Comparison of a semiclassical stochastic-master-equation approach to laser fluctuations with the Scully-Lamb theory

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The analogy of a laser with an autocatalytic chemical reaction is used to write down a macroscopic stochastic master equation of the birth-and-death type for the photon probability function in a semiclassical laser formalism. This equation may be solved exactly in the steady state using a generating function. This leads to a prescription for calculating all the higher-order moments of the photon number as well as providing a unique truncation of the hierarchy of moment equations at any given order. Further, the variance is given by a Poisson distribution, in agreement with the strong-signal Scully-Lamb theory. This master equation is also shown to lead to an exact Fokker-Planck equation by using a modified Kramers-Moyal expansion. A comparison between our results and the (microscopic) Scully-Lamb theory shows that the two approaches give the same results at large photon number.

I. INTRODUCTION

It is well known[1,2] that a typical laser exhibits a second-order phase transition at a critical value of the population inversion. Below threshold, the laser light in the cavity is chaotic and has a Planck distribution. Above threshold, the light becomes increasingly coherent until the so-called "strong-signal" regime is reached, where the photon number is very large and the light is totally coherent having a Poisson distribution. The photon statistics above threshold have been derived using the quantum Scully-Lamb master equation[2] as well as a c-number Fokker-Planck equation.[1,3] However, there does not appear to be at present a semiclassical treatment of laser fluctuations. One reason for this is that in a semiclassical theory, the photon number is always large (the photons being in coherent states) so that any semiclassical calculation would give results for the laser fluctuations and statistics only in the strong-signal region. Further, the semiclassical treatment neglects spontaneous emission (which serves as a noise source) below threshold. It would appear, however, that in the strong-signal regime, the results of a semiclassical calculation are exact and we shall consider the laser statistics in this region.

The semiclassical equations are equations for averaged quantities and neglect fluctuations. In order to include classically the effects of fluctuations, we introduce a macroscopic probability function $P(n,t)$, which gives the probability of finding n photons in the cavity at time t. The equation for P is stochastic and has the gain-loss form effectively treating the laser as a birth-and-death process. Such methods have been extensively used in the theory of fluctuations in nonlinear chemical reactions.[4,5] Our treatment will lead to a simple prescription for calculating all the higher moments of the energy density n (this being the classical analog of the photon number) as well as a simple truncation scheme for the Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy of moment equations at any given order. This is one of the chief advantages of the birth-and-death approach to fluctuation theory: If a steady-state solution of the master equation can be found from a generating function, then all the moments may be calculated in closed form. Further, in nonlinear systems such as the laser, the nonlinearity manifests itself in the transition probabilities appearing in the macroscopic master equation, a feature that is absent in microscopic theories. The moments of n obtained by this present method are compared with the corresponding Scully-Lamb results and found to agree for large photon numbers, this being the first time that such a comparison between a macroscopic stochastic formulation and the corresponding microscopic theory has been made.

It will be seen that an exact comparison between the hierarchy obtained from the stochastic master equation and the corresponding Scully-Lamb hierarchy is not possible because one cannot derive a general expression for the kth moment (even though we do present a detailed analysis based on the first three moments of n). A more exact comparison is afforded by the transformation to a nonlinear Fokker-Planck equation in the continuous variable limit. Usually, this has been done using a Kramers-Moyal expansion.[6] Physically, the discrete variables occurring in the master equation are divided by a large characteristic parameter, e.g., the volume V of the system. Then, the nth term is of $O(V^{-n+1})$ and may be neglected in the thermodynamic limit. However, the volume dependence of the higher-order derivatives is frequently not known, this point having first been raised by Van Kampen,[6] who developed an approach which assumed the distribution function to be Gaussian in the thermodynamic limit and, moreover, that the stochastic value was distributed about its deterministic mean value. This assumption, however, breaks down near a critical point where it has been demonstrated[7] that the master equation predicts large fluctuations that drive the system away from its deterministic path. Further, the nonlinear Fokker-Planck equation has been shown[7] to be an exact asymptotic representation of the master equation, predicting the same behavior for the fluctuations as obtained from the master equation.

In Sec. II, we present a derivation of the laser master equation whose steady-state solution is found in Sec. III. In Sec. IV we derive the hierarchy of moment equations for this case as well as a prescription for truncating this hierarchy at any given order, followed by similar calculations based on the Scully-Lamb microscopic theory in Sec. V, where the results of the two approaches are compared. It will be seen that the omission of spontaneous emission in the semiclassical theory does not affect the results far above threshold, where stimulated processes play the dominant role. Finally we derive an exact Fokker-Planck equation in Sec. VI by expanding[8] the probability distribution in terms of a Poisson distribution. This equation is found to be equivalent with the corresponding Fokker-Planck equation derived from the Scully-Lamb theory up to a normalization constant.

II. LASER MASTER EQUATION

We begin with the single-mode semiclassical laser equations which have the form[1,2]

$$\dot{\rho}_{12} = (i\omega - \Gamma)\rho_{12} - igb^*d\,, \tag{1}$$

$$\dot{b}^* = (i\omega - \kappa)b^* + ig^*\rho_{12}\,, \tag{2}$$

$$\dot{d} = (d_0 - d)/T + 2i(gb^*\rho_{21} - g^*b\rho_{12})\,, \tag{3}$$

where $d = N_2 - N_1$ is the population difference in the atomic levels, d_0 being the equilibrium value, T and Γ^{-1} are atomic relaxation times, κ being the cavity half-width. ρ_{12} is proportional to the atomic dipole moment, and b, b^* are complex field amplitudes. In a fully quantum description, these would be photon destruction and creation operators and $n = b^*b$ would be the photon number in the mode under consideration. However, in a semiclassical theory, n is the energy density in the cavity field. We now eliminate the atomic variables adiabatically by setting $\dot{\rho}_{12} \approx 0 \approx \dot{d}$. This yields a rate equation diagonal in the energy density n. The approximation used is reasonable, since the motion of the atoms in the cavity is generally unaffected by the photons. We then obtain the familiar result

$$d \approx d_0(1 - 4g^2Tn/\Gamma)\,, \tag{4}$$

which from (2) leads to

$$\dot{n} = k_1 n - k_2 n^2\,, \tag{5}$$

where

$$k_1 = -2\kappa + 2g^2Td_0/\Gamma\,, \tag{6a}$$

$$k_2 = -8g^4d_0/\Gamma^2\,. \tag{6b}$$

Further, it has been shown that[9]

$$k_1 = A - C\,, \quad k_2 = B\,. \tag{7}$$

Here, A, C, and B are the Scully-Lamb parameters[2] and are, respectively, the linear gain, cavity loss, and saturation parameters. The subthreshold region is characterized by $A \ll C$ and above threshold we have $A \gg C$. We shall consider the Scully-Lamb theory in greater detail in Sec. V. The identification (7) will be used to provide a connection between our results and the fully quantum (microscopic) theory of Scully and Lamb, this being the principle object of this work.

It is possible to make a chemical analog to a single-mode laser. This has been done by McNeil and Walls[10] who proposed a simple autocatalytic reaction to represent stimulated emission. The reaction was allowed to be reversible thus giving rise to the nonlinearity, which corresponds to the atom-field interaction in laser theory. They also introduced reactions corresponding to spontaneous emission and losses, thereby obtaining a complete analogy with the quantum laser model close to threshold. It must be stressed here, however, that our model, being semiclassical, will not include spontaneous emission terms. In this important respect, it differs from the McNeil-Walls model. Accordingly, let us consider the autocatalytic reaction (corresponding to stimulated emission)

$$A + X \underset{\bar{k}_1}{\overset{\bar{k}_2}{\rightleftharpoons}} 2X\,, \tag{8}$$

which has the rate equation

$$\frac{dX}{dt} = \bar{k}_1 AX - \bar{k}_2 X^2\,, \tag{9}$$

the variables in the above referring to the concentrations of the corresponding quantities appearing in (8). The analogy with the semiclassical laser model is complete with the identification

$$\bar{k}_1 A \leftrightarrow k_1, \quad \bar{k}_2 \leftrightarrow k_2, \quad X \leftrightarrow n. \tag{10}$$

The master equation corresponding to (9) has the form[4,5]

$$\frac{dP(\{X\},t)}{dt} = \sum_{\rho} W(\{X_i - \mu_{i\rho}\} \to \{X_i\}) P(\{X_i - \mu_{i\rho}\},t) - \sum_{\rho} W(\{X_i\} \to \{X_i + \mu_{i\rho}\}) P(\{X_i\},t). \tag{11}$$

Here, the W's are transition probabilities given in general by

$$W(\{X_i - \mu_{i\rho}\} \to \{X_i\}) = k \prod_i (X_i - \mu_{i\rho}). \tag{12}$$

Further, $\mu = 0, \pm 1$ in this case, since we are dealing with a one-photon process. We then find, for the case under consideration,

$$\frac{dP(X,t)}{dt} = \bar{k}_1 A (X-1) P(X-1,t) - \bar{k}_1 A X P(X,t) + \bar{k}_2 X (X+1) P(X+1,t) - \bar{k}_2 X (X-1) P(X,t). \tag{13}$$

The first two terms in (13) are seen to represent the probability flow between the "levels" $X-1$ and X, $\bar{k}_1 A(X-1)$ corresponding to the transition rate for absorbtion. The second and third terms represent the probability flow between the levels X and $X+1$. Equation (13) is Markovian and has the gain-loss structure.

III. STEADY-STATE SOLUTION OF THE MASTER EQUATION; STATISTICS

We now seek a generating function solution to (13) of the form

$$F(s,t) = \sum_{X=0}^{\infty} s^X P(X,t). \tag{14}$$

Multiplying (13) by s^X and summing over X we readily obtain the differential equation

$$\frac{\partial F}{\partial t} = s(s-1)\left(W \frac{\partial F}{\partial s} - \bar{k}_2 \frac{\partial^2 F}{\partial s^2}\right), \tag{15}$$

where $W = \bar{k}_1 A$. In the steady state, (15) reduces to

$$\frac{\partial^2 F}{\partial s^2} = \frac{W}{\bar{k}_2}\frac{\partial F}{\partial s}, \tag{16}$$

admitting of the solution

$$F(s) = \exp[(W/\bar{k}_2)(s-1)]. \tag{17}$$

Further, it may be readily seen from (14) that

$$\langle X \rangle = \frac{\partial F}{\partial s}\bigg|_{s=1}, \tag{18}$$

$$\langle X^2 \rangle - \langle X \rangle = \frac{\partial^2 F}{\partial s^2}\bigg|_{s=1}, \tag{19}$$

and thus we find

$$\langle n \rangle = (A - C)/B, \tag{20}$$

and

$$\langle (\delta n)^2 \rangle \equiv \langle n^2 \rangle - \langle n \rangle^2 = \frac{\partial^2 F}{\partial s^2}\bigg|_{s=1} - \left(\frac{\partial F}{\partial s}\right)^2_{s=1} + \frac{\partial F}{\partial s}\bigg|_{s=1} = \langle n \rangle. \tag{21}$$

Equation (20) is simply the first-order Scully-Lamb result and (21) shows the photon distribution to be Poisson. This may be checked by returning to (17), which may be cast in the form

$$F(s) = e^{-\langle X \rangle} e^{s\langle X \rangle} = e^{-\langle X \rangle} \sum_{X=0}^{\infty} \frac{s^X \langle X \rangle^X}{X!}.$$

Comparing this with (14) we find

$$P(n) = \frac{e^{-\langle n \rangle}\langle n \rangle^n}{n!}, \tag{22}$$

a Poisson distribution. Thus, with this simple semiclassical approach, we have rederived the strong-signal Scully-Lamb results for the high photon numbers under consideration here. However, Eq. (13) and the Scully-Lamb theory are certainly *not* expected to be equivalent everywhere. Slightly above threshold when the photon number is not yet compatible with a coherent state, the two approaches lead to different results. This will become apparent in Sec. IV. We close this section by obtaining a solution to (9), subject to the initial condition

$$n = n_0, \quad C = -k_1/n_0 \text{ for all } t = 0. \tag{23}$$

This condition states that a finite number n_0 of photons must be assumed to be present in the cavity at initial times to start the lasing. Then, (9) may readily be integrated to give

$$n = \langle n \rangle [1 + (\langle n \rangle / n_0) e^{-B\langle n \rangle t}]^{-1}, \tag{24}$$

where we have used (20). We see that

$$n(t \to \infty) = \langle n \rangle = (A - C)/B, \tag{25}$$

in agreement with (20).

IV. MOMENT EQUATIONS: STOCHASTIC MASTER EQUATION

The stochastic equation (13) may be written in the form

$$\frac{\partial P(n,t)}{\partial t} = (A-C)[(n-1)P(n-1) - nP(n)] + B[n(n+1)P(n+1) - n(n-1)P(n)]. \tag{26}$$

Multiplying both sides by n^k and summing over n we find

$$\frac{\partial\langle n^k\rangle}{\partial t} = (A-C)\sum_n P(n)[n(n+1)^k - n^{k+1}] + B\sum_n P(n)[n(n-1)^{k+1} - (n-1)n^{k+1}], \quad (27)$$

where we have made the simple changes of variables $n \to n+1$ and $n \to n-1$ in the first and third terms on the right-hand side of (26). Expanding (27) by the binomial theorem we readily obtain

$$\frac{\partial\langle n^k\rangle}{\partial t} = (A-C+B)\left(k\langle n^k\rangle + \frac{k(k-1)}{2!}\langle n^{k-1}\rangle - \frac{k(k-1)(k-2)}{3!}\langle n^{k-2}\rangle \cdots\right) - B\left(k\langle n^{k+1}\rangle + \frac{k(k-1)}{2!}\langle n^k\rangle - \frac{k(k-1)(k-2)}{3!}\langle n^{k-1}\rangle \cdots\right). \quad (28)$$

In particular, we find for the first and second moments,

$$\frac{\partial\langle n\rangle}{\partial t} = -B\langle n^2\rangle + (A-C+B)\langle n\rangle, \quad (29)$$

and,

$$\frac{\partial\langle n^2\rangle}{\partial t} = -2B\langle n^3\rangle + (2A-2C+3B)\langle n^2\rangle + (A-C+B)\langle n\rangle. \quad (30)$$

In order to suitably truncate these equations, we return to our analysis of Sec. III, in which it was seen that all the higher moments could be calculated from the generating function (14). In particular, $\langle n\rangle$ and $\langle n^2\rangle$ are given by (18) and (19). It may be seen that

$$\left.\frac{\partial^3 F}{\partial s^3}\right|_{s=1} = \langle n^3\rangle - 3\langle n^2\rangle + 2\langle n\rangle, \quad (31)$$

which using (19) gives

$$\langle n^3\rangle = \langle n\rangle^3 + 3\langle n\rangle^2 + \langle n\rangle. \quad (32)$$

This procedure demonstrates how any moment $\langle n^k\rangle$ may be expressed in terms of the factorized moments $\langle n\rangle^k$ for the solution (17). We write down the first five:

$$\langle n^5\rangle = \langle n\rangle^5 + 10\langle n\rangle^4 + 25\langle n\rangle^3 + 15\langle n\rangle^2 + \langle n\rangle, \quad (33a)$$

$$\langle n^4\rangle = \langle n\rangle^4 + 6\langle n\rangle^3 + 7\langle n\rangle^2 + \langle n\rangle, \quad (33b)$$

$$\langle n^3\rangle = \langle n\rangle^3 + 3\langle n\rangle^2 + \langle n\rangle, \quad (33c)$$

$$\langle n^2\rangle = \langle n\rangle^2 + \langle n\rangle. \quad (33d)$$

In general we have

$$\langle n^k\rangle = \sum_{i=0}^{k} \langle n\rangle^i S_{i,k}, \quad (34a)$$

$S_{i,k}$ being the Stirling numbers

$$S_{n,s} = (n!)^{-1}\sum_{i=0}^{n}(-)^{n-i}\binom{n}{i} i^s. \quad (34b)$$

We have thus obtained a factorization scheme which provides a truncation of the hierarchy (27) at any given order, compatible with a Poisson distribution. The equations (33c) and (33d) may be shown to satisfy (29) and (30) in the steady state.

As we have indicated earlier, Eq. (26) and the general Scully-Lamb equation (which we shall introduce in Sec. V) are not the same. The Scully-Lamb equation describes intermediate as well as high photon numbers and gives Poisson statistics only far above threshold,[2] while the truncation (33) gives a Poisson distribution at each step of the hierarchy (28). An exact calculation of the second and third moments for the Scully-Lamb case yields a possible truncation which agrees with (33) only at very large photon numbers ($A \gg C$), differing results being obtained at intermediate photon numbers. This is explored in greater detail in the next section.

V. MOMENT EQUATIONS: SCULLY-LAMB RESULTS

In the Scully-Lamb theory, the atoms and the light field are both described quantum-mechanically, cavity losses due to damping being introduced phenomenologically as nonthermal reservoirs. A microscopic master equation is obtained for the coarse-grained density operator, the atoms being eliminated adiabatically. This equation is[2]

$$\frac{d\rho_{nn}}{dt} = -A\left(1 + \frac{B}{A}(n+1)\right)^{-1}(n+1)\rho_{nn} + A\left(1 + \frac{B}{A}n\right)^{-1} n\rho_{n-1,n-1} - Cn\rho_{nn} + C(n+1)\rho_{n+1,n+1}, \quad (35)$$

where we restrict ourselves to the diagonal rate equation, the constants A, C, B having been defined earlier. We now expand this equation to first order in B/A and, analogous to the procedure used in deriving (28), we may write down an equation[11] for $\langle n^k\rangle$:

$$\frac{d\langle n^k\rangle}{dt} = \sum_{i=0}^{k-1}\binom{k}{i}\{\langle n^{i+1}\rangle[A - (-)^{k-i-1}C] + \langle n^i\rangle A - B(\langle n^{i+2}\rangle + 2\langle n^{i+1}\rangle + \langle n^i\rangle)\}, \quad (36)$$

which leads to

$$\frac{\partial\langle n\rangle}{\partial t} = -B\langle n^2\rangle + (A - C - 2B)\langle n\rangle + A - B\,, \qquad (37)$$

$$\frac{\partial\langle n^2\rangle}{\partial t} = -2B\langle n^3\rangle + (2A - 2C - 5B)\langle n^2\rangle + (3A + C - 4B)\langle n\rangle + A - B\,. \qquad (38)$$

In the classical limit we assume $\langle n^2\rangle \sim \langle n\rangle^2$ in (37), which leads to the solution (20) for $\langle n\rangle$. However, we cannot, in general, assume all the higher moments to be factorable, and a suitable truncation must be found.

We return, therefore, to the full Scully-Lamb equation (35) whose general solution is[2]

$$\rho_{nn} = Z^{-1}\left(\frac{A^2}{BC}\right)^{n+A/B} \frac{1}{(n + A/B)!}\,, \qquad (39)$$

with

$$\langle n\rangle = A^2/BC\,, \qquad (40)$$

for strong signals ($A \gg C$), Z being a normalization constant. We now consider the third moment,

$$\langle n^3\rangle = Z^{-1}\sum_n n^3\left(\frac{A^2}{BC}\right)^{n+A/B}\frac{1}{(n+A/B)!}\,,$$

$$= Z^{-1}\sum_n\left(n + \frac{A}{B} - \frac{A}{B}\right)^3\left(\frac{A^2}{BC}\right)^{n+A/B}\frac{1}{(n+A/B)!}\,,$$

$$= Z^{-1}\sum_n\left(\frac{A^2}{BC}\right)^{n+A/B}\frac{1}{(n+A/B)!}\left[\left(n+\frac{A}{B}\right)^3 - \frac{A^3}{B^3} - \frac{3A}{B}\left(n+\frac{A}{B}\right)^2 + 3\left(\frac{A}{B}\right)^2\left(n+\frac{A}{B}\right)\right]. \qquad (41)$$

Consider the term

$$Z^{-1}\sum_n 3\left(\frac{A}{B}\right)^2\frac{n + A/B}{(n+A/B)!}\left(\frac{A^2}{BC}\right)^{n+A/B}.$$

This may be written as

$$3Z^{-1}\sum_n\frac{1}{(n+A/B-1)!}\left(\frac{A^2}{BC}\right)^{n+A/B-1}\left(\frac{A^2}{BC}\right)\left(\frac{A}{B}\right)^2 = 3\left(\frac{A}{B}\right)^2\frac{A^2}{BC}.$$

The remaining terms in $\langle n^3\rangle$ are similarly treated giving finally

$$\langle n^3\rangle = \left(\frac{A^2}{BC}\right)^3 + 3\left(\frac{A^2}{BC}\right)^2 + \frac{A^2}{BC} + 3\left(\frac{A}{B}\right)^2\frac{A^2}{BC} - \frac{A^3}{B^3} - 3\frac{A}{B}\left(\frac{A^2}{BC}\right)^2 - 3\frac{A}{B}\left(\frac{A^2}{BC}\right), \qquad (42)$$

and a similar calculation gives

$$\langle n^2\rangle = \left(\frac{A^2}{BC}\right)^2 + \left(\frac{A^2}{BC}\right) + \frac{A^2}{B^2} - 2\frac{A}{B}\frac{A^2}{BC}. \qquad (43)$$

Equations (42) and (43) do not satisfy Eqs. (37) and (38) in contrast to the preceding case. This discrepancy has been observed in a numerical calculation[11] in which a different truncation,

$$\langle n^k\rangle = \left[\sum_{i=1}^{k-1}(-)^{k-i-1}\binom{k-1}{i-1}\langle n^i\rangle^{1/i}\right]^k, \qquad (44)$$

has been used. It may be seen from the above that the truncation of the hierarchy (36) does not give Poisson statistics except for large n. This is because at intermediate photon numbers, the Scully-Lamb solutions correspond to a mixture of coherent and chaotic light. This difference in the two approaches is further elucidated by a calculation of $\langle n^3\rangle_{SL}$ for the Scully-Lamb case using (44). We have from (37) in the steady state

$$\langle n^2\rangle = \frac{A-B}{B} + \frac{A-C-2B}{B}\langle n\rangle \qquad (45)$$

$$\approx \frac{A}{B} + \langle n\rangle^2\,, \qquad (46)$$

where we have used (20). Now, the prescription (44) gives

$$\langle n^3\rangle = (2\langle n^2\rangle^{1/2} - \langle n\rangle)^3\,, \qquad (47)$$

which using (46) becomes

$$\langle n^3\rangle_{SL} = 8(A/B + \langle n\rangle^2)^{3/2} - \langle n\rangle^3 - 4\langle n\rangle(A/B + \langle n\rangle^2) + 2\langle n\rangle(A/B + \langle n\rangle^2)^{1/2}. \qquad (48)$$

We now neglect A/B compared to $\langle n\rangle^2$ (a typical set of numerical values gives $A/B = 300$, $\langle n\rangle = 50$) which reduces (48) to

$$\langle n^3\rangle_{SL} \approx 3\langle n\rangle^3 + 2\langle n\rangle^2 - 4(A/B)\langle n\rangle\,. \qquad (49)$$

Comparing this with our result (33c) we find

$$\langle n^3\rangle_{SL} \approx 3\langle n^3\rangle_{Stoch}\,. \qquad (50)$$

A similar calculation gives

$$\langle n^4\rangle_{SL} \approx 185\,921\langle n^4\rangle_{Stoch}\,. \qquad (51)$$

The difference in the hierarchies (28) and (36) thus appears first at the third and fourth moments for intermediate photon numbers.

Let us now consider the Scully-Lamb moments in the strong-signal limit. Using (40), the moment equations (42) and (43) may be cast in the form

$$\langle n^3\rangle = \langle n\rangle^3 + 3\langle n\rangle^2 + \langle n\rangle + 3(C/A)^2\langle n\rangle^3 - (C/A)^3\langle n\rangle^3 - 3(C/A)\langle n\rangle^3 - 3(C/A)\langle n\rangle^2\,, \qquad (52)$$

and

$$\langle n^2\rangle = \langle n\rangle^2 + \langle n\rangle + (C/A)^2\langle n\rangle^2 - 2(C/A)\langle n\rangle^2 . \tag{53}$$

Far above threshold, we have $A \gg C$ and Eqs. (52) and (53) reduce to our Poisson equations (33c)–(33d). In fact, we may generalize this procedure by noting that the distribution function (39) may be written in the form

$$P(n) = \frac{\langle n\rangle^{n+A/B}}{(n+A/B)!}\left(\sum_n \frac{\langle n\rangle^{n+A/B}}{(n+A/B)!}\right)^{-1}, \tag{54}$$

where (40) has been used. Let N be the number of photons in the cavity far above threshold (N need not be the total number of photons), i.e., $N \gg \langle n\rangle$, $\langle n\rangle$ being the threshold photon number. Then we have

$$N + A/B = N + (C/A)\langle n\rangle \approx N , \tag{55}$$

so that

$$P(n) \underset{n\to N}{\longrightarrow} \frac{e^{-\langle N\rangle}\langle n\rangle^N}{N!} . \tag{56}$$

This is, of course, the Poisson result as has been obtained from the solution (17) of our master equation. Thus we see that at very large photon numbers, the agreement between the fully quantum microscopic theory of Scully and Lamb and our macroscopic semiclassical theory is exact, this being the first time that such a comparison has been made for a stochastic description of the type developed in Sec. II.

The above results are further illustrated by the graphs of the second, third, and fourth moments for the two theories shown in Figs. 1–3. In each figure, the semiclassical quantity is plotted together with the corresponding moment obtained from the Scully-Lamb hierarchy (36) for different values of A/B. The discontinuity in the Scully-Lamb moments occurs at threshold where $\langle n\rangle \sim A/B$. High above threshold the Scully-Lamb moments are seen to merge with the corresponding semiclassical quantities as the photon number becomes larger.

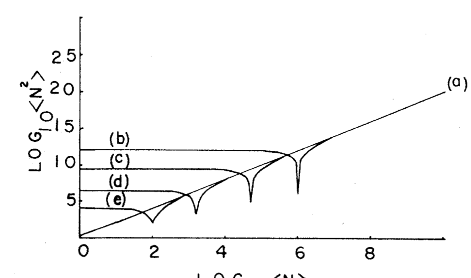

FIG. 1. Logarithmic plot of the second moment $\langle n^2\rangle$ of the photon number vs the average photon number $\langle n\rangle$ for (a) the semiclassical result [Eq. (33d)] and the corresponding Scully-Lamb expression [Eq. (46)] for (b) $A/B = 10^6$, (c) $A/B = 50\,000$, (d) $A/B = 1600$, (e) $A/B = 100$.

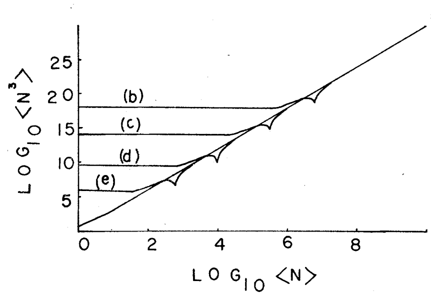

FIG. 2. A similar plot for the third moments, these being given by Eqs. (33c) and (45) (with the same values of A/B).

VI. COMPARISON OF THE FOKKER-PLANCK EQUATIONS

As mentioned in Sec. I, the Fokker-Planck equation is obtained from the discrete master equation by effecting the transition to continuous variables. In order to derive a Fokker-Planck equation from the semiclassical master equation (26), we expand[8] the probability $P(n,t)$ in a Poisson distribution,

$$P(n,t) = \int d\xi \frac{e^{-\xi}\xi^n}{n!} f(\xi,t) . \tag{57}$$

We substitute this into the master equation (26) and integrate by parts, assuming f and its derivatives to vanish on the boundaries. As an example, we shall consider in detail the term

$$Q = (n-1)P(n-1) , \tag{58}$$

which, using (57), becomes

$$Q = (n-1)\int d\xi \frac{e^{-\xi}\xi^{n-1}}{(n-1)!} f(\xi,t)$$

$$= n(n-1)\int d\xi \frac{e^{-\xi}\xi^{n-1}}{n!} f(\xi,t) . \tag{59}$$

We now integrate this by parts, obtaining

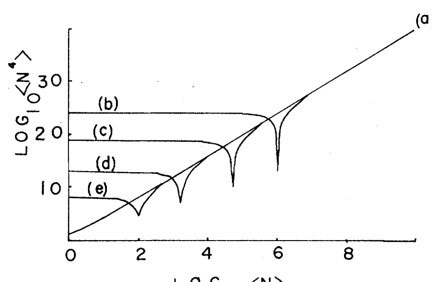

FIG. 3. Plot of the fourth moments for the same values of A/B.

$$Q=(n-1)\int d\zeta \frac{e^{-\zeta}\zeta^n}{n!}\left(f-\frac{\partial f}{\partial \zeta}\right).$$

Integrating by parts again, we find

$$Q=\int d\zeta \frac{e^{-\zeta}\zeta^n}{n!}\left(\zeta f-2\zeta\frac{\partial f}{\partial \zeta}-2f+2\frac{\partial f}{\partial \zeta}+\zeta\frac{\partial^2 f}{\partial \zeta^2}\right). \quad (60)$$

The remaining terms in (26) are similarly treated, giving finally

$$\frac{\partial f}{\partial t}=-\frac{\partial}{\partial \zeta}[(A-C-B\zeta)f\zeta]+\frac{\partial^2}{\partial \zeta^2}[(A-C-B\zeta)f\zeta]. \quad (61)$$

We now make contact with the coherent-state formalism[12] by setting

$$\zeta=\alpha^*\alpha, \quad (62)$$

so that

$$\frac{\partial}{\partial \zeta}=\alpha^{-1}\frac{\partial}{\partial \alpha^*}+\alpha^{*-1}\frac{\partial}{\partial \alpha}, \quad (63a)$$

and

$$\frac{\partial^2}{\partial \zeta^2}=\left(\alpha^{-1}\frac{\partial}{\partial \alpha^*}+\alpha^{*-1}\frac{\partial}{\partial \alpha}\right)^2. \quad (63b)$$

We then find from (61),

$$\frac{\partial f}{\partial t}=-\left(\frac{\partial}{\partial \alpha^*}[(A-C-B|\alpha|^2)\alpha^* f]+\text{c.c.}\right)+2A\frac{\partial^2 f}{\partial \alpha^*\partial \alpha}. \quad (64)$$

The steady-state solution to (64) is

$$f(\alpha)=N\exp\left(\frac{A-C}{A}\alpha^*\alpha-\frac{B}{A}(\alpha^*\alpha)^2\right). \quad (65)$$

Starting from (65) and using the normalization

$$\int d^2\alpha\, f(\alpha)=1=\pi\int_0^\infty dn\, f(n), \quad (66)$$

we readily find $N=(2/\pi)(B/\pi A)^{1/2}$ far above threshold, where we assume the quadratic term in the exponent of (65) to be the dominant one. The variance calculated from (65) is

$$\langle(\delta n)^2\rangle=\pi\int_0^\infty dn\,(n-\langle n\rangle)^2 P(n)+\langle n\rangle, \quad (67)$$

which may be shown to be Poisson far above threshold.

Let us briefly return to the Scully-Lamb theory and consider Eq. (35) expanded to first order in A/B. By expanding $\rho(t)$ in coherent states,[12]

$$\rho(t)=\int d^2\alpha P(\alpha,t)|\alpha\rangle\langle\alpha|, \quad (68)$$

we readily find an equation for $P(\alpha,t)^2$:

$$\frac{\partial P(\alpha,t)}{\partial t}=-\frac{1}{2}\left(\frac{\partial}{\partial \alpha}[A-C-B|\alpha|^2)\alpha P]+\text{c.c.}\right)+A\frac{\partial^2 P}{\partial \alpha^*\partial \alpha}. \quad (69)$$

This equation does not involve a series truncation of the type encountered in conventional Kramers-Moyal type approaches. Also, the expansion (68) in terms of coherent states is convenient since the Scully-Lamb equation (35) may be written in operator form.[2] Further, (69) is valid in the strong-signal region, since we have assumed the photons to be in coherent states. The solution to (69) is given by

$$P(\alpha)=N'\exp\left(\frac{A-C}{A}\alpha^*\alpha-\frac{B}{2A}(\alpha^*\alpha)^2\right). \quad (70)$$

A comparison of Eqs. (64) and (69), or alternatively, of solutions (65) and (70), shows them to be identical up to a normalization constant. As mentioned earlier, the Fokker-Planck equation constitutes an exact asymptotic representation of the discrete master equation, so that our original Markovian equation (26) is, indeed, completely equivalent to the Scully-Lamb equation at large numbers.

VII. DISCUSSION

We have shown in this work that a macroscopic master equation of the form (26) gives an adequate description of the laser far above threshold, the results being compatible with the strong-signal Scully-Lamb theory. It has been shown that the Scully-Lamb steady-state solution, as well as the higher-order moments, tend to the semiclassical values obtained from (26) at large photon numbers. The comparison between the two theories becomes far clearer at the level of the Fokker-Planck equations, which are seen to be the same in both cases (up to a normalization constant) and which, moreover, are exact asymptotic representations of the master equations. Further, it has been pointed out that the master equation approach of Sec. II is the only formulation that is rigorously valid near a critical point, so that we have here a very simple method which may be applied to a rigorous macroscopic treatment of systems in which a microscopic formulation is not possible.

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