(p)\langle\langle a(p)\rangle\rangle = [u(p;t)+iv(p;t)]e^{-i\Omega t} ,$$

$$\bar{g}^*(q)\langle\langle A(q)\rangle\rangle = [\bar{u}(q;t)+i\bar{v}(q;t)]e^{-i\Omega t} ,$$

where the new functions are real functions. Furthermore, we shall assume that all atoms in each cell are identical and homogeneously distributed within the cell. In that case the index p (or q) disappears and our basic set of equations becomes

$$\left(\frac{d}{dt}+\kappa\right)E(t) = Nv(t) + \bar{N}\bar{v}(t) , \tag{3.8}$$

$$E(t)(\Omega-\nu) = Nu(t) + \bar{N}\bar{u}(t) , \tag{3.9}$$

$$\left(\frac{d}{dt}+\gamma_\perp\right)u(t) = (\omega-\Omega)v(t) , \tag{3.10}$$

$$\left(\frac{d}{dt}+\gamma_\perp\right)v(t) = (\Omega-\omega)u(t) + |g|^2 D(t)E(t) , \tag{3.11}$$

$$\left(\frac{d}{dt}+\bar{\gamma}_\perp\right)\bar{u}(t) = (\bar{\omega}-\Omega)\bar{v}(t) , \tag{3.12}$$

$$\left(\frac{d}{dt}+\bar{\gamma}_\perp\right)\bar{v}(t) = (\Omega-\bar{\omega})\bar{u}(t) + |\bar{g}|^2\bar{D}(t)E(t) , \tag{3.13}$$

$$\left(\frac{d}{dt}+\gamma_\parallel\right)D(t) = \gamma_\parallel\sigma - 4v(t)E(t) , \tag{3.14}$$

$$\left(\frac{d}{dt}+\bar{\gamma}_\parallel\right)\bar{D}(t) = \bar{\gamma}_\parallel\bar{\sigma} - 4\bar{v}(t)E(t) . \tag{3.15}$$

B. Stationary solutions

We define the stationary state by the condition that all time derivatives vanish in Eqs. (3.8)–(3.15). Then the last six equations can be solved in terms of the field amplitude E:

$$D = \sigma(1+SE^2)^{-1}, \quad v = |g|^2 ED/\delta_\perp, \quad u = \Delta v , \tag{3.16}$$

and we have introduced the standard notation

$$S = 4|g|^2/\gamma_\parallel\delta_\perp, \quad \delta_\perp = \gamma_\perp(1+\Delta^2), \quad \Delta = (\Omega-\omega)/\gamma_\perp , \tag{3.17}$$

where S is the saturation parameter of the active atoms and Δ the detuning. Needless to say, a corresponding set of solutions holds for the passive atoms: one simply has to add a bar on all atomic functions and parameters in Eqs. (3.16) and (3.17). Inserting these results into Eq. (3.8), we obtain the equation which gives the intensity of the field and which we write as follows:

$$0 = I^{1/2}\left(1 - \frac{A}{1+I} - \frac{1-C}{1+aI}\right) \tag{3.18}$$

with

$$I = SE^2, \quad A = \sigma/\sigma(T) = N|g|^2\sigma/\kappa\delta_\perp ,$$
$$C = 1 - \bar{\sigma}/\bar{\sigma}(T) = 1 - \bar{N}|\bar{g}|^2\bar{\sigma}/\kappa\delta_\perp, \quad a = \bar{S}/S . \tag{3.19}$$

I is the normalized intensity and $\sigma(T)$ is the threshold inversion per atom for laser action when the cavity contains only the active atoms. Note that the SC stationary equation (3.18) for the normalized intensity depends on only the three parameters A, C, and a. The parameter A characterizes the amplifying cell ($\sigma > 0$); its relevant range is $A > 0$. The parameter C characterizes the absorbing cell ($\bar{\sigma} < 0$) and is defined in such a way that its relevant range is $C > 1$. Finally, the parameter a measures the relative saturability of the two cells: when $0 < a < 1$ it is easier to saturate the active atoms than the passive atoms, whereas when $a > 1$ we have the converse situation.

Let us stress that Eq. (3.18) only gives the mathematical solutions to our problem; these solutions can be positive, negative, or even complex. The only *admissible* solutions are the real non-negative roots of Eq. (3.18). In this last group the *physical* solutions are the roots which correspond to a stable stationary state. In this section we shall determine the admissible roots of Eq. (3.18) and in the next section we shall make the stability analysis to see which are the physical roots. To discuss the solutions of Eq. (3.18), we shall neglect the additional intensity dependence which arises from the modified damping constant $\delta_\perp$ via Eq. (3.9); in other words, we approximate Ω by ν in Eq. (3.18). In the case where there is no absorbing cell this neglect is perfectly justified[13] provided $\gamma_\perp \gg \kappa$, a property which holds in general for solid-state lasers and which we assume to hold here. The roots of Eq. (3.18) are $I_0 = 0$,

$$I_\pm = (1/2a)(a(A-1) - C \pm \{[a(A-1)-C]^2 - 4a(C-A)\}^{1/2}) . \tag{3.20}$$

The first condition to be imposed is that the roots be real. This condition can be expressed as

$$(aA - X_+)(aA - X_-) > 0 ,$$
$$X_\pm = a + C - 2 \pm 2[(a-1)(C-1)]^{1/2} . \tag{3.21}$$

For the following discussion it is useful to notice that if $a > 1$, one has $aC > X_\pm$ and $X_+ \gtrless a + C$ if $C \gtrless a/a - 1$. When the condition (3.21) is verified, three possibilities can arise:

$$\begin{aligned} &\text{(i)} \quad A > C: && I_+ > 0, \quad I_- < 0 ; \\ &\text{(ii)} \quad aA > a + C, \quad A < C: && I_+ > 0, \quad I_- > 0 ; \\ &\text{(iii)} \quad aA < a + C, \quad A < C: && I_+ < 0, \quad I_- < 0 . \end{aligned} \tag{3.22}$$

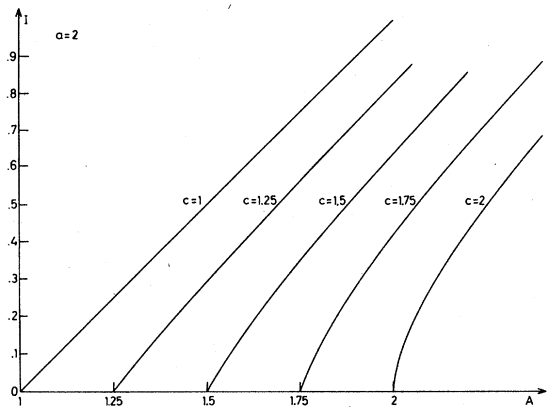

FIG. 1. Reduced intensities $I_+ = SE^2$ and $I = 0$ versus the pump parameter of the amplifying cell for $1 < a < C/C - 1$.

We can now discuss the various possible situations.

(a) $0 < a < 1$. When a is smaller than 1, the condition (3.21) is trivially satisfied because X_+ and X_- are complex conjugates. Furthermore, case (ii) is ruled out. Hence one has the following admissible roots:

$$0 < a < 1 \begin{cases} I = I_+, \quad I = I_0 \text{ if } A \geq C , \\ I = I_0 \text{ if } A \leq C . \end{cases} \tag{3.23}$$

In this case the function $I_+ = I_+(A)$ for $A \geq C$ is convex towards the A axis.

(b) $a = 1$. This case is trivial and we merely quote the result:

$$a = 1 \begin{cases} I = I_+ = A - C, \quad I = I_0 \text{ if } A \geq C , \\ I = I_0 \text{ if } A \leq C . \end{cases} \tag{3.24}$$

(c) $1 < a < C/C - 1$. In this situation only case (i) is possible. For $A > C$ condition (3.21) is satisfied

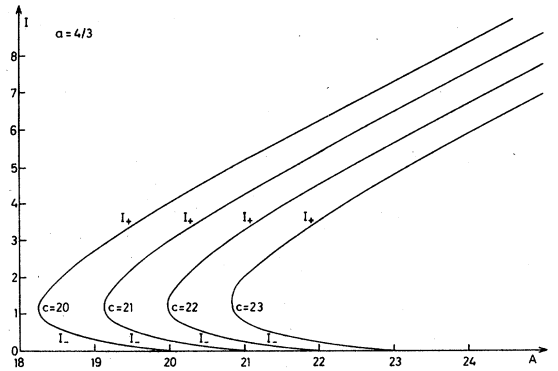

FIG. 2. Reduced intensities $I_\pm = SE_\pm^2$ and $I = 0$ versus the pump parameter of the amplifying cell for $a > C/C - 1$. The domain in which there are three solutions is $X_+ < aA < aAC$.

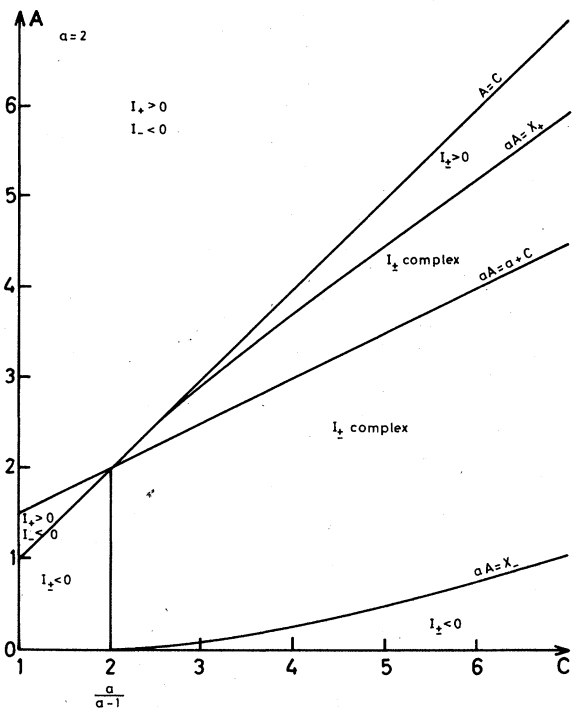

FIG. 3. Behavior of the roots I_+ and I_- given by Eq. (3.20) in the (A, C) plane. Note that the root $I=0$ is a solution everywhere in this plane.

because $aA > aC > X_\pm$. Hence we again obtain the result (3.23). The only difference is that the function $I_+ = I_+(A)$ for $A \geq C$ is concave towards the A axis (see Fig. 1).

(d) $a > C/C-1$. This is the interesting situation because now condition (ii) in (3.22) can be fulfilled. Taking into account $X_- < a+C < X_+$ and Eq. (3.21), one verifies that the admissible roots of Eq. (3.18) are (see Fig. 2)

$$a > C/C-1 \quad \begin{cases} I = I_0 \text{ if } aA < X_+ , \\ I = I_+, \quad I = I_-, \quad I = I_0 \text{ if } X_+ < aA < aC , \\ I = I_+, \quad I = I_0 \text{ if } A > C . \end{cases} \tag{3.25}$$

The new feature is the occurrence of a new domain defined by the conditions $X_+ < aA < aC$ in which all the roots of Eq. (3.18) are admissible solutions. Therefore it is important to study the stability of these roots to know which one of them is a physical root. Figure 3 shows how the roots I_- and I_+ behave in the (A, C) plane for $a > 1$. We stress that there is a striking similarity between Figs. 1–3 and the corresponding plots for a gas laser.[2(c)] However in our model these results are completely analytical.

C. Linear-stability analysis

1. General analysis

To study the stability of the admissible roots we shall make a simplifying assumption: we consider the case $\Omega = \omega = \bar{\omega} = \nu$. This is physically sound because in the experimental situations one tries very hard to minimize the differences $\omega - \nu$ and $\bar{\omega} - \nu$; furthermore, some experiments are performed with the same atoms in both cells (hence $\omega = \bar{\omega}$), the difference between the amplifying and absorbing cells being then the intensity of the pumping. We are then left with a set of five equations, namely, Eqs. (3.8), (3.11), (3.13)–(3.15). Let $X(t)$ be any of the five relevant functions. We introduce the decomposition $X(t) = X + X' \exp(-\lambda \kappa t)$ where X is the stationary value of $X(t)$. Inserting this decomposition into the five equations and linearizing them with respect to the X' leads to

$$(-\lambda + 1)\kappa E' = N v' + \bar{N} \bar{v}' , \tag{3.26}$$

$$(-\lambda + d_\perp) v' = |G|^2 (D \kappa E' + D' \kappa E) , \tag{3.27}$$

$$(-\lambda + \bar{d}_\perp) \bar{v}' = |\bar{G}|^2 (\bar{D} \kappa E' + \bar{D}' \kappa E) , \tag{3.28}$$

$$(-\lambda + d_\parallel) D' = -4 (v' \kappa E + v \kappa E') , \tag{3.29}$$

$$(-\lambda + \bar{d}_\parallel) \bar{D}' = -4 (\bar{v}' \kappa E + \bar{v} \kappa E') , \tag{3.30}$$

where $d = \gamma \kappa^{-1}$ and $|G|^2 = |g|^2 \kappa^{-2}$. In order that these five equations admit nontrivial solutions, it is necessary that the associated determinant vanishes, and this of course furnishes an equation for λ. When $I = I_0 = 0$ this algebraic equation is

$$(\lambda - d_\parallel)(\lambda - \bar{d}_\parallel)\{(\lambda - 1)(\lambda - d_\perp)(\lambda - \bar{d}_\perp) - N|G|^2 D(\lambda - \bar{d}_\perp) - \bar{N}|\bar{G}|^2 \bar{D}(\lambda - d_\perp)\} = 0 . \tag{3.31}$$

When $I = I_\pm$ the equation is somewhat more complicated:

$$C(0)\lambda^5 + C(1)\lambda^4 + C(2)\lambda^3 + C(3)\lambda^2 + C(4)\lambda + C(5) = 0 , \tag{3.32}$$

where the five coefficients are

$$C(0) = 1 ,$$

$$C(1) = -[1 + d_\perp + \bar{d}_\perp + d_\parallel + \bar{d}_\parallel] ,$$

$$C(2) = I(d_\parallel d_\perp + a \bar{d}_\parallel \bar{d}_\perp) + d_\perp \bar{d}_\perp + d_\perp + \bar{d}_\perp - d_\perp \frac{A}{1+I} - \bar{d}_\perp \frac{1-C}{1+aI} + (d_\parallel + \bar{d}_\parallel)(1 + d_\perp + \bar{d}_\perp) + d_\parallel \bar{d}_\parallel ,$$

$$C(3) = -aI\bar{d}_{\parallel}\bar{d}_{\perp}\left(1+d_{\perp}+d_{\parallel}+\frac{1-C}{1+aI}\right) - Id_{\parallel}d_{\perp}\left(1+\bar{d}_{\perp}+\bar{d}_{\parallel}+\frac{A}{1+I}\right) - d_{\parallel}\bar{d}_{\parallel}(1+d_{\perp}+\bar{d}_{\perp})$$
$$-(d_{\parallel}+\bar{d}_{\parallel})\left(d_{\perp}\bar{d}_{\perp}+d_{\perp}+\bar{d}_{\perp}-d_{\perp}\frac{A}{1+I}-\bar{d}\,\frac{1-C}{1+aI}\right) ,$$

$$C(4) = aI^2 d_{\parallel}d_{\perp}\bar{d}_{\parallel}\bar{d}_{\perp} + aI\,\bar{d}_{\parallel}\bar{d}_{\perp}\left[2d_{\perp}\frac{1-C}{1+aI}+d_{\parallel}\left(1+d_{\perp}+\frac{1-C}{1+aI}\right)\right]$$
$$+Id_{\parallel}d_{\perp}\left[2\bar{d}_{\perp}\frac{A}{1+I}+\bar{d}_{\parallel}\left(1+\bar{d}_{\perp}+\frac{A}{1+I}\right)\right]+d_{\parallel}\bar{d}_{\parallel}\left(d_{\perp}\bar{d}_{\perp}+d_{\perp}+\bar{d}_{\perp}-d_{\perp}\frac{A}{1+I}-\bar{d}_{\perp}\frac{1-C}{1+aI}\right) ,$$

$$C(5) = -2Id_{\parallel}\bar{d}_{\parallel}d_{\perp}\bar{d}_{\perp}\left(aI+a\frac{1-C}{1+aI}+\frac{A}{1+I}\right) .$$

We shall not try to solve the fifth-order equations: although this can be done analytically,[14] all we are interested in for a stability analysis is to know the sign of the real parts of the roots $\lambda(i)$, $i=1$ to 5. This can be done by means of the extended Routh-Hurwitz theorem[15] which we now state for the case $c(0)>0$ and all $c(i)$ real. Consider the polynomial

$$f(x) = c(0)x^n + c(1)x^{n-1} + \ldots + c(n-1)x + c(n) .$$

To this polynomial we associate a set of determinants $D(m)$, $m=1,\ldots,n$ defined by

$$D(m) = \begin{vmatrix} c(1) & c(0) & 0\ldots 0 \\ c(3) & c(2) & c(1)\ldots 0 \\ & \ldots & \\ c(2m-1) & c(2m-2) & c(2m-3)\ldots c(m) \end{vmatrix}$$

with the convention $c(m)=0$ for $m>n$.

Theorem: *The number of roots of $f(x)$ having a positive real part is equal to the number of changes of signs in the ordered sequence S given by* $S = c(0)$, $D(1)$, $D(2)/D(1)$, $D(3)/D(2),\ldots,D(n)/D(n-1)$. Consequently, with the notation used in this paper a root $x(i)$ of $f(x)$ will correspond to a stable state if and only if all signs alternate in the ordered sequence S.

We first apply this theorem to Eq. (3.31) for which the sequence is $S=+1,-1-d_{\perp}-\bar{d}_{\perp},-D(2)/(1+d_{\perp}+\bar{d}_{\perp}),d_{\perp}\bar{d}_{\perp}(A-C)$. A first condition is $A<C$. But when $A<C$ we also have $D(2)<0$ because

$$D(2) = -d_{\perp}(1+\bar{d}_{\perp})(1+d_{\perp}+\bar{d}_{\perp})$$
$$+Ad_{\perp}(1+d_{\perp}) - C\bar{d}_{\perp}(1+\bar{d}_{\perp}) .$$

Consequently, we see that $I=I_0=0$ is a physical solution in the domain $A<C$. This result is independent of the value assigned to a. Let us now apply the Routh-Hurwitz theorem to the more complicated equation (3.32). The sequence to be studied is $S=+1,-|c(1)|,-D(2)|c(1)|^{-1},D(3)/D(2),D(4)/D(3),c(5)$. A necessary condition for the stability of $I_{\pm}$ is $c(5)<0$. Explicitly, this condition becomes here $(1+aI)^2+(a-1)(1-C)>0$. The reader will easily verify that the inequality $\partial I/\partial A>0$ leads to exactly the same condition. Hence a necessary condition of stability for $I_{\pm}$ is that the slope of the intensity in the (I,A) plane be positive. A glance at Fig. 2 indicates at once that only I_+ verifies this condition so that I_- is always an unstable state. There only remains to study the stability condition for I_+. Unfortunately, this requires the analysis of $D(2)$, $D(3)$, and $D(4)$ which is hardly possible analytically. Therefore we have made this analysis numerically and verified that I_+ is a stable solution for a typical point in the domain $aA>X_+$ with the numerical values $a=2$, $\bar{d}_{\perp}=\bar{d}_{\parallel}=d_{\perp}=d_{\parallel}=2$, $A=5$, and $C=5.5$. Of course this is a partial result because it concerns only one point of the bistable domain and uses only one set of values for the relaxation constants. In fact when $d_{\parallel}<1$ then I_+ could become unstable.[16(b)] Furthermore, an instability of the Risken-Nummedal type[16(a)] is not *a priori* ruled out; we have not investigated this possibility which in the case of the usual laser is known to occur only for very high values of A. In the following we assume that the parameters are chosen in such a way that I_+ is stable when it is real and positive.

Figure 4 displays the distribution of the physical solutions in the (A,C) plane. The main characteristic of this figure is the occurrence of a bistable domain: when $X_+<aA<aC$ two physical solutions I_0 and I_+ coexist. This is even more transparent on Fig. 5 which shows how the intensity varies as a function of A. Let us consider this last figure in detail. Starting from a situation where the laser does not sustain any oscillation ($aA \ll X_+$), we increase A (by increasing the pumping in the amplifying cell for instance). Crossing the boundary $aA=X_+$ we reach the bistable domain in which the intensity remains zero until we reach the second boundary at $A=C$, where a sudden jump occurs and the laser begins to operate with a finite inten-

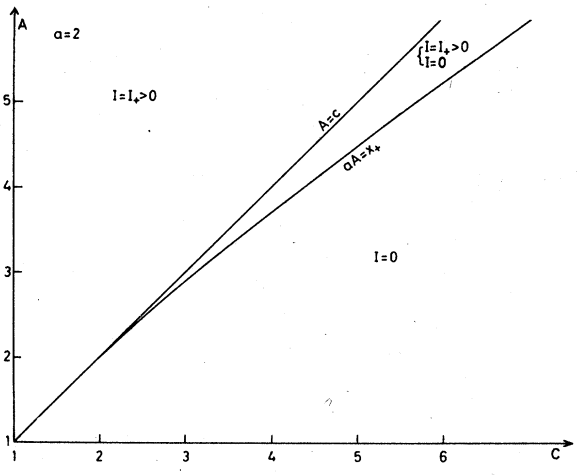

FIG. 4. Distribution of the stable roots in the (A, C) plane.

sity. For greater values of A we simply follow the curve of I_+. On the contrary, if we start with a value of A such that $A \gg C$, we are on the upper branch of Fig. 5. Decreasing gradually the value of A, we remain on the upper branch until we reach the critical value $aA = X_+$, where the laser suddenly ceases to sustain the oscillation. In other words, we are in the presence of an hysteresis cycle: the value of the intensity in the bistable region depends crucially upon the path followed to reach this value. The bistable region is characterized by the upper and lower values of A and the corresponding values of I_+:

$$A(1) = C > a/a - 1: \quad I_+ = I(1) = C - 1 - C/a \ , \tag{3.33}$$

$$aA(2) = X_+: \quad I_+ = I(2) = a^{-1}[-1 + (C-1)^{1/2}(a-1)^{1/2}] \ . \tag{3.34}$$

The width of the bistable region is

$$A(1) - A(2) = I(1) - 2I(2) = aI^2(2) \ . \tag{3.35}$$

This shows in particular that if there is any bistable region at all, then the maximum intensity $I(1)$ is at least twice the minimum intensity $I(2)$. If we increase a at constant C, the width of the bistable region and the maximum intensity $I(1)$ increase whereas $I(2)$ decreases. More specifically we have

$$a \to \infty: \quad \begin{aligned} &A(1) - A(2) \to C - 1, \\ &I(1) \to C - 1, \quad I(2) \to 0 \ . \end{aligned} \tag{3.36}$$

A point which we can show analytically on the basis of the previous linear-stability analysis is the existence of a critical slowing down when A approaches $A(1)$ from below or $A(2)$ from above. Let us consider first the latter case. As $A = A(2) + |\epsilon|$, $\epsilon \to 0$ one has $\partial I/\partial A \to 0$ which as we have shown implies that $c(5) \to 0$. Hence as A approaches the critical value $A(2)$, one of the roots of the characteristic equation (3.32) tends to vanish. We recall that the real part of the roots of Eq. (3.32) give the rate at which the system returns to the stationary state once it has been slightly removed from it. Therefore, as $A = A(2) + |\epsilon|$ and $\epsilon \to 0$, regression to the stationary state I_+ becomes slower and slower. This critical slowing down is similar to that which one finds in tunnel diodes[17] and in optical bistability.[7(b)] The existence of a critical slowing down for $A = A(1) - |\epsilon|$, $\epsilon \to 0$ follows along the same line taking into account that the constant term of Eq. (3.31) is proportional to $A - C$.

2. *Adiabatic elimination of the atomic variables*

In most lasers one has $\gamma_\perp \gg \kappa$. If we furthermore assume $\bar{\gamma}_\perp \gg \kappa$, we can adiabatically eliminate the polarizations u, $\bar{u}$, v, $\bar{v}$ from Eqs. (3.8)–(3.15) thus obtaining a reduced set of equations. This procedure is certainly correct provided A is not too large. Hence setting $du/dt = d\bar{u}/dt = dv/dt = d\bar{v}/dt = 0$ in Eqs. (3.10)–(3.13) and eliminating u, $\bar{u}$, v, $\bar{v}$, we obtain the three coupled equations

$$\begin{aligned} &\left(\frac{d}{dt} + 2\kappa\right) I(t) = I(t)\left(\frac{2N|g|^2}{\delta_\perp} D(t) + \frac{2\bar{N}|\bar{g}|^2}{\bar{\delta}_\perp} \bar{D}(t)\right) , \\ &\left(\frac{d}{dt} + \gamma_\parallel\right) D(t) = \gamma_\parallel [\sigma - D(t)I(t)] \ , \\ &\left(\frac{d}{dt} + \bar{\gamma}_\parallel\right) \bar{D}(t) = \bar{\gamma}_\parallel [\bar{\sigma} - aD(t)I(t)] \ , \end{aligned} \tag{3.37}$$

where $I = SE^2$. The system (3.37) has been studied in Ref. 16(b) where it is shown that when $\kappa \gg \gamma_\parallel$ the stationary solution I_+ can become unstable. In fact in this case the cubic characteristic equation which one obtains by linearizing Eqs. (3.37) around the stationary solution $I = I_+$, $D = \sigma(1 + I_+)^{-1}$, $\bar{D} = \bar{\sigma} \times (1 + aI_+)^{-1}$ has two complex-conjugate roots with negative real parts. In this situation the system exhibits a limit cycle behavior. We shall rule out

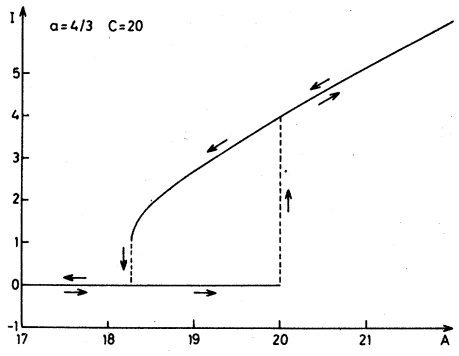

FIG. 5. Example showing the three possible domains and the hysteresis cycle.

this possibility in the following by assuming that

$$\kappa \ll \gamma_\perp, \bar{\gamma}_\perp, \gamma_\parallel, \bar{\gamma}_\parallel \ . \tag{3.38}$$

Under condition (3.38) one can also adiabatically eliminate the population inversions D and $\bar{D}$, obtaining the following closed time-evolution equation for $I(t)$ which is valid for $t \to \infty$:

$$\frac{dI(t)}{dt} = 2\kappa I(t)\left(1 - \frac{A}{1+I(t)} - \frac{1-C}{1+aI(t)}\right) . \tag{3.39}$$

D. Stationary renormalized frequency

Let us again consider the Eq. (3.8)–(3.15) in the stationary state. Until now we have discussed in great detail the properties of the intensity which is the solution of Eq. (3.8). Another quantity which may be of interest is the renormalized field frequency Ω given by Eq. (3.9). In the stationary state this equation becomes

$$\Omega = \nu - \frac{\Delta\kappa A}{1+I} - \frac{\bar{\Delta}\kappa(1-C)}{1+aI} \quad , \tag{3.40}$$

where we have used the definitions (3.17). This equation only holds for $I \neq 0$. Using Eq. (3.18) we can also write

$$\Omega = \nu - \kappa\Delta + \kappa(\Delta - \bar{\Delta})(1-C)(1+aI)^{-1}$$
$$= \nu - \kappa\bar{\Delta} + \kappa(\bar{\Delta} - \Delta)A(1+I)^{-1} .$$

These equations are very complicated because A, C, and I are still nonlinear functions of Δ and $\bar{\Delta}$. However one must note that the difference between the unperturbed and the perturbed frequencies $\Omega - \nu$ must be very small because our system is contained in a resonant cavity which can sustain only a discrete set of modes. Hence we are entitled to linearize Eq. (3.40) around $\Omega = \nu$. This yields

$$\Omega = \frac{\gamma_\perp\nu + \kappa\omega + \kappa\gamma_\perp(\bar{\omega}/\bar{\gamma}_\perp - \omega/\gamma_\perp)(1-C)/(1+aI)}{\gamma_\perp + \kappa + \kappa\gamma_\perp(1/\bar{\gamma}_\perp - 1/\gamma_\perp)(1-C)/(1+aI)}$$
$$= \frac{\bar{\gamma}_\perp\nu + \kappa\bar{\omega} + \kappa\bar{\gamma}_\perp(\omega/\gamma_\perp - \bar{\omega}/\bar{\gamma}_\perp)A/(1+I)}{\bar{\gamma}_\perp + \kappa + \kappa\bar{\gamma}_\perp(1/\gamma_\perp - 1/\bar{\gamma}_\perp)A(1+I)} \ .$$

IV. FOKKER-PLANCK DESCRIPTION

Although the SC description furnished a first and relatively simple description of the system we are studying, it is quite incomplete and unsatisfactory in many respects: (i) the linewidth of the field vanishes (the field is purely monochromatic); (ii) the true stationary state must be unique: the hysteresis cycle corresponds to the existence of metastable states whose lifetime is unusually long; (iii) because the field is described as a pure coherent state there are no fluctuations at all. It is precisely to give a more sensible answer to these three points that we now develop a purely quantum description.

A. Derivation of the Fokker-Planck equation

To derive the master equation which gives the time evolution of the Glauber P function, we shall use a method which is particularly well suited for our problem; basically, we use a slightly modified formulation of the theory developed by one of us.[10] Let us consider the full von Neumann equation (2.1) and introduce the functions

$$P(\beta, \beta^*; t) = e^{i\mathfrak{L}_F t}\mathrm{Tr}W(t) \ , \tag{4.1}$$

$$A(p; \beta, \beta^*; t) = g(p)e^{i(\mathfrak{L}_F - \nu)t}\,\mathrm{Tr}a^\dagger(p)W(t) \ , \tag{4.2}$$

$$B(p; \beta, \beta^*; t) = g^*(p)e^{i(\mathfrak{L}_F + \nu)t}\,\mathrm{Tr}a(p)W(t) \ , \tag{4.3}$$

$$C(p; \beta, \beta^*; t) = e^{i\mathfrak{L}_F t}\,\mathrm{Tr}a^\dagger(p)a(p)W(t) \ . \tag{4.4}$$

where $\hbar\mathfrak{L}_F = L_F$ and we recall that Tr stands for the trace over all atomic variables. The function P defined by (4.1) is the field distribution function in the Glauber diagonal representation: it is the function we are interested in. The functions A, B, and C are auxiliary functions related to the active atoms, and of course there is a corresponding set of functions $\bar{A}$, $\bar{B}$, and $\bar{C}$ related to the passive atoms. Using Eq. (2.1) and the definitions (4.1) to (4.4), we easily get

$$\left(i\hbar\frac{\partial}{\partial t} - i\Lambda_F\right)P(\beta, \beta^*; t) = \hbar\sum_{p=1}^{N}\left(\frac{\partial}{\partial\beta^*}A(p; \beta, \beta^*; t) - \frac{\partial}{\partial\beta}B(p; \beta, \beta^*; t)\right)$$
$$+\hbar\sum_{q=1}^{\bar{N}}\left(\frac{\partial}{\partial\beta^*}\bar{A}(q; \beta, \beta^*; t) - \frac{\partial}{\partial\beta}\bar{B}(q; \beta, \beta^*; t)\right) \ , \tag{4.5}$$

$$[i\hbar\,\partial/\partial t + i\hbar\gamma_\perp(p) - i\Lambda_F - \hbar\omega(p) + \hbar\nu]\,B(p; \beta, \beta^*; t) = \hbar\,|g(p)|^2\{\beta P(\beta, \beta^*; t) + (\partial/\partial\beta^* - 2\beta)C(p; \beta, \beta^*; t)\} \ , \tag{4.6}$$

$$[i\hbar\,\partial/\partial t + i\hbar\gamma_\perp(p) - i\Lambda_F + \hbar\omega(p) - \hbar\nu]A(p; \beta, \beta^*; t) = -\hbar|g(p)|^2\{\beta^* P(\beta, \beta^*; t) + (\partial/\partial\beta - 2\beta^*)C(p; \beta, \beta^*; t)\} \ , \tag{4.7}$$

$$[i\hbar\partial/\partial t - i\Lambda_F + i\hbar\gamma_\parallel(p)]C(p; \beta, \beta^*; t) = i\hbar\gamma_\parallel(p)\tfrac{1}{2}[1 + \sigma(p)]P(\beta, \beta^*; t) + \hbar\beta A(p; \beta, \beta^*; t) - \hbar\beta^* B(p; \beta, \beta^*; t) \ , \tag{4.8}$$

and a similar set of equations for $\bar{A}$, $\bar{B}$, and $\bar{C}$. The only approximation we have introduced to derive these

equations is to assume that there are low concentrations of active and passive atoms in their respective cells. The next approximation we introduce is the adiabatic elimination of the atomic variables. More precisely, we assume that the inequality (3.38) holds, and we consider times t such that $t\gamma_\perp(p)\gg 1$ and $t\gamma_\parallel(p)\gg 1$. Practically, this enables us to replace the operators $\hbar\partial_t-\Lambda_F+\gamma_\perp(p)$ and $\hbar\partial_t-\Lambda_F+\gamma_\parallel(p)$ by $\gamma_\perp(p)$ and $\gamma_\parallel(p)$, respectively. This leads to

$$\left[\frac{\partial}{\partial t}-\hbar^{-1}\Lambda_F-\left(\sum_{p=1}^{N}\frac{|g(p)|^2}{\delta(p)}+\sum_{q=1}^{\bar N}\frac{|\bar g(q)|^2}{\bar\delta(q)}\right)\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)\right.$$
$$\left.-i\left(\sum_{p=1}^{N}\frac{\Delta(p)|g(p)|^2}{\delta(p)}+\sum_{q=1}^{\bar N}\frac{\bar\Delta(q)|\bar g(q)|^2}{\bar\delta(q)}\right)\left(\frac{\partial}{\partial\beta^*}\beta^*-\frac{\partial}{\partial\beta}\beta\right)\right]P(\beta,\beta^*;t)$$
$$=2\sum_{p=1}^{N}\frac{|g(p)|^2}{\delta(p)}\left[\frac{\partial^2}{\partial\beta\partial\beta^*}-\frac{\partial}{\partial\beta}\beta-\frac{\partial}{\partial\beta^*}\beta^*-i\Delta(p)\left(\frac{\partial}{\partial\beta^*}\beta^*-\frac{\partial}{\partial\beta}\beta\right)\right]C(p;\beta,\beta^*;t)$$
$$+2\sum_{q=1}^{\bar N}\frac{|\bar g(q)|^2}{\bar\delta(q)}\left[\frac{\partial^2}{\partial\beta\partial\beta^*}-\frac{\partial}{\partial\beta}\beta-\frac{\partial}{\partial\beta^*}\beta^*-i\bar\Delta(q)\left(\frac{\partial}{\partial\beta^*}\beta^*-\frac{\partial}{\partial\beta}\beta\right)\right]\bar C(q;\beta,\beta^*;t)\,, \quad (4.9)$$

$$\left(\frac{1+\sigma(p)}{2}+\frac{2|g(p)|^2|\beta|^2}{\gamma_\parallel(p)\delta(p)}\right)P(\beta,\beta^*;t)=\left[1-\frac{|g(p)|^2}{\gamma_\parallel(p)\delta(p)}\left(\beta\frac{\partial}{\partial\beta}+\beta^*\frac{\partial}{\partial\beta^*}-4|\beta|^2\right)\right.$$
$$\left.-\frac{i\Delta(p)|g(p)|^2}{\gamma_\parallel(p)\delta(p)}\left(\beta\frac{\partial}{\partial\beta}-\beta^*\frac{\partial}{\partial\beta^*}\right)\right]C(p;\beta,\beta^*;t) \quad (4.10)$$

and a similar equation for $\bar C(q;\beta,\beta;t)$. In Eqs. (4.9) and (4.10) we have used the notation $\delta(p)=\gamma_\perp(p)[1+\Delta^2(p)]$, $\Delta(p)=[\omega(p)-\nu]\gamma_\perp^{-1}(p)$. As a last step towards the Fokker-Planck equation, we neglect the derivation operators in Eq. (4.10). We shall not attempt to justify this procedure here because in a future paper[18] we provide a fairly complete discussion of the "Fokker-Planck" equations with and without this last approximation. Introducing $\beta=re^{i\theta}$, we finally get

$$\frac{\partial}{\partial t}P(r,\theta;t)=\left[\frac{1}{r}\frac{\partial}{\partial r}r^2E(r^2)+\frac{\partial}{\partial\theta}F(r^2)+\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)G(r^2)\right]P(r,\theta;t)\,. \quad (4.11)$$

In the frame of the simplified atomic model in which all atoms are equivalent and homogeneously distributed in each cell, the three coefficients appearing in Eq. (4.11) are

$$E(y)=\kappa\left(1-\frac{A}{1+Sy}-\frac{1-C}{1+aSy}\right)\,,$$
$$F(y)=-\frac{\kappa A\Delta}{1+Sy}-\frac{\kappa\bar\Delta(1-C)}{1+aSy}\,,$$
$$G(y)\simeq q=\frac{|g|^2(1+\sigma)N}{4\delta}+\frac{|\bar g|^2(1+\bar\sigma)\bar N}{4\bar\delta}\,.$$

For E and F we have used the parameters A, a, C, and S defined in the semiclassical description. One sees at once that the drift term $E(r^2)$ is intimately connected to the SC intensity equation. The coefficient of the angular derivative $F(r^2)$ has also a known structure: it is related to the correction of the unperturbed frequency [see Eq. (3.40)]. The only new coefficient is the "diffusion" coefficient $G(r^2)$ for which we only retained the dominant contribution, namely $G(y)\simeq q$. The occurrence of the "diffusion" part in the Fokker-Planck equation is ultimately connected with the spontaneous emission process and enables us to take fluctuations into account. Equation (4.11) generalizes the equation first deduced in Ref. 9 for the laser with active atoms only. If we introduce the normalized intensity $\zeta=Sr^2$ and neglect the diffusion term, Eq. (4.11) becomes

$$\frac{\partial}{\partial t}P(\zeta,\theta;t)=\left(2\frac{\partial}{\partial\zeta}\zeta E(S^{-1}\zeta)+\frac{\partial}{\partial\theta}F(S^{-1}\zeta)\right)P(\zeta,\theta;t)\,.$$

This equation can be solved by the method of characteristics. One immediately verifies that the time-evolution equation for the variable ζ coincides with Eq. (3.39), whereas the time-evolution equation for the variable θ coincides with Eq. (3.9) when the inequalities (3.38) hold, i.e.,

$$\frac{d\zeta}{dt}=2\zeta E(S^{-1}\zeta)=2\kappa\zeta\left(1-\frac{A}{1+\zeta}-\frac{1-C}{1+a\zeta}\right)\,,$$
$$\frac{d\theta}{dt}=F(S^{-1}\zeta)\,.$$

B. Stationary solutions

To begin our analysis of Eq. (4.11), we seek its stationary solution which is independent of θ. The stationary equation is

$$\left(rE(r^2)+q\frac{d}{dr}\right)P(r)=0 \quad (4.12)$$

and its solution is

$$P(z) = N^{-1}\{e^{-z}(1+z)^{A}(1+az)^{(1-C)/a}\}^{\epsilon} \qquad (4.13)$$

where $z = Sr^2$, $\epsilon = \kappa(2Sq)^{-1}$, and the normalization constant N is defined by

$$N = \pi S^{-1}\int_0^\infty dz[e^{-z}(1+z)^{A}(1+az)^{(1-C)/a}]^{\epsilon} \ .$$

The extrema of $P(z)$ are given by the condition $dP(z)/dz = 0$; i.e.,

$$z\left(1 - \frac{A}{1+z} - \frac{1-C}{1+az}\right) = 0$$

which is exactly our SC equation (3.18) for the intensity. It is therefore very easy to discuss the behavior of $P(z)$ in relation with the discussion of the roots of Eq. (3.18):

$$\text{(i) } 0 < a < C/C-1 \begin{cases} I = 0, & A \leq C \ , \\ I = 0, I_+ & A \geq C \ . \end{cases}$$

This means that when $A \leq C$ the only physical root is $I = 0$ and therefore $P(z)$ has one maximum at $z = 0$. When $A \geq C$, $P(z)$ has a minimum at the origin ($z = 0$) and a maximum for $z = I_+$ [see Eq. (3.20)].

$$\text{(ii) } a > C/C-1 \begin{cases} I = 0, & aA < X_+ \ , \\ I = I_+, I_-, 0, & X_+ < aA < aC \ , \\ I = I_+, 0, & A > C \ . \end{cases}$$

In this situation $P(z)$ has one maximum at $z = 0$ when $X_+ > aA$, two maxima (at $z = 0$ and $z = I_+$) and one minimum at $z = I_-$ when $X_+ < aA < aC$, and finally $P(z)$ has a minimum at $z = 0$ and a maximum at $z = I_+$ when $A > C$. We stress that in our model one cannot make the so-called "cubic" approximation, i.e., expand the drift term $E(y)$ into a geometrical series and keep only the first two terms. In fact for $X_+ < aA < aC$ one would not obtain a two-peaked distribution function. The saturation effects arising from the denominators $1 + Sy$ and $1 + aSy$ in $E(y)$ must be fully taken into account.

A first conclusion which can be drawn from this analysis is that a stable solution of Eq. (3.18) always corresponds to a maximum of $P(z)$, whereas an unstable solution corresponds to a minimum. In other words, a stable solution is related to a most probable value, and an unstable solution is related to a least probable value in the distribution function $P(z)$. The basic difference between our present problem and the usual laser problem (in which there is only one cell) is that now there is a domain in which $P(z)$ has two maxima. Consequently, in the SC description we found two stable stationary solutions. It is obvious that in the quantum case there will be a unique mean stationary intensity despite the existence of a double-peaked distribution function. Furthermore, it becomes obvious why I_- is never a stable solution.

By means of the stationary solution $P(z)$ we may evaluate the various stationary moments of the field. In particular, we have studied the first and second moments defined by

$$\langle I \rangle = S\langle r^2 \rangle = 2\pi S\int_0^\infty r\,dr\,r^2 P(r)$$

$$= \pi S^{-1}\int_0^\infty dz\, zP(z) \ ,$$

$$\langle I^2 \rangle = S^2\langle r^4 \rangle = 2\pi S^2\int_0^\infty r\,dr\,r^4 P(r)$$

$$= \pi S^{-1}\int_0^\infty dz\, z^2 P(z) \ .$$

Note that we used reduced variables as in the SC description. Throughout this paper all integrals involving $P(z)$ are evaluated numerically.

Figure 6 displays a plot of the intensity $\langle I \rangle$ for two values of ϵ and the corresponding SC result [or equivalently, the position of the three extrema of $P(z)$]. Note that the transition region becomes much narrower when ϵ is changed from 10 to 100. As a matter of fact, our choice of $\epsilon = 10$ and 100 is rather pedagogical because a realistic estimate of ϵ for a solid-state laser is $\epsilon \sim 10^3$. But for such values of ϵ the transition is so sharp that it is hardly possible to draw, e.g., the graph of the fluctuations versus A. Anyway, the case $\epsilon = 100$ already clearly shows that $\langle I \rangle$ exhibits a first-order phase-transition-like behavior, giving a rather sharp definition of the threshold for laser action. This feature arises from the fact that in the largest part of the bistable region one of the two peaks of the distribution function completely dominates the other. In other words, increasing the pump

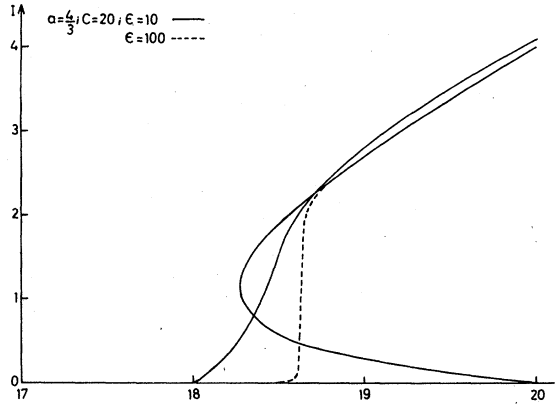

FIG. 6. Comparison between the SC intensity (parabolic curve) and the mean quantum intensity for two values of ϵ.

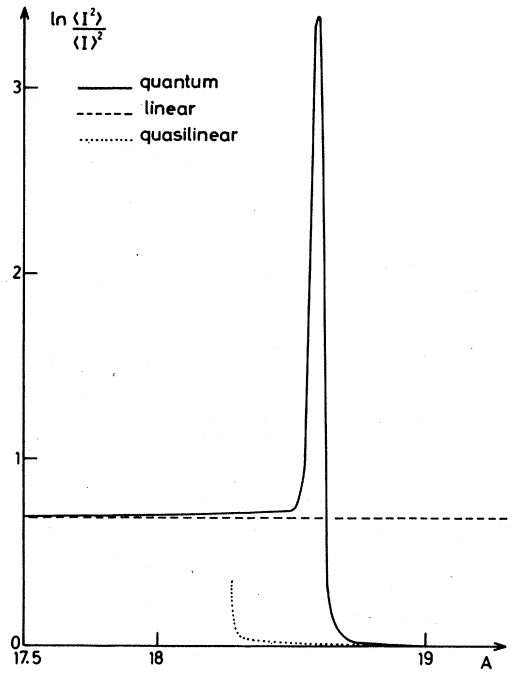

FIG. 7. Intensity fluctuations calculated from the exact and from the two asymptotic quantum theories. The parameters are $a=\frac{4}{3}$, $C=20$, and $\epsilon=100$.

parameter A from the value X_+/a one sees that $P(z=I_+)$ remains much smaller than $P(0)$ until A reaches the transition region. In this narrow region of transition, the peak at $z=I_+$ rapidly grows up, so that $P(I_+)$ becomes larger than $P(0)$.

Figure 7 shows the logarithm of the ratio $\langle I^2\rangle/\langle I\rangle^2$ which is a suitable measure of the fluctuations (the quantum evaluation to which we are referring here is drawn in full line in Fig. 7). Comparison with Fig. 6 shows that there is a drastic increase of the fluctuations in the transition region. As a matter of fact, in this transition region the two peaks of the distribution function have comparable heights, so that the mean value of the intensity is largely different both from 0 and from I_+, thereby producing extremely important fluctuations. Note further that the width of the domain in which the graph significantly deviates from ln2 (under threshold) and from zero (above threshold) is very small and practically coincides with the transition region. We shall come back to this point in Sec. V.

C. Transient solutions

To give at least a basic quantum description of the laser field there remains to find two more characteristics: the effective frequency and the linewidth of the field. Because these two functions are associated with time-dependent properties of the field, we must seek the transient solution of Eq. (4.11). Such a program cannot be carried out in an exact and analytical way. We shall therefore use an approximation scheme which proved to be successful in the case of an ordinary laser.[11] Let us consider the Fokker-Planck equations (4.11). Its general solution has the form

$$P(r,\theta;t)=\sum_{m=-\infty}^{+\infty}\sum_{n=0}^{\infty}A(n,m)e^{im\theta}P_{n,m}(r)\times e^{-\lambda(n,m)t} \tag{4.14}$$

$$0<\mathrm{Re}\lambda(n,m)<\mathrm{Re}\lambda(n+1,m)\ ,$$

where $\lambda(n,m)$ and $P_{n,m}(r)e^{im\theta}$ are the eigenvalues and eigenfunctions of the Fokker-Planck operator and the constants $A(n,m)$ depend on the initial condition. In the long-time limit one can keep for each m only the contribution corresponding to the eigenvalue $\lambda(0,m)$ with the smallest real part. Hence the solution (4.14) takes the form

$$P(r,\theta;t)\underset{t\to\infty}{\sim}\sum_{m=-\infty}^{+\infty}e^{im\theta}P_{0,m}(r)\phi(m,t)\ . \tag{4.15}$$

In particular, when E, F, and G in Eq. (4.11) are constant, one has

$$P_{0,m}(r)=r^{|m|}P(r) \tag{4.16}$$

where $P(r)$ is the stationary solution [see also Eq. (5.2)]. We shall assume that for the determination of $\phi(m,t)$ expressions (4.15) and (4.16) are reasonable approximations when E and F are no longer constants. Of course $P(r)$ is now the exact stationary solution (4.13). Inserting the ansatz (4.15) into Eq. (4.11) and integrating over r, one finds an equation for $\phi(m,t)$ whose solution is

$$\phi(m,t)=\phi(m,0)\exp[im\Omega(m)-m^2\tau(m)]t \tag{4.17}$$

with

$$\Omega(m)=\frac{\int_0^\infty dr\,r^{|m|+1}F(r^2)P(r)}{\int_0^\infty dr\,r^{|m|+1}P(r)}=\frac{\int_0^\infty dz\,z^{|m|/2}F(z)P(z)}{\int_0^\infty dz\,z^{|m|/2}P(z)} \tag{4.18}$$

and

$$\tau(m)=\frac{q\int_0^\infty dr\,r^{|m|-1}P(r)}{\int_0^\infty dr\,r^{|m|+1}P(r)}=\frac{qS\int_0^\infty dz\,z^{-1+|m|/2}P(z)}{\int_0^\infty dz\,z^{|m|/2}P(z)}\ . \tag{4.19}$$

Because the field is proportional to $\langle\beta\rangle=\langle re^{i\theta}\rangle$, its linewidth and frequency are given by $\tau(1)$ and $\Omega(1)$, respectively. In Fig. 8 we have shown two plots (corresponding to two values of ϵ) of $\ln[1+\tau(1)/qS]$ which is a measure of the linewidth. This is the result of the numerical evaluation of (4.19). We see that with increasing values of ϵ the transition from a finite value of $\tau(1)$ towards a nearly vanishing linewidth

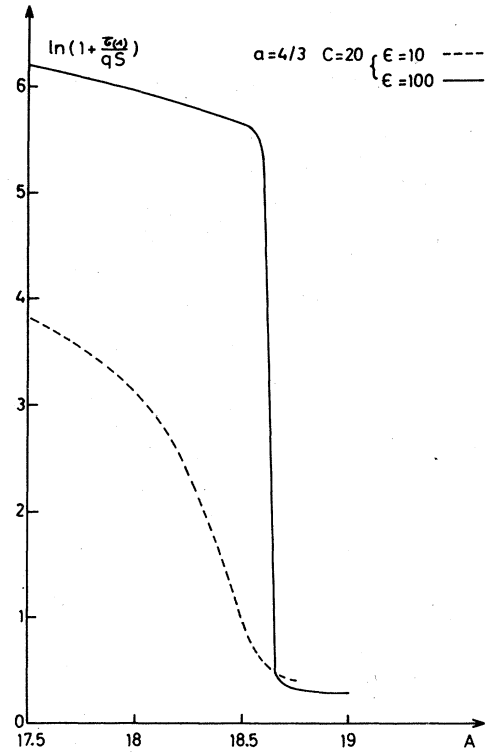

FIG. 8. For clarity we have plotted here the logarithm of 1 plus the reduced linewidth $\tau(1)/qS$. These graphs show the influence of ϵ on the sharpness of the transition.

becomes very sharp. This corresponds to an abrupt line narrowing which occurs when the laser passes from below to above the threshold. An effect of this type has been observed in the experiments of Ref. 6(b). In connection with this result we want to mention a difference which arises with respect to the usual laser problem. Let us define a linewidth factor α through $\tau(1) = \alpha qS\langle I\rangle^{-1}$. In the usual laser case (one cell) the function α decreases monotonically from 2 (well below threshold) to 1 (well above threshold). In the present case the extreme values 2 and 1 remain the asymptotic values of α, but in the transition region α is no longer a monotonic function: two maxima of α (versus A) arise in the transition region. This means that the actual dependence of the linewidth with respect to the intensity is much more complicated than a simple $\langle I\rangle^{-1}$-dependence.

Because we are working in the interaction representation [see (4.1)], the function $\Omega(1)$ is the correction to the unperturbed frequency. But this correction is relevant only above threshold where $P(z)$ is sharply peaked around one maximum and where the field is practically a pure coherent state. Hence we may approximate $\Omega(1)$ by (see also Ref. 11)

$$\Omega(1) = \Omega - \nu \simeq F(\langle r^2\rangle) ,$$

i.e.,

$$\Omega = \nu - \frac{\kappa\Delta A}{1+\langle I\rangle} - \frac{\kappa\bar{\Delta}(1-C)}{1+\alpha\langle I\rangle} . \tag{4.20}$$

This is precisely the equation (3.40), obtained in the SC description. The difference is that now $\langle I\rangle$ is the true stationary intensity.

V. ASYMPTOTIC DESCRIPTIONS

Although we now have a fully quantum description of our system, we are not yet satisfied. The trouble with an equation like (4.11) is that the considerable amount of information it contains is nearly inaccessible owing to the difficulty of obtaining a complete analytical solution. It is for this reason that we investigate in this section two asymptotic descriptions (leading to the linearized and quasi-linearized Fokker-Planck equations) in order to determine as far as possible their relative domains of application. It is quite clear that such theories cannot cover the transition region; however, there remains to know the width of the region which is not covered by these asymptotic descriptions. We shall see that this width is surprisingly small.

A. Linearized Fokker-Planck equations

The linearized theory is a weak-coupling theory: one assumes that the field-matter interaction is weak. Instead of rederiving this theory we shall simply start from Eq. (4.11) and retain in the three coefficients E, F, and G the dominant contribution in g and $\bar{g}$. Hence we have

$$\frac{\partial}{\partial t}P(r,\theta;t) = \left[\frac{E}{r}\frac{\partial}{\partial r}r^2 + F\frac{\partial}{\partial\theta} + q\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\right]P(r,\theta;t) \tag{5.1}$$

where

$$E = \kappa(C-A) ,$$

$$F = -\kappa[\Delta A + \bar{\Delta}(1-C)] ,$$

$$q = \frac{|g|^2 N(1+\sigma)}{4\delta} + \frac{|\bar{g}|^2\bar{N}(1+\bar{\sigma})}{4\bar{\delta}} .$$

Equation (5.1) is a typical Fokker-Planck equation and can be solved analytically. The solution is

$$P(r,\theta;t) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{+\infty} A(n,m)e^{im\theta}e^{-\lambda(n,m)t}r^{|m|} \times e^{-r^2E/2q}L_n^{|m|}(r^2E/2q) . \tag{5.2}$$

In this formula $A(n,m)$ are constants determined by the initial condition, $\lambda(n,m)$ is given by

$$\lambda(n,m) = (2n + |m|)E - imF \ , \tag{5.3}$$

and $L_n^{|m|}(y)$ are the associated Laguerre polynomials.[19] The stationary solution of (5.1) is

$$P(z) = S\epsilon\pi^{-1}(C-A)e^{-z\epsilon(C-A)} \tag{5.4}$$

with the notation used for Eq. (4.13). Let us note that $P(z)$ is normalizable if and only if $E>0$ which means $A<C$. This gives a mathematical upper bound to the domain of validity of the linearized description. The first stationary moments are

$$\langle I\rangle = [\epsilon(C-A)]^{-1} \ , \tag{5.5}$$

$$\langle I^2\rangle = 2[\epsilon(C-A)]^{-2} \ , \tag{5.6}$$

so that $\langle I^2\rangle/\langle I\rangle^2 = 2$, whereas the linewidth of the field is given by

$$\tau(1) = \mathrm{Re}\lambda(0,1) = E = 2qS/\langle I\rangle \ . \tag{5.7}$$

We shall comment on these results after having derived the quasilinearized equation.

B. Quasilinearized Fokker-Planck equation

In Sec. IV we have seen that when the laser is above threshold, the intensity fluctuations are very small. This feature suggests immediately the possibility of linearizing the Fokker-Planck equation (4.11) around the value I_+ of the intensity. This procedure is well known in the case of the usual laser[20] and is called quasilinearization. To illustrate it let us first rewrite Eq. (4.11) as follows:

$$\frac{\partial}{\partial t}P(\zeta,\theta;t) = \left[2\frac{\partial}{\partial\zeta}\zeta E(S^{-1}\zeta) + \frac{\partial}{\partial\theta}F(S^{-1}\zeta) + qS\left(4\frac{\partial}{\partial\zeta}\zeta\frac{\partial}{\partial\zeta} + \frac{1}{\zeta}\frac{\partial^2}{\partial\theta^2}\right)\right]P(\zeta,\theta;t) \tag{5.8}$$

with $\zeta = Sr^2$. Introducing the variable $x = \zeta - I_+$, one develops the terms of Eq. (5.8) into a Taylor expansion around $x=0$ and for each of them keeps only the first nonvanishing contribution. One obtains

$$\frac{\partial}{\partial t}P(x,\theta;t) = \left[8Sqb^2I_+^{-1}\frac{\partial}{\partial x}x + F(S^{-1}I_+)\frac{\partial}{\partial\theta} + qS\left(4I_+\frac{\partial^2}{\partial x^2} + \frac{1}{I_+}\frac{\partial^2}{\partial\theta^2}\right)\right]P(x,\theta;t) \ , \tag{5.9}$$

where we have introduced the notation

$$b = I_+\left[\frac{\epsilon}{2}\left(\frac{A}{(1+I_+)^2} + \frac{a(1-C)}{(1+aI_+)^2}\right)\right]^{1/2} . \tag{5.10}$$

Equation (5.9) is a Fokker-Planck equation; its analytic solution is

$$P(x,\theta;t) = \sum_{n=0}^{\infty}\sum_{m=-0}^{+\infty} A(n,m)e^{im\theta}e^{-\lambda(n,m)t} \times e^{-(bx/I_+)^2}H_n(bx/I_+) \ , \tag{5.11}$$

where $A(n,m)$ are constants determined by the initial condition, $\lambda(n,m)$ is given by

$$\lambda(n,m) = (qS/I_+)(8b^2n + m^2) - imF(S^{-1}I_+) \ , \tag{5.12}$$

and $H_n(y)$ are the Hermite polynomials.[19] The stationary solution of Eq. (5.8) is

$$P(x) = N\exp[-(bx/I_+)^2] = N\exp\{-(b(\zeta-I_+)/I_+]^2\} \ , \tag{5.13}$$

$$N = 2bS[\pi^{3/2}I_+\,\mathrm{erfc}(-b)]^{-1} \ ,$$

$$\mathrm{erfc}(y) = \frac{2}{\sqrt{\pi}}\int_y^{\infty} dt\, e^{-t^2} \ .$$

Note that (5.13) is the Gaussian approximation of the exact stationary distribution (4.13) around $\zeta = I_+$. The first stationary moments are

$$\langle I\rangle = I_+\left(1 + \frac{e^{-b^2}}{\sqrt{\pi}\,b\,\mathrm{erfc}(-b)}\right), \tag{5.14}$$

$$\langle I^2\rangle = I_+^2/2b^2 + I_+\langle I\rangle \ . \tag{5.15}$$

For $b \gg 1$ Eqs. (5.14) and (5.15) practically reduce to

$$\langle I\rangle = I_+ \ , \tag{5.14'}$$

$$\langle I^2\rangle = I_+^2(1 + 1/2b^2) \ , \tag{5.15'}$$

and $\ln(\langle I^2\rangle/\langle I\rangle^2) \simeq 1/2b^2 \ll 1$. The linewidth of the field is given by

$$\tau(1) = \mathrm{Re}\lambda(0,1) = qS/\langle I\rangle \tag{5.16}$$

while $\mathrm{Im}\lambda(0,1)$ gives the usual phase shift [see Eq. (4.20)].

C. Comparison with the complete Fokker-Planck equation

From Eq. (5.14') and Fig. 6 we see that above threshold the intensity obtained in the quasilinear approximation practically coincides with the intensity calculated (numerically) with the stationary solution (4.13): they both coincide with I_+. The situation under threshold is illustrated in Fig. 9 in which we have plotted the ratio of the intensity calculated with Eq. (4.13) and the linearized intensity (5.5). A comparison of the linewidths is shown in Fig. 10 where the dotted (dashed) curve gives the ratio of the exact and linearized (quasilinearized) linewidth [see Eqs. (5.7) and (5.16)]. From the Figs. 9 and 10 one sees that the situation is very clear: either the linearized theory gives a very good result (as it does well below threshold) or it entirely fails to give any sensible result (from the transition region upwards). The same holds for

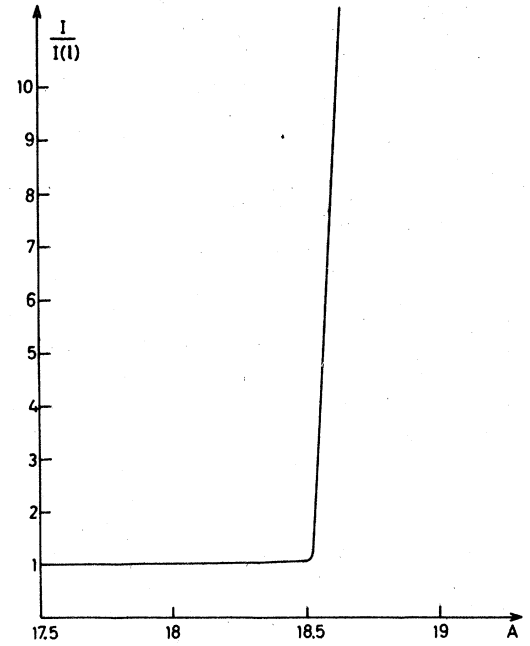

FIG. 9. Ratio of the exact and the linearized intensities versus A for $a=\frac{4}{3}$, $C=20$, and $\epsilon=100$.

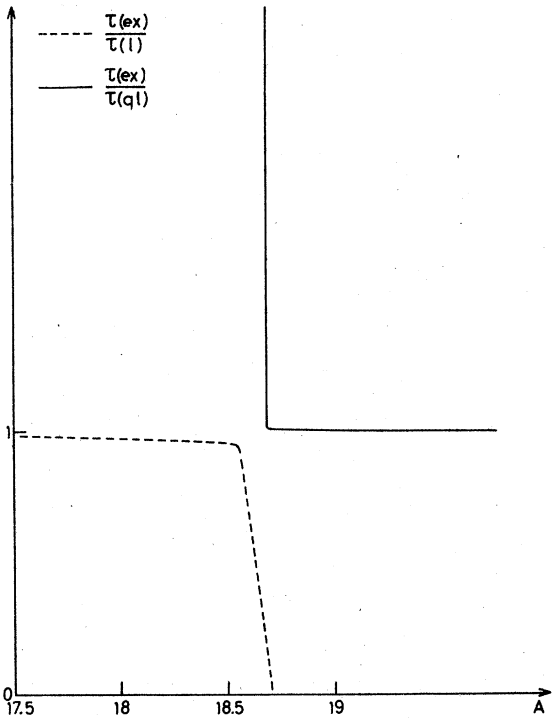

FIG. 10. $\tau(l)$ and $\tau(ql)$ are the linewidth derived from the linearized and the quasilinearized theories, respectively. The parameters are $a=\frac{4}{3}$, $C=20$, and $\epsilon=100$.

the quasilinearized theory which works excellently only above threshold. Hence it is only in the transition region that the exact solution (4.13) is really necessary to get meaningful results. This conclusion is confirmed by Fig. 7 where we have plotted $\ln(\langle I^2\rangle/\langle I\rangle^2)$ for the exact stationary solution (full line), for the linearized theory (horizontal line), and for the quasilinearized theory (dotted line). The plot for the quasilinear theory begins of course at $A=X_+/a$ and is obtained by use of Eqs. (5.15) and (5.16). Note that when A approaches X_+/a from above, one has $b\to 0$ so that $\langle I^2\rangle/\langle I\rangle^2\to(\pi-2)/2$.

D. Metastable states

On the basis of the stationary distribution (4.13), one sees that if one excludes the narrow transition region, then for each A in the range $X_+<aA<aC$ only one of the two steady-state solutions $I=I_+$ and $I=0$ is absolutely stable. However, the other solution still has a physical meaning as a metastable state. To illustrate this point let us consider the function

$$V(z)=-\ln[P(z)/P(0)]\,, \tag{5.17}$$

where $P(z)$ is given by (4.13). In our problem $V(z)$ plays the role of a generalized thermodynamic potential. It has two minima at $I=0$ and $I=I_+$ separated by a "barrier" centered on $I=I_-$. The stable state corresponds to the absolute (i.e., the lowest) minimum. The problem is similar to that of a Brownian particle moving in the potential profile (5.17). If the particle is initially in the well centered at the relative minimum, it remains there until a large fluctuation makes it jump over the barrier. Hence the relative minimum corresponds to a metastable state. Let us come back to our system. Since it remains in the metastable state for a relatively long time, it makes sense to ask which are the properties of the system when it is in this state. To answer this question one should, in principle, solve Eq. (4.11) for an initial distribution function peaked around the metastable state and analyze the solution in the time interval in which the distribution function remains peaked there. However, an approximate evaluation can be obtained by making the reasonable assumption that the behavior of the system as long as it remains in the metastable state is determined by the part of the potential profile which surrounds the relative minimum. On the basis of this assumption we can study the properties of the metastable state using the linearized or the quasilinearized Fokker-Planck equation.

To be more precise let us consider first the SC solution $I=0$ for $A\leq C$. Under threshold, $I=0$ is the absolutely stable state and we already know

that the linearized equation (5.1) describes well the properties of the system in this situation. For values of A varying from the transition region to $A=C$, the state $I=0$ becomes metastable but it is still described by the linearized equation (5.1). Hence one associates to the quasistationary metastable state the stationary solution (5.4) which describes the system as long as it remains in the metastable state. Of course this procedure makes sense only as long as the distribution (5.4) remains narrow, i.e., as long as A remains somewhat smaller than C. In such a way Eqs. (5.5) to (5.7), for $A<C$ but above threshold, are reinterpreted as describing the properties of the metastable state $I=0$. In particular since $E\sim C-A$ one sees that this metastable state exhibits a line narrowing as A approaches C. A completely parallel discussion can be repeated for the state $I=I_+$. Namely, for $A>X_+/a$ but below threshold, we associate the quasilinear equation (5.9) to the *metastable* state $I=I_+$. Hence in this range of values of A, Eqs. (5.14)–(5.16) are reinterpreted as describing the properties of this metastable state. In particular, one finds a line broadening as A approaches X_+/a from above. In conclusion, the linearized and quasilinearized Fokker-Planck equations give a meaningful description in the whole range of their definition either referring to the stable or referring to the metastable state.

VI. CONCLUSIONS

We have introduced a simple model to describe a solid-state laser with an absorbing cell, and we have studied it both at the semiclassical and at the quantum-mechanical level.

The SC equation which gives the stationary values of the intensity I has been solved analytically. A remarkable feature of this equation is that it depends only on three parameters a, A, and C despite the large number of parameters introduced in the model: a measures the relative saturability of the absorbing atoms with respect to the active atoms, A is the pump parameter of the active atoms, and $C=1-B$ where $B<0$ is the pump parameter of the absorbing atoms. This reduction in the number of variables allows us to give a full classification of the SC steady-state solutions. In particular, one can explicitly specify the range of values of a, A, and C for which one finds three different stationary solutions $I=0$, $I=I_-$, $I=I_+$. The stability of these solutions has been investigated by means of the extended Routh-Hurwitz theorem. We showed analytically that the solution $I=I_-$ is always unstable, whereas the solution $I=0$ is stable only for $A<C$. Our stability analysis for the solution $I=I_+$ is incomplete because the inequalities arising from the Routh-Hurwitz determinants are too complicated to be studied analytically. These determinants have been evaluated numerically for a suitable choice of the parameters in the bistable region for which the solution $I=I_+$ turned out to be stable. On the basis of this analysis, it is shown that in the bistable case the SC intensity versus the pump parameter A exhibits a hysteresis cycle. We give explicit expressions for the maximum and minimum values of A and I which fix the vertices of the cycle. In particular, it turns out that the maximum intensity is at least twice the minimum intensity and that in the limit of large a the width and the height of the cycle both become equal to $C-1$. Moreover, we can show analytically that approaching the maximum and the minimum values of A in the cycle, one finds a critical slowing down of the macroscopic mean values (intensity, polarizations, and atomic inversions).

The quantum-mechanical description is based on a suitable Fokker-Planck equation for the Glauber quasiprobability distribution function of the field. This equation is deduced under the assumption of low density of active and passive atoms and by performing the adiabatic elimination of the atomic variables. It can be solved exactly in the stationary case. The solution depends only on four parameters: the three SC parameters a, A, and C and a new parameter ϵ which appears in the diffusion term of the Fokker-Planck equation and is therefore linked to the fluctuations. In the bistable domain the stationary distribution function has two maxima at $I=0$ and $I=I_+$ and one minimum at $I=I_-$. This means that the two stable SC solutions $I=0$ and $I=I_+$ correspond to most probable values, whereas the unstable solution $I=I_-$ corresponds to a least probable value. The plot of the mean value of the intensity versus A shows that it practically coincides with zero or with I_+ except in a transition region. In this region the two peaks of the probability distribution have comparable heights and the intensity undergoes extremely high fluctuations. The width of the transition region is determined by the parameter ϵ: if ϵ is large, the transition region is very narrow, and in the limit $\epsilon\to\infty$ one gets an infinitely sharp transition recovering a true first-order phase transition. The transient properties (e.g., line width and phase shift) have been studied by means of a suitable procedure which allows us to evaluate approximately the solution of the Fokker-Planck equation in the long-time limit. Furthermore, we have investigated the possibility of using asymptotic but exactly solvable equations. It turns out that the linearized Fokker-Planck equation reproduces quite well the exact results for A below the transition region.

The same holds for the quasilinearized Fokker-Planck equation when A is above the transition region. The interest in this result arises from the fact that the transition region is extremely small. On the other hand, the full nonlinear Fokker-Planck equation is essential to define the transition region, as well as to describe the properties of the system in the transition region. To this extent the nonlinearity must be fully taken into account because an approximation of the "cubic" type changes completely the characteristics of the solution. Finally, the linear and the quasilinear descriptions are reinterpreted in connection with the metastable states. In fact, for $A \leq C$ but above threshold, the state $I=0$ is metastable and we describe the properties of this metastable state by means of the linearized Fokker-Planck equation. Likewise, below threshold, the state $I=I_+$ is metastable and we associate to it the solution of the quasilinearized Fokker-Planck equation.

Practically all the results of this paper should be subjected to experimental test. The experiment should concern both the metastable states (appearance of an hysteresis cycle, dependence of the size of this cycle on the relevant parameters, presence of a critical slowing down when the boundaries of the cycle are approached, fluctuation and linewidth in the metastable states) and the stable states (dependence of the mean intensity and of the linewidth on the pump parameter, position and width of the transition region, fluctuations in the transition region). Experimentally, a metastable state can be achieved, e.g., by starting from a stable state and rapidly varying the pump parameter until the metastable region is reached. Then the system will remain temporarily "trapped" in that metastable state which can then be observed and studied.

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†On leave of absence from Istituto di Fisice dell'Universita, 20133 Milano, Italy.

‡Chercheur qualifié F.N.R.S.

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