Level-degeneracy effects in super-radiance theory. Calculations for $j = 1/2$ to $j' = 1/2$ dipole transition

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We study here level-degeneracy effects in super-radiance, using the quantum "small-system" model whose interest and limitations are discussed. First, we show that this model allows one to understand the physical origin of these effects and to describe their qualitative influence on the properties of the emitted light. We also present and analyze numerical results obtained for the case of a dipole transition between two degenerate levels of angular momenta 1/2. We show that in this particular case one can define a basis of collective states which is well adapted to the invariance properties of the master equation. It is then possible to solve the master equation and to compute expectation values for the radiated field (intensity radiated with a given polarization, quantum fluctuations of this intensity) for various initial conditions and for small but already significant values of the number of individual systems. A detailed analysis of these results and a comparison with analogous calculations in the case where the degeneracy is supposed to be removed allows us to estimate the individual influence of the different types of effects (interference effects and the two sorts of competition effects, inhibition and initiation). We also compare the total intensity radiated to the intensity radiated by the same number of two (nondegenerate) level atoms. This comparison can be simply understood if one studies the partition of the population of the collective energy levels between the different states of these levels, and it appears that the most populated states are these which are able to radiate the most. Finally, it is shown how level degeneracy affects the quantum fluctuations of the intensity.

I. INTRODUCTION

Cooperative emission of light by a collection of identical initially excited systems— super-radiance—has been until recently extensively studied, theoretically,[1-3] and experimentally.[4-7] However, even in the most sophisticated models, the individual systems are always considered as two (nondegenerate) level atoms and level degeneracy is ignored. An important interference effect can however be expected for degenerate levels and experimental evidence for the existence of such an effect can already be found in a recent work[7]: quantum beats have been observed on super-radiant signals between quasidegenerate levels even if no coherent superposition of states has been initially prepared.

In a previous paper,[8] we have described a theoretical approach to the problem of the influence of level degeneracy on super-radiance. Using the small system model,[1] we have been able to analyze the differences between the case of two (nondegenerate) level atoms and the case of atoms with two degenerate levels of angular momenta j and j' (connected by an electric or magnetic dipole transition) The specific influence of level degeneracy consists in interference effects, but they are added to another type of effect which would also appear if level degeneracy was removed: an effect of competition between the different Zeeman transitions sharing a common upper or lower state. We have also shown that the use of tensorial formalism should simplify considerably the solution of the master equation but the labeling of collective states for any j and j' remains very complicated and no general conclusions on the influence of level degeneracy on the emitted light have been derived.

We will show here[9] that such problems can still be solved in the particular case of two degenerate levels of angular momenta $\frac{1}{2}$. In this case, it is possible to label the collective states using the formalism of Ref. 8 and, finally, the master equation can be numerically solved for values of the number of atoms which are already significant.

The numerous limitations of the small-system model are well known in the two (nondegenerate) level case. Let us recall that since the atoms are confined in a small volume, the dipole-dipole interaction would in fact destroy super-radiance[10]; moreover it has been shown[11] that the Markovian approximation is not valid (except at the very beginning of the phenomenon) in all the experiments which have been performed up to now. Concerning the influence of level degeneracy a supplementary limitation appears, due to the geometry of the system, which is different from the pencil-shaped geometry used in all experiments. In the small system model, three independent polarizations can super-radiate, instead of two for pencil-shaped systems, and this difference will clearly affect both interference and competition effects.

Nevertheless, the small system model provides a simple description of the phenomenon of super-radiance, in which the influence of level degener-

acy clearly appears. It is thus well adapted to analyze the different effects, to understand their physical origin and to qualitatively discuss their influence on the pulse shape and on the polarization properties. Moreover, a careful discussion of the numerical results obtained for a $j=\frac{1}{2}\rightarrow j'=\frac{1}{2}$ transition has allowed us to give a more precise estimate of the individual influence of the different types of level degeneracy effects. In view of these results, one can think that these effects cannot in general be neglected; in particular, important polarization effects are to be expected, which depend in a nontrivial way on the excitation light polarization properties; moreover even if no polarization analysis is made, the maximum intensity and the corresponding time delay can be significantly different from the results one would obtain if level degeneracy was ignored.

In the second part of the present paper (Sec. II), level-degeneracy effects in collective spontaneous emission are described in a general and qualitative way. The basic equations for collective spontaneous emission between two degenerate levels of angular momenta j and j' are recalled and a very simple analysis of the physical origin of the different level-degeneracy effects is given (Sec. II A). Then the specific influence of competition effects (Sec. II B) and of interference effects (Sec. II C) is qualitatively discussed. The third part (Sec. III) is devoted to the particular case of a $j=\frac{1}{2}\rightarrow j'=\frac{1}{2}$ transition. A basis for collective states is chosen (Sec. III A) and the principle of the evaluation of the matrix elements of the collective dipole operator if given (Sec. III B); for this latter point we have used group-theoretical methods and the details of the calculation are reported in the Appendix. A conservation equation which is the generalization of the Bloch vector length conservation is also given. Then it is shown how the use of tensorial formalism simplifies the solution of the master equation and the calculations which have been done are described (Sec. III C). The obtained results are finally presented and analyzed (Sec. III D).

II. GENERAL AND QUALITATIVE DESCRIPTION OF LEVEL-DEGENERACY EFFECTS

A. Basic equations

In the small system model, the master equation describing collective spontaneous emission of N atoms with two degenerate levels of angular momenta j and j' (j for the upper level) connected by an electric or magnetic dipole transition has been derived in Ref. 8. It reads

$$\dot{\rho}(t)=\Gamma\sum_{q}\left[R_q\rho(t)R_q^{\dagger}-\tfrac{1}{2}(R_q^{\dagger}R_q\rho(t)+\rho(t)R_q^{\dagger}R_q)\right], \quad (1)$$

with

$$R_q=(2j+1)^{1/2}\sum_{\substack{\alpha=1,N\\ m'-m=q}}(-1)^{j'-m'}\times\begin{pmatrix} j' & 1 & j\\ -m' & q & m\end{pmatrix}|j'm')_{\alpha\alpha}(jm|. \quad (2)$$

As the sums in Eq. (2) run over all possible values of α, m, and m', it means that the origin of the emitted photons of given polarization cannot be known, so that interferences appear. These interferences are the manifestation of two distinct effects. The first one is due to the fact that it is indeed quite impossible to say which atom has emitted a photon, at it is the case for two (nondegenerate) level atoms. As shown in Ref. 11, the corresponding interatomic interferences are responsible for the building up of the super-radiant signal. The second effect, on the contrary, does not appear for two (nondegenerate) level atoms; it comes from the fact that, for a photon of given polarization, it is impossible to say on which Zeeman transition it has been emitted and the consequence is the existence of supplementary interference effects[12] which are essentially specific to the level degeneracy.

These effects are not the only difference between the two (nondegenerate) level case and the considered one; they are added in fact to another type of effect which would also appear if the degeneracy were removed. In effect, for atoms with more than two nondegenerate levels the master equation can be written

$$\dot{\rho}(t)=\sum\Gamma_{ij}[R_{ij}\rho(t)R_{ij}^{\dagger}-\tfrac{1}{2}(R_{ij}^{\dagger}R_{ij}\rho(t)+\rho(t)R_{ij}^{\dagger}R_{ij})]; \quad (3)$$

the sum runs over all transitions $i\rightarrow j$ (i for the upper level) of transition probabilities Γ_{ij} and the operator R_{ij} is given by

$$R_{ij}=\sum_{\alpha=1,N}|j)_{\alpha\alpha}(i|, \quad (4)$$

$|i)$ and $|j)$ being monatomic states. If the different transitions do not share any common level, $\rho(t)$ can be written as a sum of terms with independent evolution. Otherwise, super-radiance on the different transitions sharing a common upper or lower level will influence one another. These "competition" effects are obviously present in the $j\rightarrow j'$ case and are inseparable from the level-degeneracy interference effects: these two types of effects and their qualitative influence on the emitted light will be discussed in Secs. II B and II C.

The expressions of expectation values for the

radiated field in the $j \rightarrow j'$ case are also given in Ref. 8. We recall here the expression of the intensity radiated in all directions with a polarization $\vec{\epsilon}$:

$$I_{\vec{\epsilon}}(t) = I_0 \langle (\vec{R} \cdot \vec{\epsilon})^\dagger (\vec{R} \cdot \vec{\epsilon}) \rangle (t); \tag{5}$$

we also give the expression of the expectation value for the total intensity, radiated in all directions and with all polarizations

$$I(t) = I_0 \langle (\vec{R}^\dagger \cdot \vec{R}) \rangle (t), \tag{6}$$

which is proportional to the mean value of the scalar operator

$$X = (\vec{R}^\dagger \cdot \vec{R}) = \sum_q R_q^\dagger R_q . \tag{7}$$

Notice that, because of the geometrical symmetry of the small system model and due to the isotropic nature of spontaneous emission, Eq. (1) is invariant under rotations. For this reason the system generally radiates in all directions and with all polarizations. Therefore this equation does not apply to the experimental situation, in which the system radiates in two opposite directions and thus on two polarizations at most. As a consequence both interference and competition are expected to be different from those described by Eq. (1). However this equation is quite useful for understanding the physical origin of these effects and for discussing their qualitative influence on the pulse properties; let us add that the rotational invariance itself allows important formal simplifications, so that quantitative results can be obtained in a particular case.

B. Competition effects

As said above, competition effects appear if a system super-radiates on at least two transitions sharing a common level. The situation is different if the common level is the upper one, the lower one, or the intermediate one. In this discussion we shall consider three cases of three level atoms (case *a*: common upper level; case *b*: common lower level; case *c*: common intermediate level). The energy level diagram for one single atom in each case can be found in Figs. 1(a), 1(b), and 1(c). The master equation is given by Eq. (3) and, for the sake of simplicity, we restrict ourselves to symmetric collective states (the symmetric character is obviously conserved during the evolution). These collective states can be written in the occupation number representation

$$|N_0 N_1 N_2\rangle,$$

with $N_0 + N_1 + N_2 = N$, and the collective state diagrams corresponding to the three cases are represented in Figs. 1(a), 1(b), and 1(c). The evolution of the system starting from a symmetric collective state consists in cascading emission between the states of these diagrams. The corresponding transition probabilities are easily derived using the expression of R_{ij} in terms of annihilation and creation operators and they are precisely stated on separate diagrams in Figs. 1(a) 1(b), and 1(c). The consequences of the competition between the two transitions are rather different in the three cases.

1. Common upper level: Inhibition

The branching ratio between the two transition probabilities corresponding to the emission of a photon on $0 \rightarrow 1$ or $0 \rightarrow 2$ at a given point of the main diagram of Fig. 1(a) is $(N_1+1)\Gamma_1/(N_2+1)\Gamma_2$. If $\Gamma_1 > \Gamma_2$ and if the system starts from near the top of the diagram, this ratio is always favorable to the emission on $0 \rightarrow 1$, and the process is cumulative since the emission of each photon on this transition increases the branching ratio. It is thus expected that the emission on $0 \rightarrow 2$ will be much less important than it would be if this transition was alone, the effect increasing with N. In fact super-radiance on the most probable transition empties the upper level before the appearance of super-radiance, on the other one, which is therefore "inhibited."

2. Common lower level: Initiation

In this case [see Fig. 1(b)] the branching ratio between the emission probabilities of the two sorts of photons is $N_1\Gamma_1/N_2\Gamma_2$. If $\Gamma_1 > \Gamma_2$, this ratio is favorable to the emission on $1 \rightarrow 0$ only if N_2/N_1 is not too large: but the emission of each photon on this transition decreases N_1 and leaves N_2 unaltered, so that the branching ratio decreases. It is thus expected that super-radiance on $2 \rightarrow 0$ will be "initiated": it will appear sooner and larger than if alone.[13] The importance of this effect depends on the ratio of the initial populations of levels 1 and 2 and it obviously vanishes if one of these populations is zero. Moreover each emission of a photon on whatsoever transition increases N_0 and consequently both transition probabilities: it is thus expected that both super-radiant pulses will appear sooner and larger than if alone.

3. Intermediate common level: Cascade

In this case [see Fig. 1(c)] the branching ratio between the emission probabilities of the two sorts of photons is $N_1(N_0+1)\Gamma_1/N_0(N_2+1)\Gamma_2$. If the system starts nearly from the top of the diagram, this branching ratio is very favorable to the emission

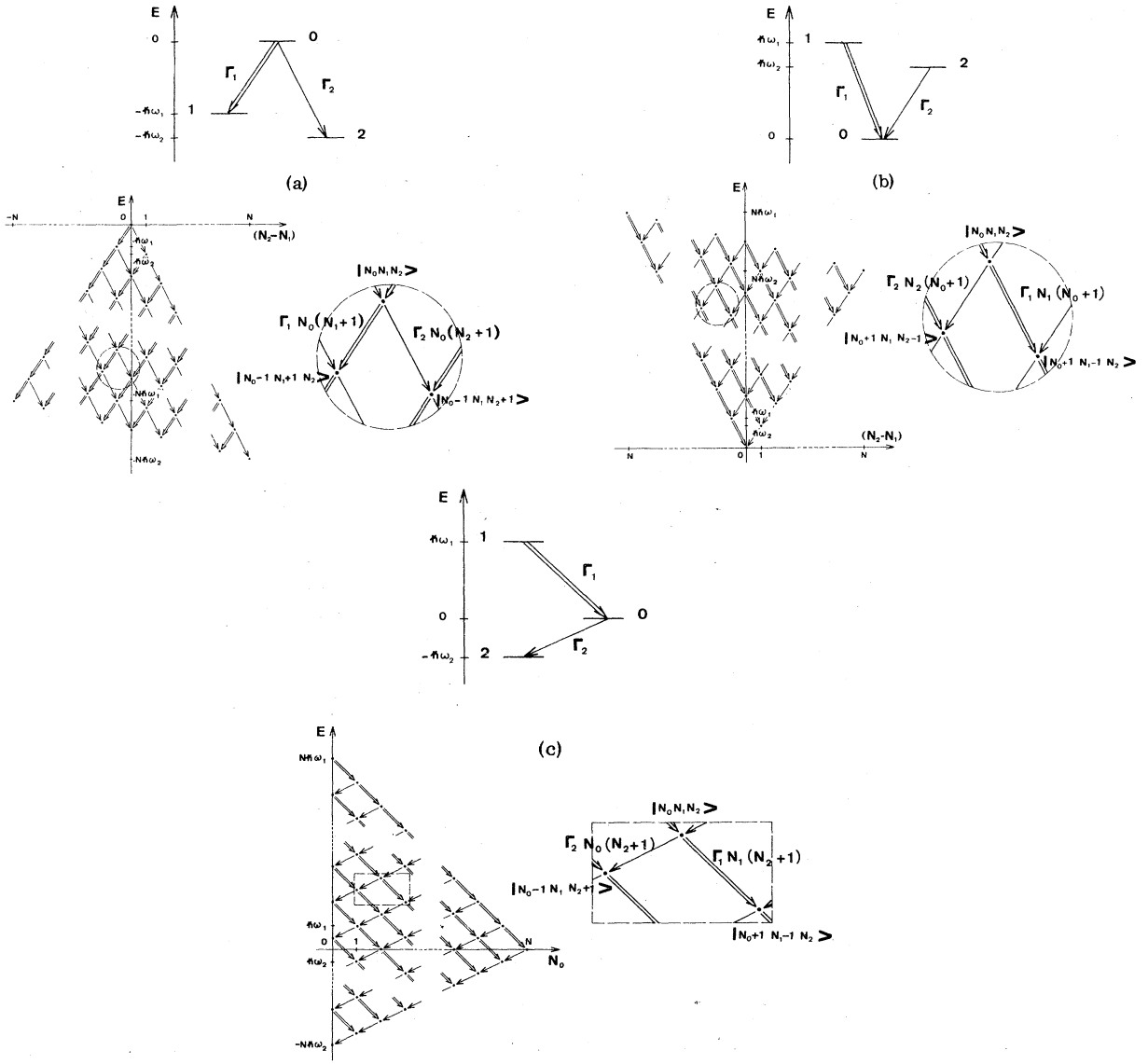

FIG. 1. (a) Competition for super-radiance on two transitions sharing a common upper level (inhibition). The figure includes three diagrams. First, the energy-level diagram of one single atom (case *a*). Second, a diagram representing cascading emission between the symmetric collective states of N such atoms; each point represents a state $|N_0N_1N_2\rangle$ and the double and single arrows correspond, respectively, to the emission of photons of energy $\hbar\omega_1$ and $\hbar\omega_2$. Third, the general expressions for the transition probabilities between the states $|N_0N_1N_2\rangle$. (b) Competition for super-radiance on two transitions sharing a common lower level (initiation). Same as (a) with case a replaced by case *b*. (c) Super-radiance on two transitions sharing an intermediate common level (cascade). Same as (a) with case *a* replaced by case *b*.

on $1 \rightarrow 0$, for any Γ_1/Γ_2 ratio, and this remains true as long as N_1 is large enough. Thus, during a first period, photons are emitted almost only on $1 \rightarrow 0$. At the end of this period, levels 1 and 2 are almost empty and the population of level 0 is almost equal to N: the branching ratio is then favorable to the emission on $0 \rightarrow 2$. Finally it is expected to observe two successive pulses, practically identical to the super-radiant pulses emitted by N initially excited two-level atoms with transition probabilities Γ_1 and Γ_2; the second pulse is simply delayed and begins at the end of the first one.

C. Interference effects

As shown before, the specific consequence of level degeneracy consists in interference effects

due to the indiscernibility of the photons emitted on the Zeeman transitions corresponding to a given polarization. These interferences manifest their influence not only on the emitted real photons but also on the emission and reabsorption of virtual photons which represent, in the small system model, the interaction of the atoms between themselves.

The emitted real photons consist in cascading emission between the collective states. Interferences appear between all collective transitions of the same polarization. The only peculiar point concerning these effects—compared to other quantum interferences—is that the collective energy levels are highly degenerate; in particular for a given energy level there is a great number of states corresponding to a same eigenvalue of J_z. The number of interfering paths is therefore very large (as an example, see Fig. 4, which shows the π cascades in the case of four atoms with two degenerate levels of angular momentum $\frac{1}{2}$, in a basis which will be explained later) and the global influence of the interferences is hard to estimate.

Formally, the consequence of the interferences is that the emission of a real photon leads generally from one state of a given basis to a superposition of several states of the same basis. Let us take as an example the case of atoms with two degenerate levels of angular momentum $\frac{1}{2}$. The states of such an atom and the different Zeeman transitions are represented in Fig. 2 and one can see that the interferences concern π photons only. We again consider symmetric collective states, which are labeled in the occupation number representation as $|N_1N_2N_3N_4\rangle$ (the indices of monatomic states can be found in Fig. 2). The emission of a π photon is represented by

$$R_0\,|N_1N_2N_3N_4\rangle \propto [N_1(N_3+1)]^{1/2}\,|N_1-1N_2N_3+1N_4\rangle + [N_2(N_4+1)]^{1/2}\,|N_1N_2-1N_3N_4+1\rangle \tag{8}$$

(the particular case of two atoms is represented in Fig. 3). As a consequence, the evolution of the populations is mixed with the evolution of some coherences, at least in the considered basis, and the solution of the master equation (1) is therefore complicated. This problem is studied in more detail in Ref. 8, where it is shown that such a mixing occurs whatever the choice of the state basis is.

Concerning the emission and reabsorption of virtual photons, the formal consequence of the interferences is the same as for the emission of real photons: this process generally leads from one basis state to a superposition of several basis states. In the case of atoms with two degenerate levels of angular momentum $\frac{1}{2}$, the emission and reabsorption of a virtual π photon is represented by

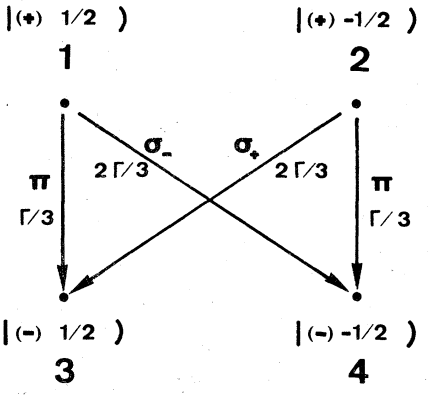

FIG. 2. State diagram of one single atom with two degenerate levels of angular momentum $\frac{1}{2}$; the states are either written as $|(\pm)m)$, the symbols (+) and (–) referring, respectively, to the upper and lower level, or simply numbered from 1 to 4. The transition probabilities of the different Zeeman components are given.

$$R_0^{\dagger}R_0\,|N_1N_2N_3N_4\rangle \propto [N_1(N_3+1)+N_2(N_4+1)]\,|N_1N_2N_3N_4\rangle + [N_1(N_2+1)(N_3+1)N_4]^{1/2}\,|N_1-1N_2+1N_3+1N_4-1\rangle + [(N_1+1)N_2N_3(N_4+1)]^{1/2}\,|N_1+1N_2-1N_3-1N_4+1\rangle \tag{9}$$

(see also Fig. 3 for the two-atom case). It appears here an interesting phenomenon: because of the interferences, the exchange of virtual photons can modify some mean values. In particular, the population of an upper state may increase, although only Markovian processes are considered and no real absorption is taken into account. Let us notice that the mixing of states through an exchange of virtual photons can be suppressed [in order to formally simplify the solution of the master equation (1)]: in effect, it is always possible (see Ref. 8) to choose a basis in which the operator X defined by Eq. (7) is diagonal (such a basis will be exhibited in Sec. III A for the $\frac{1}{2}\rightarrow\frac{1}{2}$ case).

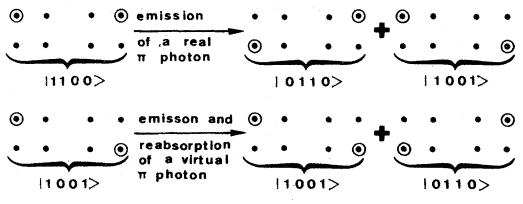

FIG. 3. Elementary-level-degeneracy interference processes for two atoms with two degenerate levels of angular momentum $\frac{1}{2}$; a symmetric two-atom state is represented by a nonordered pair of monatomic state diagrams in which the occupied states are circled.

III. APPLICATION TO $A\ j = \frac{1}{2} \rightarrow j' = \frac{1}{2}$ TRANSITION

For a quantitative evaluation of the influence of level degeneracy effects on the properties of the super-radiant pulse, one has to solve the master equation, and preliminarily to choose a basis for the collective states. This choice is important because the formal complexity of the equations depends on the basis. In the general case of atoms with two degenerate levels of angular momenta j and j', the problem of the choice of a "good" collective state basis is complex and remains unsolved. However in the particular case $j = j' = \frac{1}{2}$, and as long as symmetric states are considered, it is possible to define with the help of group theory a basis which is well adapted to the solution of the master equation: it is then possible to solve it numerically for numbers of atoms which are already significant.

A. Collective state basis

Let us consider a collection of N atoms with two degenerate levels of angular momentum $\frac{1}{2}$; the monatomic states are represented in Fig. 2. It is easy to show[8] that the whole set of collective states forms a basis for irreducible representations of the SU(4) group whose infinitesimal operators are sums of monatomic operators

$$P_{(\pm)m_1,(\pm)m_2} = \sum_{\alpha=1,N} |(\pm)m_{1\alpha}\rangle\langle\alpha(\pm)m_2| \tag{10}$$

(our notation is made explicit in Fig. 2). The importance of this group—the cooperation group—in the study of superradiance is due to the fact that the operators R_q and $R_q^\dagger$ belong to its Lie algebra and that consequently the evolution of the system by collective spontaneous emission does not mix different irreducible representations. Moreover the components of the total angular momentum of the atoms $\vec{J}$, which are

$$\begin{aligned} J_z &= \sum_m m[P_{(+)m,(+)m} + P_{(-)m,(-)m}], \\ J_\pm &= P_{(+)\pm 1/2,(+)\mp 1/2} + P_{(-)\pm 1/2,(-)\mp 1/2}, \end{aligned} \tag{11}$$

belong also the the Lie algebra of the cooperation group; the group $SU^J(2)$ generated by $\vec{J}$, which is an invariance group for the master equation, is a subgroup of SU(4). Another invariance property of the master equation is to be considered: the invariance under permutations of atoms, which expresses that, in the small system model, the atoms are indiscernible for the radiated field. Consequently the evolution does not mix different irreducible representations of the permutation group S_N. In the following, we shall restrict ourselves to collective states which are invariant under permutations and which form a single irreducible representation of SU(4), the so-called symmetric representation $\{N\}$. In the two (nondegenerate) level case this restriction would correspond to the value $\frac{1}{2}N$ of the Dicke's cooperation number r, that is to complete population inversion. In the two (degenerate) level case there exist nonsymmetric states even for complete population inversion. However, the assumption that the initial density matrix is defined on symmetric states only provides important formal simplifications (we would at the present time be unable to solve the master equation for other representations of S_N). Moreover this is a rather natural restriction, as long as the atoms can be really considered as indiscernible—in particular for the excitation field. A more rigorous discussion of this problem remains to be done. It is not impossible that this restriction might, in some cases, affect the description of the collective phenomenon itself. However we notice here that such a discussion would require a precise knowledge of the excitation process. If the initial conditions are described only by the mean values of the populations of the upper states and of the coherence between them, it is always possible to use an initial density matrix defined on symmetric states only. The peculiarity of such a density matrix will not appear in the expectation values of the intensity radiated with the different polarizations but only on properties such as the correlations between photons of different polarizations (see Sec. III D 5).

Since dealing with symmetric states, one is allowed to use the second quantization formalism. A pair of boson creation and annihilation operators,

$$a_{(\pm)m},\ a^\dagger_{(\pm)m}, \tag{12}$$

is associated to each monatomic state $|(\pm)m\rangle$ and they obey the usual commutation relations. The sums of monatomic operators given by Eq. (10) have to be replaced by

$$a^\dagger_{(\pm)m_1} a_{(\pm)m_2}\ .$$

An obvious basis for symmetric collective states would be the occupation number representation, but this basis does not permit to take advantage of the rotational invariance of the master equation. It is more convenient to label the collective states according to a chain of subgroups of the cooperation group which includes $SU^J(2)$.

First it is straightforward to show that the annihilation and creation operators (12) have tensorial properties[14] with respect to $\vec{J}$; they are

the components of four tensor operators of rank $\frac{1}{2}$

$$a^{\dagger}_{(\pm)m} = (a^{\dagger}_{(\pm)})^{(1/2)}_{m},$$
$$a_{(\pm)m} = (-1)^{1/2-m}(a_{(\pm)})^{(1/2)}_{-m}. \tag{13}$$

Operators commuting with $\vec{J}$ are thus obtained by constructing scalar products of two of these operators; in particular we define the following operators:

$$K_z = \frac{1}{\sqrt{2}}\{[(a^{\dagger}_{(+)})^{(1/2)} \cdot (a_{(+)})^{(1/2)}]^{(0)}_0 - [(a^{\dagger}_{(-)})^{(1/2)} \cdot (a_{(-)})^{(1/2)}]^{(0)}_0\}$$
$$= \frac{1}{2}\sum_m [a^{\dagger}_{(+)m}a_{(+)m} - a^{\dagger}_{(-)m}a_{(-)m}], \tag{14}$$
$$K_{\pm} = \sqrt{2}[(a^{\dagger}_{(\pm)})^{(1/2)} \cdot (a_{(\mp)})^{(1/2)}]^{(0)}_0$$
$$= \sum_m a^{\dagger}_{(\pm)m}a_{(\mp)m};$$

they commute with $\vec{J}$ and belong to the Lie algebra of SU(4); the operator K_z is closely related to the Hamiltonian H_0 of the N atoms and one has (see Ref. 8)

$$H_0 = E_0 K_z, \tag{15}$$

where E_0 is the energy difference between the two levels of an isolated atom. In fact it is easy to show that K_z, K_+, and K_- span a Lie algebra whose commutation relations and unitarity conditions,

$$[K_z, K_{\pm}] = \pm K_{\pm},$$
$$[K_+, K_-] = 2K_z,$$
$$(K_z)^{\dagger} = K_z, \tag{16}$$
$$(K_{\pm})^{\dagger} = K_{\mp},$$

are those of an angular momentum $\vec{K}$, commuting with $\vec{J}$. Thus the cooperation group SU(4) contains the direct product of the two SU(2) groups corresponding to $\vec{J}$ and $\vec{K}$[15]:

$$\mathrm{SU}(4) \supset \mathrm{SU}^J(2) \times \mathrm{SU}^K(2). \tag{17}$$

The Casimir operators $\vec{J}^2$ and $\vec{K}^2$ of the two SU(2) groups can be shown to be identical and the reduction of the irreducible representation $\{N\}$ of SU(4) according to the chain (17) is given by:

$$\sum_J D^J \times D^J$$
$$\text{with } J = \tfrac{1}{2}N, \tfrac{1}{2}N-1, \tfrac{1}{2}N-2, \ldots, \begin{cases} 0 \text{ if } N \text{ even} \\ \tfrac{1}{2} \text{ if } N \text{ odd.} \end{cases} \tag{18}$$

The symmetric collective states are thus characterized by two commuting angular momenta $\vec{J}$ and $\vec{K}$ such that the eigenvalues of $\vec{J}^2$ and $\vec{K}^2$ are both equal to $J(J+1)$; if M and μ are the eigenvalues of J_z and K_z, respectively, one has, as usual:

$$M, \mu = -J, -J+1, -J+2, \ldots, J. \tag{19}$$

The states can finally be written $|\mu JM)$; J and M characterize the total angular momentum of the N atoms and μ characterizes their total energy [which is, cf. Eq. (15), $E = E_0\mu$]. As an example, all the states corresponding to $N = 4$ are represented in Fig. 4. An important remark can be made here: the operator X defined by Eq. (7) is a scalar operator with respect to $\vec{J}$ and it obviously commutes with H_0 or K_z; it is thus diagonal in the chosen basis so that, as shown in Sec. II C, interferences manifest themselves on real emission only. In fact the basis $|\mu JM)$ is exactly of the type defined in Ref. 8 and all the formalism described in this reference can be used.

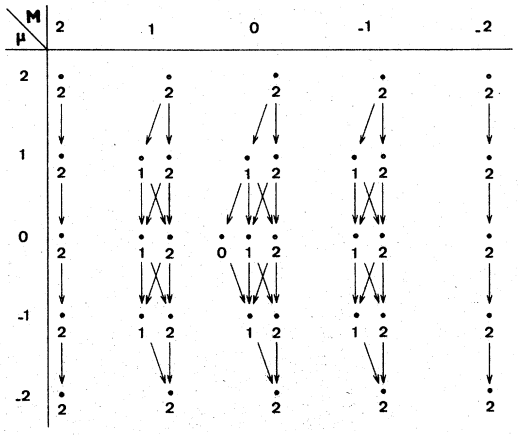

FIG. 4. Collective symmetric states of four atoms with two degenerate levels of angular momentum $\frac{1}{2}$ in the basis $|\mu JM)$; the energy of a state is proportional to μ, the eigenvalue of J_z is M and the values of J [the eigenvalues of $\vec{J}^2$ are $J(J+1)$] are indicated under each point which represents one state; only π photon cascades are represented.

B. Matrix elements of the collective dipole operator

Before solving the master equation, one has to evaluate the matrix elements of the collective dipole operator. Since operators R_q are components of a tensor operator $R^{(1)}$, the Wigner-Eckart theorem can be used but the evaluation of the reduced matrix elements of $R^{(1)}$ still remains complicated, In order to do it we have found convenient to use some more group theory. This part of the work is quite unessential for understanding the following; it is reported in the Appendix and here we just outline the procedure.

The annihilation and creation operators have

tensorial properties with respect not only to $\vec{J}$ but also to $\vec{K}$. Using these properties it is possible to construct a Lie algebra commuting with $\vec{J}$ and $\vec{K}$, which is the Lie algebra of a noncompact SU(1,1) group. In fact this group is quite analogous to a "quasispin" group.[16] One introduces then a larger noncompact group [containing the direct product $SU(1,1)\times SU^J(2)\times SU^K(2)$] such that the symmetric collective states corresponding to all possible values of N form the basis of only two irreducible representations of this group.[17] Finally for the evaluation of the matrix elements of R_q one may use three times the Wigner-Eckart theorem [once in each group: $SU^J(2)$, $SU^K(2)$, and SU(1,1)] and the evaluation of the corresponding reduced matrix elements is then straightforward.

From the matrix elements of R_q it is easy to derive the eigenvalues of the operator X, $X(\mu J)$, which are proportional to the total intensity radiated (in all directions and with all polarizations) by the system in a state $|\mu JM\rangle$; one has

$$X(\mu J)=\tfrac{1}{3}[\tfrac{1}{2}N(N+4)-J(J+1)-\mu(\mu-3)]. \tag{20}$$

The analogous quantity in the two (nondegenerate) level case is[1]

$$x(m)=\tfrac{1}{2}N(\tfrac{1}{2}N+1)-m(m-1), \tag{21}$$

which is proportional to the intensity radiated by a symmetric state ($r=\frac{1}{2}N$). The quantum numbers m and μ are quite analogous since they characterize the collective energy levels. In the two (nondegenerate) level case $x(m)$ depends on m only and is maximum for $m=0$ [$x(0)\simeq\frac{1}{4}N^2$]. In the $\frac{1}{2}\to\frac{1}{2}$ case, $X(\mu J)$ depends on J too and, for a given energy level, it varies from its maximum value, obtained with the smallest value of $J(J=|\mu|)$, to its minimum value, obtained with the largest value of $J(J=\frac{1}{2}N)$; for $\mu=0$, in particular, it varies approximately from $\frac{2}{3}(\frac{1}{4}N^2)$ to $\frac{1}{3}(\frac{1}{4}N^2)$.

Expressions (20) and (21) correspond to operator equations which are, respectively,

$$(\vec{\mathcal{R}}^\dagger\cdot\vec{\mathcal{R}})=\tfrac{1}{3}[\tfrac{1}{2}\mathfrak{N}(\mathfrak{N}+4)-(\vec{J}\cdot\vec{J})-K_z(K_z-3)], \tag{22}$$

$$\mathcal{R}_+\mathcal{R}_-=\tfrac{1}{2}\mathfrak{N}(\tfrac{1}{2}\mathfrak{N}+1)-\mathcal{R}_z(\mathcal{R}_z+1), \tag{23}$$

where $\mathfrak{N}$ stands in each case for the total number operator, of eigenvalue N; $\mathcal{R}_+$, $\mathcal{R}_-$, and $\mathcal{R}_z$ are the spherical components of the Bloch vector in the two- (nondegenerate) level case. These equations can in fact be derived from more fundamental conservation equations. In the two-nondegenerate-level case, it is known that the length of the Bloch vector is conserved during the evolution:

$$\frac{d}{dt}\left(\tfrac{1}{2}\langle\mathcal{R}_+\mathcal{R}_-+\mathcal{R}_-\mathcal{R}_+\rangle+\langle\mathcal{R}_z^2\rangle\right)=0. \tag{24}$$

From a group-theoretical point of view this equation expresses that the mean value of the Casimir operator of the cooperation group SU(2) is conserved. Of course the same property holds in the $\frac{1}{2}\to\frac{1}{2}$ case. The Casimir operator of SU(4) can be written, in terms of annihilation and creation operators,

$$G=\sum_{(\epsilon)m,(\epsilon')m'}a^\dagger_{(\epsilon)m}a_{(\epsilon')m'}a^\dagger_{(\epsilon')m'}a_{(\epsilon)m}, \tag{25}$$

where the sums run over indices of monatomic states. It is easy to verify the operator relationship

$$G=\tfrac{1}{2}\mathfrak{N}(\mathfrak{N}+2)+(\vec{J}\cdot\vec{J})+K_z^2+\tfrac{3}{2}[(\vec{\mathcal{R}}^\dagger\cdot\vec{\mathcal{R}})+(\vec{\mathcal{R}}\cdot\vec{\mathcal{R}}^\dagger)]. \tag{26}$$

The conservation equation which is the analog of Eq. (24) is thus

$$\frac{d}{dt}\left[\tfrac{3}{2}\langle(\vec{\mathcal{R}}^\dagger\cdot\vec{\mathcal{R}})+(\vec{\mathcal{R}}\cdot\vec{\mathcal{R}}^\dagger)\rangle+\langle K_z^2\rangle+\langle(\vec{J}\cdot\vec{J})\rangle\right]=0. \tag{27}$$

This equation can be written in a form closer to Eq. (24) if one defines, for each linear polarization, a vector which is the analog of a Bloch vector. Let us put

$$\begin{aligned}R_\lambda&=\vec{\mathcal{R}}\cdot\vec{\epsilon}_\lambda,\\ R_\lambda^\dagger&=\vec{\mathcal{R}}^\dagger\cdot\vec{\epsilon}_\lambda,\\ R_{z\lambda}&=\tfrac{1}{2}[R_\lambda^\dagger,R_\lambda],\end{aligned} \tag{28}$$

$\vec{\epsilon}_\lambda$ being a unit vector. It follows that, for any $\vec{\epsilon}_\lambda$,

$$R_{z\lambda}=\tfrac{1}{3}K_z \tag{29}$$

Equation (27) becomes

$$\frac{d}{dt}\left(\sum_\lambda\left[\tfrac{1}{2}\langle R_\lambda^\dagger R_\lambda+R_\lambda R_\lambda^\dagger\rangle+\langle R_{z\lambda}^2\rangle\right]+\tfrac{1}{3}\langle(\vec{J}\cdot\vec{J})\rangle\right)=0, \tag{30}$$

where the sum runs over three orthogonal polarizations. The conserved quantity is thus the sum of the squared lengths of the "generalized" Bloch vectors corresponding to three orthogonal polarizations and of one third of the squared length of the total angular momentum.

C. Solution of the master equation

Because of the invariance of the master equation (1) under the $SU^J(2)$ group, the use of tensorial formalism is very convenient. In particular the density operator can be written as a sum of irreducible tensor operators; this is described in detail for the $j\to j'$ case in Ref. 8: one has

$$\rho(t)=\sum_{\substack{k=0,1,2,\dots\\ q=-k,-k+1,\dots,k}}a_{kq}\rho^{(kq)}(t), \tag{31}$$

where the a_{kq} are constants, depending on initial conditions only. The evolution of operators $\rho^{(kq)}(t)$ with different values of k are independent and one

has, in the $\frac{1}{2} \rightarrow \frac{1}{2}$ case,

$$\left(\frac{d}{dt} + \tfrac{1}{2}\Gamma[X(\mu'J') + X(\mu J)]\right)(\mu'J'||\rho^{(k)}(t)||\mu J) = \Gamma \sum_{J_0, J_0'} (-1)^{J+J_0'+k+1} \begin{Bmatrix} J_0' & k & J_0 \\ J & 1 & J' \end{Bmatrix} (\mu'J'||R^{(1)}||\mu' + 1J_0')$$
$$\times (\mu J||R^{(1)}||\mu + 1J_0)(\mu' + 1J_0'||\rho^{(k)}(t)||\mu + 1J_0). \tag{32}$$

The great interest of development (31) is due to the fact that the mean value of a tensor operator of rank k,

$$\langle V_q^{(k)}\rangle(t) = \sum_{\mu', J', \mu, J} (2k+1)^{-1}(-1)^{J-J'+q} \times a_{k-q}(\mu'J'||\rho^{(k)}(t)||\mu J) \times (\mu J||V^{(k)}||\mu'J'), \tag{33}$$

involves reduced matrix elements of $\rho^{(k)}(t)$ only. The evaluation of the mean value of a tensor operator requires therefore the solution of Eq. (32) for one value of k. As shown in Ref. 8, the expressions of the expectation values for the radiated field can be written as linear combinations of mean values of tensor operators. For the radiated intensity the corresponding ranks are 0, 1, and 2 and for the quantum fluctuations of this intensity the rank varies from 0 to 4. From the computational point of view this is a great simplification since it reduces drastically the number of coupled equations one has to solve. It has been thus possible to compute, for various initial conditions, the quantum fluctuations for N up to 40, the radiated intensity corresponding to the different polarizations for N up to 50 and the total intensity (which involves $k=0$ only) for N up to 110.

As a comparison, we have made the corresponding calculations for the case where the level degeneracy is supposed to be removed, that is using the master equation (3) and the occupation number representation. However in this case, the tensorial formalism cannot be used and, for a given number of atoms, the number of coupled equations one has to solve is much larger than for the degenerate case: for this reason, the values of N have been limited to $N=20$.

The initial conditions for the density matrix are determined by the excitation conditions, in particular by the polarization properties of the excitation light. We have considered cases of initial complete population inversion and, as mentioned before, we have assumed that the initial density matrix is defined on symmetric states only. More precisely we have assumed that the distribution of the atoms in the two states of the upper level is given by boson statistics, that is,

$$\rho(0) = \sum_{N_1, N_2} \alpha^{N_1}\beta^{N_2} \frac{N!}{N_1!N_2!} |N_1N_200\rangle\langle N_1N_200| \tag{34}$$

(the states are written in the occupation number representation, $N_1+N_2=N$ and $\alpha+\beta=1$). It is easy to show that the mean values of the populations of the upper states and of the coherence N_{12} between them are, at $t=0$,

$$\langle N_1\rangle(0) = \alpha N, \quad \langle N_2\rangle(0) = \beta N, \quad \langle N_{12}\rangle(0) = 0. \tag{35}$$

Consequently three particular pairs of values of α and β can be interpreted as resulting from simple excitation conditions: (i) $\alpha=1, \beta=0$; this situation could be obtained by pumping in σ_+ circularly polarized light from a third level of angular momentum $\frac{1}{2}$; (ii) $\alpha=\frac{3}{4}, \beta=\frac{1}{4}$; this could be obtained by pumping in σ_+ light also, but from a level of angular momentum $\frac{3}{2}$; (iii) $\alpha=\beta=\frac{1}{2}$; this could be obtained by pumping in linearly polarized light from a third level of angular momentum $\frac{1}{2}$ or $\frac{3}{2}$.

The states appearing in Eq. (34) are also states of the $|\mu JM)$ basis

$$|N_1\,N_2\,00\rangle \equiv |\tfrac{1}{2}N\,\tfrac{1}{2}N\,M), \quad \text{with} \quad M = \tfrac{1}{2}(N_1 - N_2). \tag{36}$$

The values of a_{kq} and of the reduced matrix elements of $\rho^{(k)}(0)$ are easily derived from Eq. (34) [see Ref. 8, Eq. (29)]. In particular, a_{kq} is zero except for $q=0$ and the nonzero reduced matrix elements are $(\frac{1}{2}N\,\frac{1}{2}N||\rho^{(k)}(0)||\frac{1}{2}N\,\frac{1}{2}N)$. Since the dependence on k of a_{kq} is arbitrary, these reduced matrix elements can be taken as equal to 1. With these conditions, the reduced matrix elements of $\rho^{(k)}(t)$ are the same for the various considered initial conditions and only the a_{k0} depend on α and β. Let us remark that, because of the trace conservation of the density matrix, one has

$$a_{00} = (N+1)^{-1/2}. \tag{37}$$

The mean values of any scalar operator is therefore the same for the different initial conditions considered; this is in particular the case of the

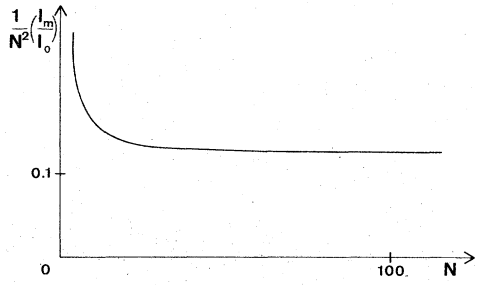

FIG. 5. Evolution with N of the maximum I_m of the total intensity emitted by N atoms with two degenerate levels of angular momentum $\frac{1}{2}$. As soon as N goes above 30, I_m becomes proportional to N^2 a good approximation.

total intensity radiated in all directions and with all polarizations.

D. Discussion of numerical results

1. General features

First of all, note that, although the computation concerns small values of N, it appears that the evolution of the various interesting quantities with respect to N reaches an asymptotic behavior as soon as N is of the order of 50 or 100. As an example, Figs. 5 and 6 show the evolution with N of the maximum of the total intensity radiated in the $\frac{1}{2} \rightarrow \frac{1}{2}$ case and of the corresponding time delay.

Figures 7–9 show typical results corresponding to initial conditions (i), (ii), and (iii). Let us emphasize that the relative values of the maxima of the pulses corresponding to the different polarizations are very different in the three cases and also different from the values for non collective spontaneous emission (which are given by the values at $t=0$). Conversely, in all cases time delays corresponding to the different polarizations are close

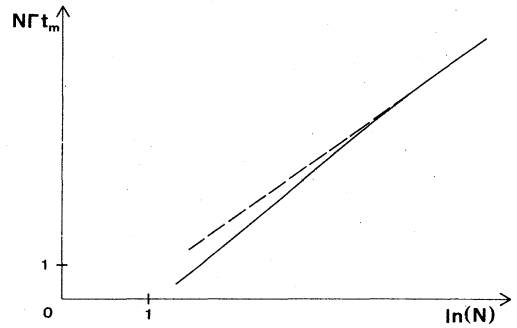

FIG. 6. Evolution with N of the time delay t_m of the super-radiant pulse emitted by N atoms with two degenerate levels of angular momentum $\frac{1}{2}$. As soon as N goes above 50, t_m varies as $(a\log N+b)/N$ to quite a good approximation.

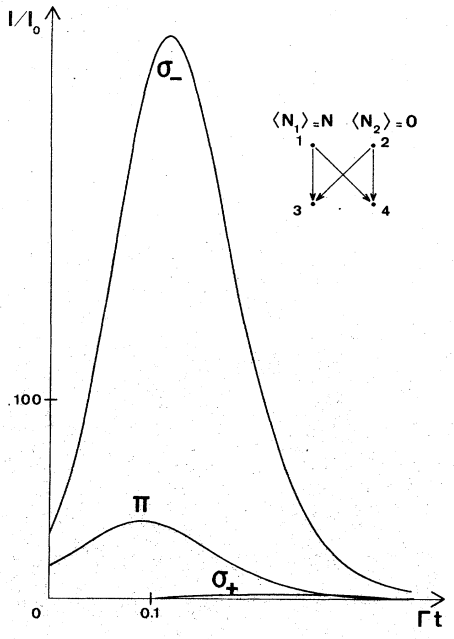

FIG. 7. Intensity radiated with π, σ_+, and σ_- polarizations by $N=50$ atoms with two degenerate levels of angular momentum $\frac{1}{2}$ for initial conditions (i). The state diagram of one single atom and the initial mean values of the populations of the upper states are recalled. The values at $t=0$ correspond to noncollective spontaneous emission: the σ_- intensity is twice the π intensity. Note that the maximum of the σ_- pulse is much larger than the maximum of the π pulse (by a factor of 7). The corresponding delay times are not very different. Note also the presence of a very small σ_+ pulse, of larger time delay.

together and the differences between them decrease with N.

2. Competition effects

In order to discuss competition effects alone (that is without interferences), we consider the case where level degeneracy is supposed to be removed. Interferences completely disappear if the frequency difference $\delta\nu$ between the two π Zeeman components is larger than the frequency width of any π collective transition. This condition can be written as $\delta\nu \gg \frac{1}{3}(\frac{1}{2}N)^2\Gamma$ and is very severe for large N. The "nondegenerate" case considered here would be in fact difficult to observe: it is however quite useful for the discussion.

For initial conditions (i) the upper state $m=-\frac{1}{2}$ is initially empty so that it is a case of competition between two transitions sharing a common upper state, one transition (σ_-) being twice as probable as the other one (π). As expected superradiance on the less probable transition is inhibited and the ratio between the σ_- and π pulse max-

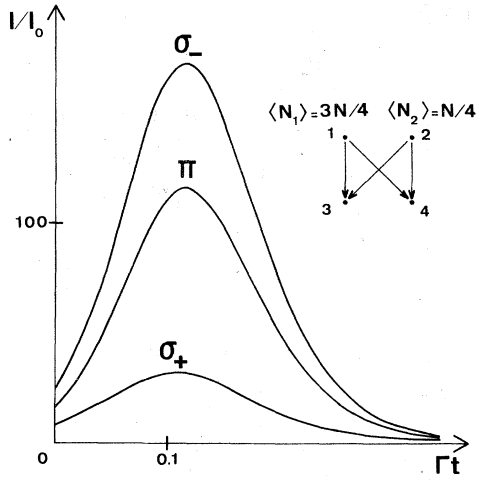

FIG. 8. Intensity radiated with π, σ_+, and σ_- polarizations by $N=50$ atoms with two degenerate levels of angular momentum $\frac{1}{2}$ for initial conditions (ii). As in Fig. 7, these initial conditions are recalled in a separate diagram. Notice that the relative values of the maxima of the different pulses are quite different from those of Fig. 7 and also different from the relatives values at $t=0$, which correspond to non collective spontaneous emission. The three time delays are quite close together.

ima is much larger than the ratio of the transition probabilities [see Fig. 7 (Ref. 18)]. In Fig. 10 is shown a comparison of the σ_- and π pulses and of the respective shapes they would have without competition. The two pulses appear sooner and are smaller than if alone. For the σ_- pulses these differences are small and decrease with N but for the π pulse it is the contrary. In fact, the π intensity decreases as soon as the σ_- pulse appears and empties the upper common state. Let us add that the inhibition increase with N. Precisely the ratio between the σ_- and π maxima increases as $\sqrt{N}$, at least in the explored range of N ($N \leq 50$).

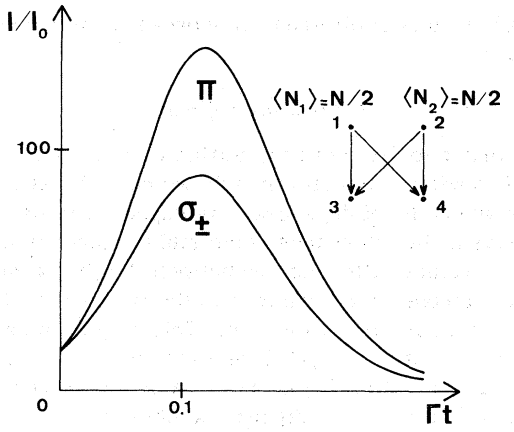

FIG. 9. Intensity radiated with π and $\sigma_\pm$ polarizations by $N=50$ atoms with two degenerate levels of angular momentum $\frac{1}{2}$ for initial conditions (iii). As in Figs. 7 and 8, these initial conditions are recalled in a separate diagram. The different intensities are equal to $t=0$ (noncollective spontaneous emission) but the maximum of the π pulse is noticeably larger than the maximum of the $\sigma_\pm$ pulses. As in Figs. 7 and 8, the time delays are almost equal.

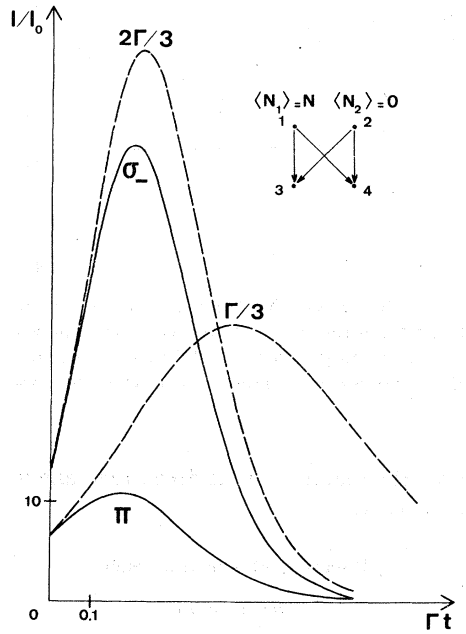

FIG. 10. Inhibition. Intensity radiated, in the $\frac{1}{2} \rightarrow \frac{1}{2}$ "nondegenerate" case, by $N=20$ atoms for initial conditions (i) (these conditions are recalled on the state diagram of one single atom). The solid lines represent the σ_- and π pulses. The broken lines represent the shapes these pulses would have if each transition were alone, that is the pulses emitted by 20 two- (nondegenerate) level atoms with transition probabilities equal to $\frac{1}{3}2\Gamma$ (σ_- transition) and $\frac{1}{3}\Gamma$ (π transition). Note that the two pulses appear sooner and are smaller than if alone. This effect is especially important for the π pulse: its maximum is reduced by a factor of 2.5 and the time delay is a little shorter than, instead of twice as large as that of the σ_- pulse.

For the other two initial conditions, (ii) and (iii), both inhibition and initiation effects are expected. Inhibition appears always [as for initial conditions (i)] between one σ component and one π component and the ratio of their transition probabilities is always 2. Initiation appears also between one σ component and one π component but the importance of the effect depends not only on the ratio of the transition probabilities, but also on the ratio of the initial populations of the upper states. The role of initiation is thus different in the two cases [let us recall that this effect does not appear for initial conditions (i)].

For initial conditions (ii) the situation is com-

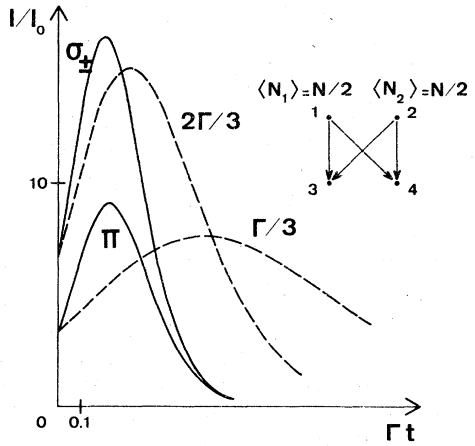

FIG. 11. Initiation. Intensity radiated, in the $\frac{1}{2}\rightarrow\frac{1}{2}$ "nondegenerate" case, by $N=20$ atoms for initial conditions (iii) (these conditions are recalled on the state diagram of one single atom). The solid lines represent the pulses corresponding to one π and one σ_+ or σ_- Zeeman component. The broken lines represent the shapes these pulses would have if the corresponding transitions were alone, that is the pulses emitted by 10 two- (nondegenerate) level atoms with transition probabilities equal to $\frac{2}{3}\Gamma$ (σ transition) and $\frac{1}{3}\Gamma$ (π transition). Note that the two pulses appear sooner and higher than if alone. In particular the time delay of the π pulse is reduced by a factor of 3.

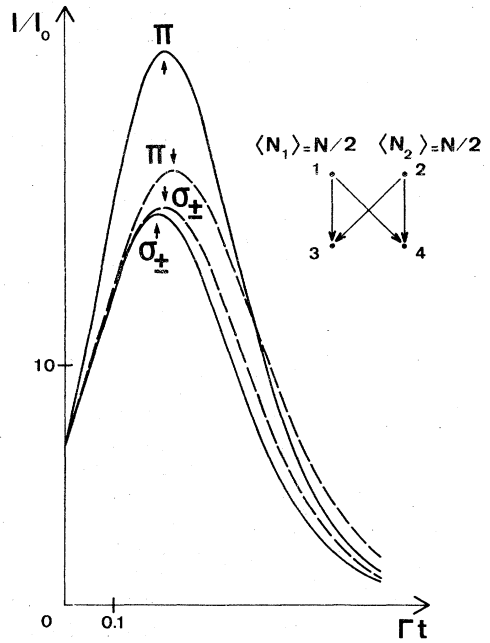

FIG. 12. Interferences. Comparison between the $\frac{1}{2}\rightarrow\frac{1}{2}$ case (solid lines) and the corresponding "nondegenerate" case (broken lines), for $N=20$ atoms and for initial conditions (iii) (the initial conditions are recalled on the state diagram of one single atom). The differences between the two cases are uniquely due to the interference effects. Note that these interferences increase the maximum of the π pulse (about 30%) and decrease a little the maximum of the σ_+ and σ_- pulses. They decrease very little the two times delays.

plicated. Inhibition and initiation are mixed together and neither one nor the other is preponderant. Their relative influence is different for each Zeeman transition and nothing more will be said about it.

For initial conditions (iii), it appears that initiation effects are important and prevail upon the inhibition ones (and all the more as N increases): the ratio between the maxima of one σ and one π component is smaller than the ratio of the transition probabilities and decreases with N. Figure 11 shows a comparison of the pulses corresponding to one π and one σ component with the respective shapes they would have without competition. One recognizes the main features of the initiation effect: the two pulses appear sooner and higher than if alone; in particular the time delay of the "initiated" π pulse is greatly reduced and tends, as N increases, to be equal to the time delay of the σ pulse.

3. *Interference effects*

The specific effect of level-degeneracy interferences can be seen in the differences between the $\frac{1}{2}\rightarrow\frac{1}{2}$ degenerate and "nondegenerate" cases.

First, for initial conditions (i), although the upper state $m=-\frac{1}{2}$ is initially empty, a small σ_+ pulse appears (see Fig. 7). Consequently the mean value of the population of this state must have increased. This effect was expected and is a manifestation of the interferences on the emission and reabsorption of virtual π photons. It is very small, probably due to the inhibition of super-radiance on the π transition; for the same reason the influence of interferences on the π and σ_- pulses is also very small for initial conditions (i).

For the other initial conditions the interferences appear to be mainly constructive: the maximum of the π pulse is larger and the corresponding time delay is a little shorter in the degenerate case than in the "nondegenerate" one (see Fig. 12). Moreover these effects increase with N; as an example in case (iii) the ratio between the π and σ maxima increases much more rapidly in the degenerate case than in the "nondegenerate" case.

4. *Comparison with the two- (nondegenerate) level case*

As shown before, interference and competition effects are combined in such a way that the total intensity has the same shape in the time domain for all considered initial conditions. It is interesting to compare this intensity with the intensity

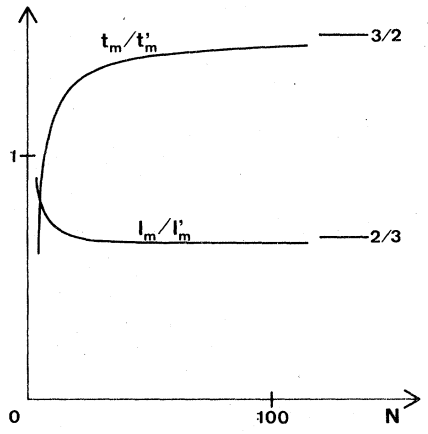

FIG. 13. Ratios I_m/I'_m and t_m/t'_m vs N; I_m and t_m are defined as in Figs. 5 and 6, and I'_m and t'_m are the value and position of the maximum of the intensity radiated by N two- (nondegenerate-) level atoms. I_m/I'_m reaches quickly (as soon as N goes above 30) a value close to $\frac{2}{3}$ and t_m/t'_m tends approximately to $\frac{3}{2}$.

radiated by the same number of two (nondegenerate) level atoms[19]: the maximum is smaller, the time delay is larger and the ratios between the two maxima and the two delays tend, as N increases, to be approximately equal to, respectively, $\frac{2}{3}$ and $\frac{3}{2}$ (see Fig. 13). [Note that these differences between the $\frac{1}{2} \to \frac{1}{2}$ and the two-(nondegenerate) level cases are important: the level degeneracy cannot be merely ignored.] Besides a comparison between the pulse shapes in the two cases shows that they almost coincide if a suitable normalization is made for one of them (see Fig. 14). In particular the full widths at half-maximum (FWHM) of the two curves are approximately equal (in the two cases, the FWHM tends, as N increases to be nearly equal to the time delay). Finally this comparison shows that, as far as the total intensity is concerned, the $\frac{1}{2} \to \frac{1}{2}$ case can be well approximated by the two (nondegenerate) level case provided the transition probability is changed from Γ to $\frac{2}{3}\Gamma$.

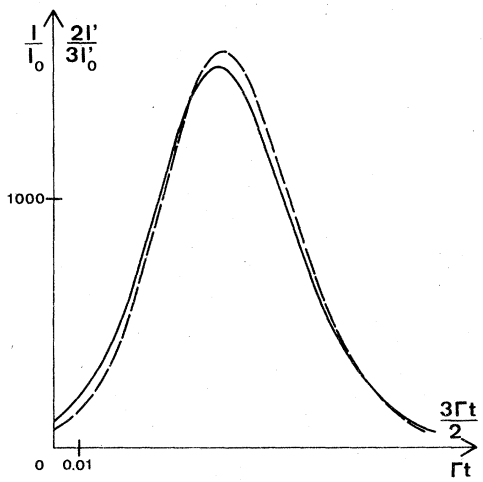

FIG. 14. The solid line represents the total intensity $I(t)$ radiated by 110 atoms in the $\frac{1}{2} \to \frac{1}{2}$ case and the broken line represent the intensity $I'(t')$ radiated by 110 two- (nondegenerate) level atoms, normalized this way: I' is multiplied by $\frac{2}{3}$ and t' by $\frac{3}{2}$. Notice that the two curves are very close together everywhere; in particular the maxima and time delays are almost equal.

This result is not surprising if initial conditions (i) are considered. The larger N, the more the super-radiance on the π transition is inhibited and, for large N, the total intensity is expected to consist in a σ_- pulse which has the same shape as if alone. For the other initial conditions, when competition and interference effects are mixed together, the previous result is less obvious. A general explanation can still be given, which is based on an analysis of the partition of the population of a given energy level between the different states of this level.

Let $P_{\mu J}(t)$ be the average value of the population of a state $|\mu JM)$ with given μ and J, that is

$$P_{\mu J}(t) = \frac{1}{2J+1} \sum_M (\mu JM|\rho(t)|\mu JM)$$

$$= (2J+1)^{-1/2} \ (\mu J||\rho^{(0)}(t)||\mu J); \tag{38}$$

the expectation value for the total intensity $I(t)$ can be written

$$I(t) = I_0 \sum_{\mu J} (2J+1) X(\mu J) P_{\mu J}(t). \tag{39}$$

The quantities $P_{\mu J}(t)$ obey the differential equations

$$\left(\frac{d}{dt} + \Gamma X(\mu J)\right) P_{\mu J}(t)$$

$$= \Gamma \sum_{J'} (2J+1)^{-1} (\mu J||R^{(1)}||\mu+1 J')^2 \ P_{\mu+1 J'}(t); \tag{40}$$

the sum runs over three values of J': $J+1$, J, and $J-1$. If N is large and if the variation of $P_{\mu J}(t)$ with J is smooth enough one may introduce the average value of $P_{\mu+1 J'}(t)$ over these three values of Eq. (40) becomes

$$\left(\frac{d}{dt} + \Gamma X(\mu J)\right) P_{\mu J}(t) \simeq \Gamma Y(\mu J) \langle P_{\mu+1 J}(t)\rangle_{\text{av}}, \tag{41}$$

where $Y(\mu J)$ is the eigenvalue of the operator

$$Y = \sum_q R_q R_q^\dagger = X - 2K_z. \tag{42}$$

$X(\mu J)$ and $Y(\mu J)$ vary analogously with J: they

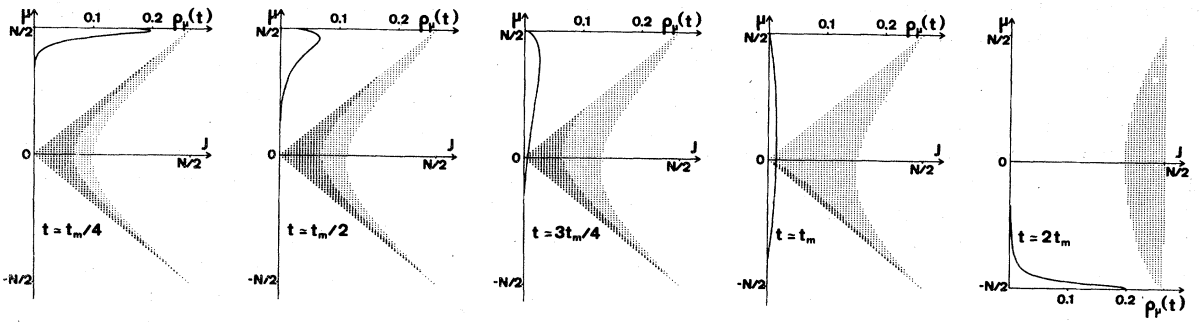

FIG. 15. Evolution of populations, for $N=110$ atoms with two degenerate levels of angular momentum $\frac{1}{2}$. The curves represent the total population $\rho_\mu(t)$ of a given energy level μ vs μ, for various values of t. The diagrams show a comparison, for the same values of t, between the average population $P_{\mu J}(t)$ of a state $|\mu JM)$ with given μ and J and the average population of a state of the same energy level μ which is given by $P_\mu(t)=\rho_\mu(t)/[\frac{1}{2}N+1)^2-\mu^2]$. Each point of the diagram corresponds to a pair (μ,J) and the set of all pairs occupies a triangle $|\mu|\leq J\leq\frac{1}{2}N$. The dark areas correspond to the inequality $P_{\mu J}(t)\geq 5P_\mu(t)$, the gray areas to $P_{\mu J}(t)\geq P_\mu(t)$, and the white ones to $P_{\mu J}(t)<P_\mu(t)$. Concerning the total population of the different energy levels, one can see that the population is progressively transferred from the highest level ($\mu=\frac{1}{2}N$), in which all the population is concentrated at $t=0$, to the lowest one ($\mu=-\frac{1}{2}N$) which will be the only occupied level for infinite t. Note that the shape of the curves $\rho_\mu(t)$ varies considerably with t: at the beginning or at the end of the pulse ($t\ll t_m$ or $t\gg t_m$), very few levels are populated but for $t\simeq t_m$ all levels are almost equally populated [the evolution of the population of the energy levels in the two-nondegenerate-level case is qualitatively the same (Ref. 19)]. Concerning the distribution of the population of a given energy level between the different values of J, note that except for values of t larger than t_m, the states for which J is close to $|\mu|$ are by far the most populated.

are maximum when J is minimum ($J=|\mu|$) and minimum when J is maximum ($J=\frac{1}{2}N$), the ratio between maximum and minimum being approximately equal to 2. Thus, for a given energy level μ, on the one hand $P_{\mu J}(t)$ relaxes faster when J is small, due to the $X(\mu J)$ term; on the other hand, it increases when J is small also, due to the $Y(\mu J)$ term. This latter effect is cumulative since the average value of the right-hand side of Eq. (41) is taken over values of J' close to J. Finally the average populations of states $|\mu JM)$ with given μ and J corresponding to small values of $J(J\simeq|\mu|)$ are expected to be much more significant and more rapidly evolving than the average populations corresponding to the same value of μ and large values of J; moreover this phenomenon will increase with N. This result can be verified for rather small values of N by an exact calculation of the populations. Figure 15 shows a comparison between the average population $P_{\mu J}(t)$ and the average population of a state of energy level μ. For values of t smaller or a little larger than the time delay t_m, the most populated states of a given energy level are those for which $J\simeq|\mu|$. Since the corresponding populations are the most rapidly evolving, it is not surprising that the reverse result becomes true for larger values of t. However the value of the total population of the different levels is then very small except for $\mu\simeq-\frac{1}{2}N$ and the system does not radiate any longer: the distribution of the population in the different states for large values of t will not affect the pulse shape.

From Eqs. (40) it is straightforward to derive

$$\sum_J\left(\frac{d}{dt}+\Gamma X(\mu J)\right)P_{\mu J}(t)=\Gamma\sum_{J'}X(\mu+1\,J')P_{\mu+1\,J'}(t), \tag{43}$$

where the sums run over all values of J and J' compatible with the values μ and $\mu+1$. One introduces the average value of $X(\mu J)$ for the level μ,

$$\bar{X}(\mu)=\sum_J(2J+1)X(\mu J)P_{\mu J}(t)\Big/\sum_J(2J+1)P_{\mu J}(t), \tag{44}$$

and the total population of the level μ,

$$\rho_\mu(t)=\sum_J(2J+1)P_{\mu J}(t). \tag{45}$$

Equations (43) and (39) become then

$$\left(\frac{d}{dt}+\Gamma\bar{X}(\mu)\right)\rho_\mu(t)=\Gamma\bar{X}(\mu+1)\rho_{\mu+1}(t),$$
$$I(t)=I_0\sum_\mu\bar{X}(\mu)\rho_\mu(t). \tag{46}$$

As shown before, for large N and as long as the radiated intensity is important, the states for which $J\simeq|\mu|$ are by far the most populated: it follows that

$$\bar{X}(\mu)\simeq X(\mu\,|\mu|)=\tfrac{2}{3}[\tfrac{1}{4}N^2+N-\mu^2+\tfrac{1}{2}(3\mu-|\mu|)]. \tag{47}$$

If one compares Eqs. (46) and (47) with the corre-

sponding ones in the two-nondegenerate-level case it appears that the only important differences consists in a factor $\frac{2}{3}$ in the expression of $\bar{X}(\mu)$. This accounts, with a good approximation, for the differences between the total intensity radiated in the $\frac{1}{2} \rightarrow \frac{1}{2}$ case and the intensity radiated in the two-nondegenerate-level case.

Let us finally emphasize that the super-radiant emission of the system keeps it in states for which the mean value of $\vec{J}^2$ is close to its minimum value. Equivalently one can say that the collective spontaneous emission populates the states that are able to radiate the most: this phenomenon is quite analogous to the creation of super-radiant states starting from complete population inversion in Dicke's interpretation of the two-nondegenerate-level case.

5. *Quantum fluctuations*

In a quantum model, the knowledge of the field properties does not reduce to the knowledge of its intensity. It is also interesting to investigate the statistical properties of the field and the evolution of these properties during the super-radiant emission. For this aim one can in particular evaluate the quantum fluctuations of the intensity[20]; it is recalled that these quantities involve the instantaneous value of a second-order correlation function[21] and that they can in principle be measured in a photon-count experiment, by detecting photon coincidences.

We have first calculated, for the various considered initial conditions, the squared relative dispersion of the quantum fluctuations of the intensity radiated with π, σ_+, or σ_- polarization. The expressions of these quantities, σ_ϵ^2, in terms of mean values of atomic operators are given in Ref. 8. Since their calculation requires the solution of Eqs. (32) for k varying from 0 to 4, it has not been possible to consider values of N larger than 40.

First one observes that the variation with t of σ_ϵ^2 has qualitatively the same shape as the variation of the squared relative dispersion of the quantum fluctuations of the intensity in the two (nondegenerate) level case σ_{nd}^2.[19] At $t=0$, the emission is noncollective and the photon statistics are Gaussian ($\sigma_\epsilon^2 \simeq 1$). During the buildup of the super-radiant pulse, σ_ϵ^2 decreases and has a minimum in the high-intensity region. This is characteristic of the cooperative effect: in this region, the atoms radiate coherently and the state of the emitted field corresponding to each polarization is in fact close to a Glauber coherent state,[21] for which one would have $\sigma_\epsilon^2 = 0$. In the two- (nondegenerate-) level case, the minimum of σ_{nd}^2 reaches an asymptotic value (about 0.09) as soon as N is approximately equal to 30.[19] In the $\frac{1}{2} \rightarrow \frac{1}{2}$ case, the asymptotic behavior is not yet reached for $N=40$. A few qualitative conclusions can however be derived from these results.

First, for initial conditions (i), the minimum of σ_ϵ^2 for the σ_- polarization is of the same order as the minimum of σ_{nd}^2 (0.13 for $N=40$, and still decreasing with higher values of N) For the π polarization, the minimum of σ_ϵ^2 is very shallow (0.85 for $N=40$ and still increasing with higher values of N). It appears thus that the emission on the π transition has almost no cooperative character in this case and this confirms the fact that superradiance on this transition is inhibited. For the other initial conditions, (ii) and (iii), the minima of σ_ϵ^2 are generally rather small (between 0.13 and 0.41 for $N=40$, and all are decreasing with higher values of N). This means that, in these cases, the influence of initiation and interference effects increases the cooperative character of the emission and compensates, at least partially, the influence of inhibition. One observes in each case, (i), (ii), and (iii), that the order according to which the minima of the different σ_ϵ^2 are increasing is the order according to which the corresponding intensity maxima are decreasing: the same cooperative effects are indeed responsible for the height of the pulses and of the depth of the coresponding σ_ϵ^2 maxima.

We have also calculated the squared relative dispersion, $\sigma^2(t)$, of the quantum fluctuations of the total intesntiy $I(t)$ radiated in all directions and with all polarizations. This quantity can be measured, in principle, by detecting photon coincidences without putting any polarizer before the photodetector and averaging over all observation directions. One has

$$\sigma^2(t) = [Q(t) - I^2(t)]/I^2(t), \tag{48}$$

with

$$Q(t) = I_0^2 \sum_{\vec{\epsilon},\vec{\epsilon}'} \langle (\vec{R}\cdot\vec{\epsilon})^\dagger (\vec{R}\cdot\vec{\epsilon}')^\dagger (\vec{R}\cdot\vec{\epsilon}')(\vec{R}\cdot\vec{\epsilon})\rangle(t), \tag{49}$$

where the sums run over three orthogonal polarizations. It is easy to show that $Q(t)$ can be written as a linear combination of mean values of scalar operators. As for $I(t)$, $Q(t)$ is the same for the various considered initial conditions and it has been possible to compute the values of σ^2 for N up to 110. It appears that $\sigma^2(t)$ has also a minimum in the high-intensity region, but the value at $t=0$ (0.56) is much smaller than 1 and the minimal value is a little smaller than that of σ_{nd}^2 (0.07 for $N=110$ and only very slightly increasing with higher values of N).

Concerning the value at $t=0$, the explanation is

simple as long as initial conditions (i) are considered: the state of the field can then be represented by two independent Gaussian distributions, corresponding to both π and σ_- polarizations, with two different average values of the number of photons. For the other initial conditions, the situation is less simple. The calculated value of $\sigma^2(0)$ implies, in these cases, the existence of a correlation between the emission of photons of different polarizations; more precisely one may have

$$\langle(\vec{R}\cdot\vec{\epsilon})^\dagger(\vec{R}\cdot\vec{\epsilon}')^\dagger(\vec{R}\cdot\vec{\epsilon}')(\vec{R}\cdot\vec{\epsilon})\rangle(0) \neq \langle(\vec{R}\cdot\vec{\epsilon})^\dagger(\vec{R}\cdot\vec{\epsilon})\rangle(0)\,\langle(\vec{R}\cdot\vec{\epsilon}')^\dagger(\vec{R}\cdot\vec{\epsilon}')\rangle(0), \quad (50)$$

for two orthogonal polarizations, $\vec{\epsilon}$ and $\vec{\epsilon}'$. We stress here that the existence of such correlations, at $t=0$ that is when the emission is noncollective, is in fact due to our choice of an initial density matrix defined on symmetric states only. In any case a study of the evolution during the super-radiant emission of such correlations would be quite interesting. In particular we already mention here that a significant correlation between π and σ_- photons exists in the high-intensity region even for initial conditions (i), although in this case no such correlation is present at $t=0$.

IV. CONCLUSION

In the present work it has been shown how the small-system model gives a simple analysis of the influence of level degeneracy on super-radiant emission. It has allowed us to understand the origin of the different types of level-degeneracy effects and to discuss their qualitative influence on the properties of the emitted light. Moreover, thanks to a formalism based on the invariance properties of the master equation, the intensity radiated on a $\frac{1}{2}\rightarrow\frac{1}{2}$ transition has been computed for various initial conditions and for small but already significant values of the number of atoms. These results have shown how both competition and interference effects can be important. Concerning competition the two different expected effects, inhibition and initiation, have appeared to be preponderant for different initial conditions. Concerning interferences, we have noted that, through an exchange of virtual photons by the atoms, they are able to increase the population of an upper state, although no real absorption is considered. These interferences are in general constructive so that they increase the maximum intensity and decrease the time delay of the concerned pulses. More generally it has appeared that all level-degeneracy effects are always combined in such a way that, for a given energy level, the most populated states are those which are able to radiate the most. As a consequence we have shown that the maximum of the total intensity is smaller (factor $\frac{2}{3}$) and the corresponding time delay larger (factor $\frac{3}{2}$) than if level degeneracy is merely ignored. Quantum fluctuations of the radiated intensity have been computed too and they also show the influence of level-degeneracy effects; for example, the emission on an inhibited transition is no more cooperative and the statistics remains approximately Gaussian in the high-intensity region. In addition, these calculations permit one to study the correlations between photons of different polarizations. A detailed study of the evolution during the super-radiant emission of these correlations remains to be done, but it can already be mentioned that such correlations may appear in the high-intensity region.

The possibility of using these results for interpreting an actual experiment is limited by the validity of the small-system model itself. As is well known it is not realistic, it ignores dipole-dipole interactions and it is purely Markovian. Now, as shown in Ref. 11, the observation of pure Markovian super-radiance would imply quite severe conditions on the total number of initially excited atoms. Moreover the geometry of the system in this model is quite different from the geometry encountered in all the experiments performed. For pencil-shaped systems, super-radiance appears with two different polarizations only; in the $j=\frac{1}{2}\rightarrow j'=\frac{1}{2}$ case for example it would be possible, with a suitable choice of the quantization axis, to calculate the radiated intensity without considering competition nor interference effects. However, even for pencil-shaped systems, these effects have surely to be taken into account for other values of j and j' and their individual influence on the super-radiant emission can be estimated from the analysis of the results obtained here. This allows us to predict that level degeneracy can play an important role and that it should be taken into account in a detailed theoretical interpretation of experiments. In particular, one has to expect polarization effects, concerning mainly the heights (more than the time delays) of the pulses corresponding to different polarizations and depending upon the polarization properties of the excitations light. Besides the maximum and time delay of the total intensity radiated can be quite different from those one would obtain ignoring level degeneracy.

Let us finally recall that a quantum model only describes rigorously how the super-radiant emission starts. As shown in Ref. 11, the Markovian approximation is always valid as long as few photons are emitted. This work should thus provide a good starting point for a rigorous study of

the early Markovian stage of super-radiance between degenerate levels.

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APPENDIX: GROUP THEORETICAL EVALUATION OF MATRIX ELEMENTS OF THE COLLECTIVE DIPOLE OPERATOR

A. Preliminary group-theoretical study

Annihilation and creation operators (12) have also tensorial properties with respect to the angular momentum $\vec{K}$; they are the components of two double tensor operators of rank $\frac{1}{2}$ with respect to both (commuting) angular momenta $\vec{J}$ and $\vec{K}$

$$\begin{aligned} a^{\dagger}_{(+)m} &= (a^{\dagger})^{(1/2\,1/2)}_{m\,1/2}, \\ a^{\dagger}_{(-)m} &= (a^{\dagger})^{(1/2\,1/2)}_{m-1/2}, \\ a_{(+)m} &= (-1)^{1/2-m}(a)^{(1/2\,1/2)}_{-m-1/2}, \\ a_{(-)m} &= (-1)^{1/2+m}(a)^{(1/2\,1/2)}_{-m\,1/2}, \end{aligned} \tag{A1}$$

(the first rank refers to $\vec{J}$, the second one to $\vec{K}$). Starting from the different double scalar products of these tensor operators, a Lie algebra commuting with $\vec{J}$ and $\vec{K}$ can be constructed. Precisely, one has

$$\begin{aligned} Q_z &= \tfrac{1}{2}[\{(a^{\dagger})^{(1/2\,1/2)}(a)^{(1/2\,1/2)}\}^{(00)}_{00} \\ &\quad + \{(a)^{(1/2\,1/2}\,(a^{\dagger})^{(1/2\,1/2}\}^{(00)}_{00}] \\ &= \frac{1}{4}\sum_m [a^{\dagger}_{(+)m}a_{(+)m} + a_{(+)m}a^{\dagger}_{(+)m} \\ &\quad + a^{\dagger}_{(-)m}a_{(-)m} + a_{(-)m}a^{\dagger}_{(-)m}], \\ Q_+ &= -\{(a^{\dagger})^{(1/2\,1/2)}(a^{\dagger})^{(1/2\,1/2)}\}^{(00)}_{00} \\ &= -\sum_m (-1)^{1/2-m}a^{\dagger}_{(+)m}a^{\dagger}_{(-)m}, \\ Q_- &= \{(a)^{(1/2\,1/2)}(a)^{(1/2\,1/2)}\}^{(00)}_{00} \\ &= \sum_m (-1)^{1/2-m}a_{(+)m}a_{(-)m}, \end{aligned} \tag{A2}$$

and the commutation relations of the Lie algebra spanned by Q_z, Q_+, and Q_- are these of an angular momentum. Q_z is closely related to the total number operator

$$\mathfrak{N} = \sum_m [a^{\dagger}_{(+)m}a_{(+)m} + a^{\dagger}_{(-)m}a_{(-)m}] \tag{A3}$$

and its eigenvalues are $Q = \frac{1}{2}N + 1$; Q_+ and Q_- clearly do not belong to the Lie algebra of the cooperation group: they do not conserve the total number of atoms. The unitarity conditions of this Lie algebra,

$$\begin{aligned} (Q_z)^{\dagger} &= Q_z, \\ (Q_{\pm})^{\dagger} &= -Q_{\mp}, \end{aligned} \tag{A4}$$

are different from these of an angular momentum and show that it is the Lie algebra of a noncompact SU(1, 1) group; the pseudoangular momentum $\vec{Q}$ defined by Eqs. (A2) is quite analogous to a quasispin operator.[16]

The annihilation and creation operators appear finally to be the components of one triple tensor operator of rank $\frac{1}{2}$ with respect to the three (commuting) angular (or pseudoangular) momenta $\vec{J}$, $\vec{K}$, and $\vec{Q}$; it means that they form a basis for a product of three irreducible representations $D^{1/2} \times D^{1/2} \times \mathfrak{D}(\frac{1}{2})$ of, respectively, the groups $SU^J(2)$, $SU^K(2)$, and SU(1, 1) (the last of the three representations[22] is defined by the same commutation relations as the first two but is nonunitary since SU(1, 1) is noncompact); this can be written

$$\begin{aligned} a^{\dagger}_{(+)m} &= (a)^{(1/2\,1/2\,1/2)}_{m\,1/2\,1/2}, \quad a^{\dagger}_{(-)m} = (a)^{(1/2\,1/2\,1/2)}_{m-1/2\,1/2}, \\ a_{(+)m} &= (-1)^{1/2-m}(a)^{(1/2\,1/2\,1/2)}_{-m-1/2-1/2}, \\ a_{(-)m} &= (-1)^{1/2+m}(a)^{(1/2\,1/2\,1/2)}_{-m\,1/2-1/2}. \end{aligned} \tag{A5}$$

We then define coupled triple-tensor operators

$$X^{(k\ell\kappa)} = \{(a)^{(1/2\,1/2\,1/2)} \cdot (a)^{(1/2\,1/2\,1/2)}\}^{(k\ell\kappa)}; \tag{A6}$$

the commutation relations of these operators show that they span, for $k + l + \kappa$ odd, the Lie algebra C_4 of a noncompact Sp(8) group[23] (with the notation of Ref. 24). The collective symmetric states of all possible numbers N of atoms form the basis of two unitary irreducible infinite dimensional representations of this group, one for N even, one for N odd. The three groups defined above are subgroups of Sp(8) and one can use the chain

$$\mathrm{Sp}(8) \supset \mathrm{SU}^J(2) \times \mathrm{SU}^K(2) \times \mathrm{SU}(1,1). \tag{A7}$$

The three Casimir operators of these subgroups prove to be identical and the reduction of the irreducible representation of Sp(8) corresponding to N even or odd leads to

$$\sum_J D^J \times D^J \times {}^{\dagger}J + 1, \tag{A8}$$

J taking all nonnegative integer values if N is even and all positive half-integer values if N is odd; the notation ${}^{\dagger}Q_0$, due to Miller,[25] represents a unitary infinite-dimensional irreducible representation of SU(1, 1) which is bounded below: the eigenvalues of Q_z for such a representation are

$$Q = Q_0, \quad Q_0 + 1, \quad Q_0 + 2, \ldots, +\infty, \tag{A9}$$

and the eigenvalue of the Casimir operator is $Q_0(Q_0 - 1)$.

According to this reduction, a symmetric state of N atoms can be labeled by J and its projections in the three subgroups, that is M, μ, and $Q=\frac{1}{2}N+1$, with

$$M \text{ or } \mu = J, J-1, J-2, \ldots, -J+1, -J, \qquad Q=\tfrac{1}{2}N+1=J+1, J+2, \ldots, +\infty . \tag{A10}$$

We shall write these states $|D^J M, D^J \mu, \dagger J+1\tfrac{1}{2}N+1)$ or, more simply, $|N\mu JM)$.

B. Evaluation of matrix elements of the collective dipole operator

The components of the energy-decreasing part of the collective dipole operator R_q can be written as components of a triple-tensor operator

$$R_q = (1/\sqrt{3})X^{(1\,1\,1)}_{q-1\,1\,0} . \tag{A11}$$

The matrix elements of these operators can thus be evaluated by using the Wigner-Eckart theorem three times: in $SU^J(2)$, in $SU^K(2)$ and in SU(1, 1). For the last group, since it is noncompact the finite dimensional representation $\mathfrak{D}(1)$ to which operators R_q belong is nonunitary. Clebsch-Gordan coefficients of SU(1, 1) have been studied[26]—and shown to be closely related to these of SU(2)—but in the case of three unitary representations only. In the considered case, however, it is not difficult to obtain by recurrence the general expressions of matrix elements of the type

$$(\dagger Q_0' Q' \,|\, \mathfrak{D}(k) Q'-Q \,|\, \dagger Q_0 Q), \tag{A12}$$

with k non-negative integer (see Ref. 27, for example). It appears from these expressions that the Q and Q' dependence can be formally derived from the expressions of 3-j symbols

$$\begin{pmatrix} J' & k & J \\ -M' & M'-M & M \end{pmatrix}, \tag{A13}$$

with the following correspondence

$$J, J' \to Q_0 - 1, Q_0' - 1, \qquad M, M' \to Q, Q', \tag{A14}$$

however, in the normal 3-j symbol one has $|M| \leq J$, whereas the relationship between Q_0 and Q gives $Q \geq Q_0$: the formal expressions of the 3-j symbols contain square root of some terms which can become negative when correspondence (A14) is made, and one has to replace these terms by their absolute value. Finally the Wigner-Eckart theorem in SU(1, 1) can be written in the following form (which fixes our definition of the reduced-matrix element)

$$(\dagger Q_0' Q' \,|\, \mathfrak{D}(k) Q'-Q \,|\, \dagger Q_0 Q) = (-1)^{Q_0'-Q_0+Q'-Q} \left[\left| \begin{pmatrix} Q_0'-1 & k & Q_0-1 \\ -Q' & Q'-Q & Q \end{pmatrix}^2 \right| \right]^{1/2} \times (\dagger Q_0' \,\|\, \mathfrak{D}(k) \,\|\, \dagger Q_0) \tag{A15}$$

(k being a nonnegative integer). Consequently the matrix elements of R_q can be written

$$\begin{aligned} (N\mu - 1 J' M+q \,|\, R_q \,|\, N\mu JM) &= (-1)^{J'-M-q+J'-\mu+1+J'-J} \begin{pmatrix} J' & 1 & J \\ -M-q & q & M \end{pmatrix} \\ &\times \begin{pmatrix} J' & 1 & J \\ -\mu+1 & -1 & \mu \end{pmatrix} \\ &\times \left[\left| \begin{pmatrix} J' & 1 & J \\ -\frac{1}{2}N-1 & 0 & \frac{1}{2}N+1 \end{pmatrix}^2 \right| \right]^{1/2} (J' \,\|\, X^{(111)} \,\|\, J), \end{aligned} \tag{A16}$$

and one has only to compute the reduced-matrix elements (formulas for 3-j symbols containing one angular momentum equal to 1 are given, for example, in Ref. 28). We have performed this calculation by evaluating the matrix elements of the operator

$$X^{(111)}_{000} = (1/\sqrt{2}) \{ a^\dagger_{(+)1/2} a_{(+)1/2} - a^\dagger_{(+)-1/2} a_{(+)-1/2} - a^\dagger_{(-)1/2} a_{(-)1/2} + a^\dagger_{(-)-1/2} a_{(-)-1/2} \}, \tag{A17}$$

between particular states written both in the occupation number representation $|N_1 N_2 N_3 N_4\rangle$ (see Fig. 2 for the notation) and in the basis $|N\mu JM)$; one has

$$|2JJJJ) \equiv |2J000\rangle, \tag{A18}$$

and the developments of other states $|N\mu JM)$ in the occupation-number representation are obtained by operating J_-, K_-, and Q_+ on both members of Eq. (A18). Finally the reduced-matrix elements appearing in Eq. (A16) are

$$\begin{aligned} (J+1 \,\|\, X^{(111)} \,\|\, J) &= (J \,\|\, X^{(111)} \,\|\, J+1) \\ &= [(2J+1)^2 (2J+2)(2J+3)^2]^{1/2}, \\ (J \,\|\, X^{(111)} \,\|\, J) &= [2(2J)(2J+1)^3(2J+2)]^{1/2}, \end{aligned} \tag{A19}$$

and the reduced (with respect to $\vec{J}$ only) matrix elements of $R^{(1)}$ are obtained by explicitly writing the first two 3-j symbols of Eq. (A16):

$$(\mu J \| R^{(1)} \| \mu+1 J+1) = -\left(\frac{2(J+\mu+1)(J+\mu+2)(\frac{1}{2}N-J)(\frac{1}{2}N+J+2)}{3(2J+2)}\right)^{1/2},$$

$$(\mu J \| R^{(1)} \| \mu+1 J) = (\tfrac{1}{2}N+1)\left(\frac{(J-\mu)(J+\mu+1)(2J+1)}{3J(J+1)}\right)^{1/2}, \tag{A20}$$

$$(\mu J \| R^{(1)} \| \mu+1 J-1) = -\left(\frac{2(J-\mu)(J-\mu-1)(\frac{1}{2}N-J+1)(\frac{1}{2}N+J+1)}{6J}\right)^{1/2}.$$

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[12]These interferences generally appear as interatomic interferences, but they can be intra-atomic too, if a coherent superposition of the upper states has been initially prepared.

[13]Let us remark that this effect does not imply an initial coherent superposition of the two upper states, for a symmetric state $|0N_1N_2\rangle$, the mean value of the coherence between these states is zero.

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[18]For initial conditions (i) the influence of interference effects is quantitatively very small and the π and σ_- intensities are almost the same in the degenerate and "nondegenerate" cases.

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[22]The notation is from A. O. Barut and C. Fronsdal, Proc. R. Soc. A 287, 532 (1965).

[23]Another noncompact structure $R(3,2)$ is found starting from the same set of annihilation and creation operators by, S. Goshen and H. J. Lipkin, in *Spectroscopic and Group-Theoretical Methods in Physics*, edited by F. Bloch *et al.* (North-Holland, Amsterdam, 1968).

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