Kinetic theory of the shear viscosity of a strongly coupled classical one-component plasma

Jean Wallenborn and Marc Baus

Chimie-Physique II (Association Euratom-Etat Belge) C.P. 231, Université Libre de Bruxelles, 1050 Bruxelles, Belgium
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We present an approximation to the linearized collision operator or memory function of the exact kinetic equation obeyed by the correlation function of the phase-space density of a classical one-component plasma. This approximate collision operator generalizes the well known Balescu-Guernsey-Lenard (BGL) operator to finite wavelengths, finite frequencies, and finite coupling constants. It, moreover, satisfies the necessary symmetry relations, leads to appropriate conservation laws, and fulfills its first sum rule exactly. Next we use this operator to compute the shear viscosity η for a series of coupling constants spanning the whole fluid phase. For weak coupling we make contact with the BGL theory, while for strong coupling we confirm, at least qualitatively, the results of Vieillefosse and Hansen, who predicted a minimum in η as a function of temperature. We also demonstrate the important role played by the sum rules in the quantitative evaluation of a transport coefficient such as η.

I. INTRODUCTION

In the past decades, the kinetic theory of plasmas[1] has been concerned mainly with weakly coupled systems where λ, the inverse of the number of particles in a Debye cube, $\lambda = k_D^3/n$, is small ($\lambda \ll 1$). More recently, an impressive number of numerical results, mainly due to Hansen *et al.*,[2] have been obtained for the simplest plasma model, the classical one-component plasma (OCP) with an inert neutralizing background. These results cover the whole range of λ values belonging to the dense phase ($\lambda \gtrsim 1$) and as such have stimulated a number of recent theoretical works on strongly coupled Coulomb systems.[3-5] In a series of papers, Ichimaru *et al.*[3] have considered a number of approximate calculations of the static and dynamic properties of a classical OCP which extend some of the known results to second order in the plasma parameter λ. Gould and Mazenko[4] have presented a fairly general theory for the single-particle motions of the OCP in relation to the molecular-dynamics results for the self-diffusion constant.[2(c)] One of us (M.B.) obtained[5] a number of exact long-wavelength results for a variety of Coulomb systems, substantiating some of the previous numerical and theoretical works.

In this paper, we present a kinetic theory for the collective motions of the OCP and apply it to the calculation of the shear viscosity η of a strongly coupled OCP. We present a simple approximation which qualitatively describes the macroscopic variation of the shear viscosity over the whole fluid phase including the minimum of η as a function of temperature obtained by Vieillefosse and Hansen.[2(f)]

In Sec. II, we have summarized the main kinetic theoretical steps, referring the reader to the existing literature for the details, but clearly exhibiting the intrinsic simplicity of the theory as compared, for instance, to the traditional Chapman-Enskog scheme for weakly coupled systems. Our approximation to the collision operator is presented in Sec. III, while its properties are reviewed in Sec. IV. The present kinetic theory is applied to the study of the shear viscosity in Sec. V. Our numerical results are presented in Sec. VI where we also compare them with the known results. Section VII contains our conclusions.

II. KINETIC THEORY

As our starting point, we consider the exact kinetic equation obeyed[5-7] by the two-point equilibrium correlation function

$$S(\vec{r} - \vec{r}', t - t'; \vec{p}, \vec{p}') = \langle \delta f(\vec{r}, \vec{p}, t) \delta f(\vec{r}', \vec{p}', t') \rangle$$

of the equilibrium fluctuation $\delta f = f - \langle f \rangle$ of the space density

$$f(\vec{r}, \vec{p}, t) = \sum_{j=1}^{N} \delta(\vec{r} - \vec{x}_j(t)) \delta(\vec{p} - \vec{p}_j(t)),$$

$$zS(\vec{k}, z; \vec{p}, \vec{p}') - \int d\vec{p}_1 \Sigma(\vec{k}, z; \vec{p}, \vec{p}_1) S(\vec{k}, z; \vec{p}_1, \vec{p}')$$

$$= iS(\vec{k}, t = 0; \vec{p}, \vec{p}'), \quad (2.1)$$

where we have used a Fourier-Laplace transform defined as follows:

$$S(\vec{k}, z; \vec{p}, \vec{p}')$$

$$= \int_{\Omega} d\vec{r} \int_0^\infty dt\, e^{izt - i\vec{k}\cdot\vec{r}} S(\vec{r}, t; \vec{p}, \vec{p}'); \quad \text{Im}\, z > 0. \quad (2.2)$$

The kinetic equation (2.1) is an exact property of the system and has been the object of a number of recent studies.[5-7] Here, we have adopted the notation of Ref. 5(a) where more detailed expressions of the known initial condition $S(\vec{k}, t = 0; \vec{p}, \vec{p}')$ and the

general memory function $\Sigma(\vec{k}, z; \vec{p}, \vec{p}')$ can be found together with a derivation of Eq. (2.1) from first principles. The general memory function Σ can be split into three parts, $\Sigma = \Sigma^0 + \Sigma^s + \Sigma^c$, the free-streaming term $\Sigma^0(\vec{k}, z; \vec{p}, \vec{p}') = \vec{k} \cdot \vec{v} \delta(\vec{p} - \vec{p}')$, the self-consistent field term

$$\Sigma^s(\vec{k}, z; \vec{p}, \vec{p}') = -\vec{k} \cdot \vec{v} \varphi(\vec{p}) c(k)$$

[where, as in Ref. 5(a), $\varphi(\vec{p})$ is the Maxwellian normalized to one and $c(k)$ the dimensionless direct-correlation function], and the genuine memory function [see Ref. 5(a) for the details of the notation which will not be needed here]

$$\Sigma^c(\vec{k}, z; \vec{p}, \vec{p}') = (\delta f(\vec{k}, \vec{p}) | LQ(z - QLQ)^{-1}QL | \delta f(\vec{k}, \vec{p}'))$$
$$\times [n\varphi(\vec{p}')]^{-1}, \qquad (2.3)$$

which will also be referred to as the nonlocal collision operator, for reasons which we will explain in Sec. III.

The objects containing the macroscopic information about the system are the hydrodynamic correlation functions

$$G_{ij}(\vec{k}, z) = \langle i | S(\vec{k}, z)(n\varphi)^{-1} | j \rangle$$

which obey the hydrodynamic transport equations[5(a)]

$$zG_{ij}(\vec{k}, z) - \sum_{j'=1}^{5} \Omega_{ij'}(\vec{k}, z) G_{j'j}(\vec{k}, z) = iG_{ij}^0(\vec{k}), \qquad (2.4)$$

where i and j run over the five hydrodynamical states (the density $i = n$, the energy $i = \epsilon$, the longitudinal momentum $i = l$, and the two transverse momentum states $i = t_1$ and t_2 jointly designed as $i = \perp$). Here we will be especially interested in the (normalized) transverse momentum correlation $G_\perp(\vec{k}, z) \equiv G_{t_i t_i}(\vec{k}, z)$,

$$G_\perp(\vec{k}, z) = \frac{\beta}{nm} \int d\vec{p}\, d\vec{p}'\, \vec{\epsilon}_{\vec{k}} \cdot \vec{p}\, \vec{\epsilon}_{\vec{k}} \cdot \vec{p}' S(\vec{k}, z; \vec{p}, \vec{p}'), \qquad (2.5)$$

where $\vec{\epsilon}_{\vec{k}}$ denotes a unit vector perpendicular to $\vec{k}$, nm is the mass density, and $\beta^{-1} = k_B T$. Because of the rotational symmetry of the system, $G_\perp$ is completely uncoupled from the remaining correlation functions G_{ij} and Eq. (2.4) yields immediately $G_\perp(\vec{k}, z) = i[z - \Omega_\perp(\vec{k}, z)]^{-1}$. Furthermore, because of momentum conservation, we can write

$$\Omega_\perp(\vec{k}, z) = -ik^2(nm)^{-1}\eta(\vec{k}, z),$$

introducing a nonlocal shear viscosity $\eta(\vec{k}, z)$ from which the ordinary shear viscosity η can be obtained as $\eta \equiv \eta(\vec{k} = 0, z = 0)$. The kinetic-theoretical expression for η obtained in this way is seen[5(a)] to split naturally into two parts, $\eta = \eta_{dir} + \eta_{ind}$,

$$\eta_{dir} = nm \lim_{k \to 0} (1/k^2) \langle \perp | i\Sigma^c(\vec{k}, z = 0) | \perp \rangle, \qquad (2.6a)$$

$$\eta_{ind} = nm \lim_{k \to 0} (1/k^2) \langle \perp | [\Sigma^0 + \Sigma^c(\vec{k}, 0)] \overline{Q} [i\overline{Q} \Sigma^c(\vec{0}, 0)\overline{Q}]^{-1} \overline{Q} [\Sigma^0 + \Sigma^c(\vec{k}, 0)] | \perp \rangle, \qquad (2.6b)$$

which, following Ref. 6, we designate as the direct [Eq. (2.6a)] and indirect [Eq. (2.6b)] part of the shear viscosity η. This separation of the viscosity is not to be confused with the separation of η into a kinetic and potential part, familiar from the Green-Kubo expression for η. Indeed, whereas the kinetic part of η is completely contained in η_{ind}, both η_{ind} and η_{dir} contribute to the potential part of η. To conclude this introductory section we make the important remark[6] that when the collision operator $i\Sigma^c$ is positive definite both η_{dir} and η_{ind} will be separately positive as seen from Eq. (2.6).

III. APPROXIMATE COLLISION OPERATOR

The general expression given for Σ^c in Eq. (2.3) involves N-body constructs like the Liouville operator L and, as such, is convenient when invoking, for instance, the conservation laws of the system, but rather inconvenient as a starting point of an approximate kinetic theory. We prefer instead to start from the following alternative but equivalent form of Σ^c completely expressed in

terms of few-body constructs[7,8]:

$$i\Sigma^c(1, 2; t)n\varphi(\vec{p}_2)$$
$$= \int d1'\, d2'\, L_I(11')L_I(22')C(11', 22', t), \qquad (3.1)$$

where $1 \equiv (\vec{r}_1, \vec{p}_1)$, $d1 \equiv d\vec{r}_1 d\vec{p}_1$, etc., while $L_I(1, 2)$ is the two-body interaction operator

$$L_I(12) = -\frac{\partial}{\partial \vec{r}_1} V(|\vec{r}_1 - \vec{r}_2|) \cdot \left(\frac{\partial}{\partial \vec{p}_1} - \frac{\partial}{\partial \vec{p}_2}\right), \qquad (3.2)$$

V being the Coulomb potential. In Eq. (3.1), $C(11', 22'; t)$ represents a contracted four-point correlation function whose detailed expression will not be needed here but which can be constructed with the aid of the three- and four-point generalizations of the two-point correlation function $S(1, 2; t)$ of Eq. (2.2). The details of the derivation of the alternative expression (3.1) can be found in the literature.[7,8] The basic justification for calling Σ^c either the memory function[6] or the collision operator[5] also stems from Eq. (3.1). Indeed, after substitution of expression (3.1) for Σ^c into Eq. (2.1), we can integrate the momentum variables

by parts and freely transform the function, Eq. (3.1), into an operator acting on the momentum variables. As such, Eq. (3.1) also forms a convenient starting point for a kinetic theory. Indeed, following Mazenko[7] we split the contracted four-point correlation function $C(11', 22'; t)$ into a connected (C_C) and a disconnected (C_D) part $C = C_D + C_C$. The disconnected part C_D can be simply written in terms of the ordinary two point function S:

$$C_D(11', 22'; t) = S(1, 1'; t)S(2, 2'; t)$$
$$+ S(1, 2'; t)S(2, 1'; t), \qquad (3.3)$$

while no such relation exists for C_C. Now substituting $C = C_C + C_D$ into Eq. (3.1) we obtain accordingly $\Sigma^c = \Sigma_C^c + \Sigma_D^c$ each term of Σ^c having a simple physical meaning. Indeed, from Eq. (3.3) we see that Σ_D^c describes a collision process in which the interacting particles propagate, during the collision process, through the medium with their *exact* propagator S, while Σ_C^c, on the contrary, describes a collision process during which the colliding particles stay close together. As indicated previously[5(b),5(c)] $\Sigma_D^c(\vec{k}, z)$ is already an interesting approximation to $\Sigma^c(\vec{k}, z)$ as it generalizes the linearized Balescu-Guernsey-Lenard (BGL) collision operator[1] to finite values of $\vec{k}$, z and the coupling λ. We will come back to this point in Sec. IV.

We are now ready to introduce our basic approximation to the collision operator. First, we will assume that we can neglect the connected contributions to the dynamics, i.e., we take $C(t) \simeq C_D(t)$ in Eq. (3.1). Since such a neglect of the close-collision contributions to $C(t)$ will lead to defects for strong coupling, we try to remedy this by keeping their contribution to the statics and the short-time behavior. Hence, as a second condition, we will require that the approximate collision operator still satisfies its first sum rule,[6] a property which is known to imply[6] that the first three sum rules for S will be satisfied. We now look therefore for an approximation to $\Sigma^c(t)$ which starts from the exact initial value $\Sigma^c(t = 0)$, so as to guarantee its first sum rule, but which neglects the close-collision processes at the later times.

In order to realize this approximation scheme, we first renormalize the collision operator Σ^c along the lines developed in Mazenko's fully re-

normalized kinetic theory (FRKT).[7] In a nutshell, this renormalization procedure consists in replacing $C(t)$ in $\Sigma^c(t)$, Eq. (3.1), by, omitting notational details,

$$C(t) \equiv \tfrac{1}{2}[AA^{-1}C(t)B^{-1}B + BB^{-1}C(t)A^{-1}A], \qquad (3.4)$$

where A and B are some, as yet unspecified, static correlation functions. After substitution of Eq. (3.4) into Eq. (3.1), $L_I A, L_I B$ and $A^{-1}C(t)B^{-1}$, $B^{-1}C(t)A^{-1}$ can be interpreted, respectively, as renormalized vertices and renormalized propagators.

In his FRKT Mazenko did take $A = B = C(t = 0)$ in which case the renormalized propagator becomes $\overline{C}(t) = C^{-1}(0)C(t)C^{-1}(0)$, while the renormalized vertex $L_I C(t = 0)$ has the interesting property of being entirely expressible in terms of static correlation functions eliminating thereby any explicit reference to the bare interaction potential. The kinetic equation (2.1) becomes then an exact relation between dynamic and static correlation functions. Although this choice of A and B is very appealing from an aesthetic point of view, it does lead to some difficulties if we approximate the dynamics $\overline{C}(t)$ by its disconnected part $\overline{C}_D(t) = C_D^{-1}(0)C_D(t)C_D^{-1}(0)$. Indeed the approximation of $\overline{C}(t)$ by $\overline{C}_D(t)$ amounts to replacing $C(t)$ in Eq. (3.1) by $C(0)\overline{C}_D(t)C(0)$. Since $C(0)\overline{C}_D(0)C(0) \neq C(0)$, the approximate collision operator $\Sigma^c(t)$ violates its first sum rule because the exact and approximate expressions for $\Sigma^c(t)$ tend to different initial values as $t \to 0$. In order to restore the sum rule, we will make the less symmetric choice $A = C(0)$ and $B = I$ in Eq. (3.4). In the disconnected approximation, we now replace $C(t)$ in Eq. (3.1) by

$$\tfrac{1}{2}[C(0)C_D^{-1}(0)C_D(t) + C_D(t)C_D^{-1}(0)C(0)],$$

corresponding to the disconnected approximation of the renormalized propagators $C^{-1}(0)C(t) \simeq C_D^{-1}(0)C_D(t)$ and $C(t)C^{-1}(0) \simeq C_D(t)C_D^{-1}(0)$. With this choice of A and B we cannot eliminate the potential completely since now one of the vertices $(L_I B)$ remains unrenormalized $(B = I)$. This choice has, however, the advantage that both the approximate and exact expression of $\Sigma^c(t)$ do have the same initial value $\Sigma^c(t = 0)$ guaranteeing thereby that our approximate $\Sigma^c(t)$ satisfies its first sum rule. Our choice of A and B and the disconnected approximation leads to the following expression for the collisional part of the memory function

$$\Sigma^c(\vec{k}, t; \vec{p}_1, \vec{p}_2)m\varphi(\vec{p}_2) \simeq \frac{i}{2\beta n}\int \frac{d\vec{1}}{8\pi^3}\int d\vec{p}_3 \int d\vec{p}_4 \left[\vec{1}\cdot\vec{\partial}_1\vec{1}\cdot\vec{\partial}_2\, c(\vec{1})V_{\vec{1}}\,\, S(\vec{k}-\vec{1}, t; \vec{p}_1, \vec{p}_2)S(\vec{1}, t; \vec{p}_3, \vec{p}_4)\right.$$

$$+\vec{1}\cdot\vec{\partial}_1(\vec{k}-\vec{1})\cdot\vec{\partial}_2\, c(\vec{1})V_{\vec{k}-\vec{1}}\,\, S(\vec{k}-\vec{1}, t; \vec{p}_1, \vec{p}_3)S(\vec{1}, t; \vec{p}_4, \vec{p}_2)$$

$$+\vec{k}\cdot\vec{\partial}_1\vec{1}\cdot\vec{\partial}_2\, a(\vec{1}, \vec{k}-\vec{1})V_{\vec{1}}\,\, S(\vec{k}-\vec{1}, t; \vec{p}_3, \vec{p}_2)S(\vec{1}, t)\varphi(\vec{p}_1)\varphi(\vec{p}_4)$$

$$\left. +(1 \to 2)\right], \qquad (3.5)$$

where $V_{\vec{k}} = 4\pi e^2/k^2$ and $c(\vec{k})$ are the Fourier transforms of, respectively, the Coulomb potential and the direct correlation function, in the notation of Ref. 5(a). In Eq. (3.5) we have also introduced

$$S(\vec{1},t) = \int d\vec{p} \int d\vec{p}' S(\vec{1},t;\vec{p},\vec{p}')$$

and $a(\vec{1},\vec{k}-\vec{1})$ defined as

$$a(\vec{1},\vec{k}-\vec{1}) = D(\vec{k},-\vec{1})[1 - c(\vec{1})][1 - c(\vec{k}-\vec{1})]$$
$$- c(\vec{1})c(\vec{k}-\vec{1}), \qquad (3.6)$$

where

$$D(\vec{k},-\vec{1}) = n^2[1 - c(\vec{k})]$$
$$\times \int d\vec{r} \int d\vec{r}' B(\vec{r};\vec{r}',\vec{0}) \exp{-i(\vec{k} \cdot \vec{r} - \vec{1} \cdot \vec{r}')}, \qquad (3.7)$$

with n the number density and

$$B(\vec{r}_1;\vec{r}_2,\vec{r}_3) = g_3(\vec{r}_1,\vec{r}_2,\vec{r}_3)$$
$$- g_2(\vec{r}_2 - \vec{r}_3)[g_2(\vec{r}_2 - \vec{r}_1) + g_2(\vec{r}_3 - \vec{r}_1) - 1], \qquad (3.8)$$

g_2 and g_3 being the two- and three-body equilibrium correlation functions.[7]

Substitution of the Laplace transform of expression (3.5) into Eq. (2.1) yields an approximate closed kinetic equation for $S(\vec{k},z;\vec{p},\vec{p}')$ which constitutes the basic kinetic equation of this paper. This equation improves the existing (linearized) kinetic theories in various aspects as can be learned from its properties reviewed in Sec. IV, where we also indicate its relation to the unrenormalized kinetic equation we used previously.[5(b),5(c)]

IV. PROPERTIES OF THE APPROXIMATE COLLISION OPERATOR

The general properties of the *exact* collision operator, Eq. (3.1), have already been reviewed by Forster.[6] It is convenient to express them in terms of the Fourier transform in time of $i\Sigma^c(t)$,

$$\Gamma(\vec{k},\omega;\vec{p},\vec{p}') = i \int_{-\infty}^{+\infty} dt\, e^{i\omega t} \Sigma^c(\vec{k},t;\vec{p},\vec{p}'). \qquad (4.1)$$

For the *approximate* collision operator given by Eq. (3.5), some of the general properties are still trivially satisfied while others need a more careful investigation. We now review the properties of Eq. (3.5)

A. Invariance and symmetry properties

Equation (3.5) leads to a function $\Gamma(\vec{k},\omega;\vec{p}_1,\vec{p}_2)$, defined by Eq. (4.1), which is (i) real, (ii) symmetric under a permutation of $\vec{p}_1$ and $\vec{p}_2$, (iii) even under the transformation of $(\vec{k},\omega)$ into $(-\vec{k},-\omega)$, and (iv) invariant under rotations and reflections. The proof of these properties follows immediately from the fact that similar properties hold[6] for the $S(\vec{k},t;\vec{p}_1,\vec{p}_2)$ functions entering the right-hand side of Eq. (3.5).

B. Collisional invariants

Direct computation of $\int d\vec{p}\, u(\vec{p})\Sigma^c(\vec{k},z;\vec{p},\vec{p}')$ from Eq. (3.5) for $u(\vec{p}) = \{1,\vec{p},\vec{p}^2\}$ shows that 1 is a collisional invariant for all $\vec{k}$ and z values, $\vec{p}$ is a collisional invariant for all z but vanishing $\vec{k}$, while $\vec{p}^2$ is a collisional invariant only when both $\vec{k}$ and z vanish. These properties insure then that the adequate conservation laws of the number, momentum, and *total* energy of the particles will be satisfied.[6]

C. Sum rule

The first sum rule states that

$$\int \frac{d\omega}{2\pi}\, \Gamma(\vec{k},\omega;\vec{p}_1,\vec{p}_2)\varphi(\vec{p}_2)$$
$$= i\Sigma^c(\vec{k},t=0;\vec{p}_1,\vec{p}_2)\varphi(\vec{p}_2), \qquad (4.2)$$

where the initial value $\Sigma^c(t=0)$ can be computed exactly[5(b)] in terms of the equilibrium properties:

$$i\Sigma^c(\vec{k},t=0;\vec{p}_1,\vec{p}_2)\varphi(\vec{p}_2)$$
$$= \frac{1}{\beta} \int \frac{d\vec{1}}{8\pi^3}\, V_{\vec{1}}\pi h(l)\vec{1} \cdot \vec{\partial}_1 \vec{1} \cdot \vec{\partial}_2 [\varphi(\vec{p}_1) - \delta(\vec{p}_1 - \vec{p}_2)]\, \varphi(\vec{p}_2)$$
$$+ \left(\frac{c(k)}{\beta^2} + \frac{n}{\beta} V_{\vec{k}}\right)\vec{k} \cdot \vec{\partial}_1\vec{k} \cdot \vec{\partial}_2\varphi(\vec{p}_1)\varphi(\vec{p}_2), \qquad (4.3)$$

where $h(k)$ and $c(k)$ are related to the static structure factor[5(a)] by $S(k) = 1 + h(k) = [1 - c(k)]^{-1}$. By direct computation one can show that Eq. (3.5) satisfies this first sum rule exactly, i.e., computing the right-hand side of Eq. (4.2) from the approximation, Eq. (3.5), one finds the right-hand side of Eq. (4.3). As a matter of fact, this property is guaranteed here by the very construction of the approximation given by Eq. (3.5). The first sum rule for $\Sigma^c(t)$, Eqs. (4.2) and (4.3), guarantees then, in turn, that the first three sum rules for $S(\vec{k},t;\vec{p},\vec{p}')$ will be satisfied and hence that the statics $S(k)$ and the short-time behavior of $S(\vec{k},t)$ remain exact.

It is important to note that if we had used Mazenko's symmetric vertex renormalization, together with the disconnected approximation for $\overline{C}$, the sum rule Eqs. (4.2) and (4.3) would have been violated (see for instance Eqs. (3.11) and (3.12) of

Ref. 4(b)]. For problems where the short-time dynamics is important, this will constitute a major defect.

D. Positivity

The system's stability is related to the positiveness of $\Gamma(\vec{k}, \omega; \vec{p}_1, \vec{p}_2)$ defined in Eq. (4.1). Therefore we would like, for an arbitrary function $f(\vec{p})$, that

$$\langle f | \Gamma(\vec{k}, \omega) | f \rangle$$

$$= \int d\vec{p}_1 \int d\vec{p}_2 f(\vec{p}_1) \Gamma(\vec{k}, \omega; \vec{p}_1, \vec{p}_2) \varphi(\vec{p}_2) f^*(\vec{p}_2) \geq 0,$$

(4.4)

would still hold for $\Gamma(\vec{k}, \omega)$ as obtained from Eqs. (3.5) and (4.1). We have not succeeded in treating the general case, Eq. (4.4), but instead have considered the local and Markovian case $\vec{k} = 0, \omega = 0$ for which we obtain from Eq. (3.5)

$$\langle f | \Gamma(\vec{0}, 0) | f \rangle = \int \frac{d\omega}{2\pi} \int \frac{d\vec{1}}{8\pi^3} V_l \frac{-c(l)}{n\beta} [\langle F_{\vec{1}} | \tilde{S}(\vec{1}, \omega) | F_{\vec{1}} \rangle \langle 1 | \tilde{S}(\vec{1}, \omega) | 1 \rangle - \langle F_{\vec{1}} | \tilde{S}(\vec{1}, \omega) | 1 \rangle \langle 1 | \tilde{S}(\vec{1}, \omega) | F_{\vec{1}} \rangle],$$

(4.5)

where we have put $F_{\vec{1}} = \vec{1} \cdot (\partial/\partial\vec{p}) f(\vec{p})$,

$$\tilde{S}(\vec{1}, \omega; \vec{p}_1, \vec{p}_2) = S(\vec{1}, \omega; \vec{p}_1, \vec{p}_2) [n\varphi(\vec{p}_2)]^{-1},$$

$S(\vec{1}, \omega; \vec{p}_1, \vec{p}_2)$ being the Fourier transform in time of $S(\vec{1}, t; \vec{p}_1, \vec{p}_2)$. Since $\langle f | \tilde{S}(\vec{1}, \omega) | f \rangle \geq 0$, for all f, Schwartz's inequality for the scalar product (4.4) ensures that the factor between brackets in the right-hand side of Eq. (4.5) is positive. Hence the condition

$$\langle f | \Gamma(\vec{0}, 0) | f \rangle \geq 0, \qquad (4.6)$$

for all f, will be satisfied as long as the direct correlation function $c(l)$ does not change sign, or else does so in a region of l values which contributes little to Eq. (4.5). Note that both for weak coupling and for small l, $c(l)$ tends to $-n\beta V_l$, i.e., $c(l) < 0$. When one of these conditions on $c(l)$ is not satisfied no general statement about the validity of Eq. (4.6) can be made and a fortiori, about Eq. (4.4).

E. Vertex and propagator renormalization

The relation of Eq. (3.5) to our previous unrenormalized kinetic theory[5(b),5(c)] is quite straightforward. Indeed, without the vertex renormalization the $c(l)$ factor in the first two terms on the right-hand side of Eq. (3.5) would return to their bare-potential values: $c(l) \rightarrow -n\beta V_l$. Moreover, without the propagator renormalization the third term in the right-hand side of Eq. (3.5), which describes the necessary readjustment of the initial condition, will also be absent and Eq. (3.5) is seen

to reduce then to the unrenormalized collision operator of Eq. (3.1a) of Ref. 5(b).

In Ref. 4, Gould and Mazenko consider the single-particle motions of a classical one-component plasma and use the symmetric vertex renormalization and the disconnected approximation. It is shown that their approximation leads to incorrect short-time dynamics. If we adapt the approach of Ref. 4 to the present case, the third term in the right-hand side of Eq. (3.5) would be slightly modified, while the bare potential V_l in the first two terms of the right-hand side of Eq. (3.5) would be renormalized to $-c(l)/n\beta$.

F. Relation to the BGL collision operator

The linearized BGL theory is concerned[1] with the local Markovian collision operator $\Sigma^c(\vec{k} = 0, z = 0)$ for weakly coupled systems. Our approximate collision operator of Eq. (3.5) is seen to depend on the one-particle propagator $S(\vec{k}, t; \vec{p}, \vec{p}')$ which is given self-consistently in terms of the collision operator by Eq. (2.1). In order to obtain Σ^c for a weakly coupled system, we first compute Σ^c with the aid of the expression for S obtained from the collisionless kinetic equation, i.e., Eq. (2.1) with the collision term Σ^c deleted from Σ. This collisionless kinetic equation is nothing but a linearized Vlassov equation including, however, the exact statics. Substituting this result for $S(\vec{k}, t; \vec{p}, \vec{p}')$ into Eq. (3.5) we obtain in the limit of vanishing $\vec{k}$ and z

$$i\Sigma^c(\vec{k} = 0, z = 0; \vec{p}_1, \vec{p}_2) n\varphi(\vec{p}_2) = \frac{n}{\beta} \int \frac{d\vec{1}}{8\pi^3} \vec{1} \cdot \vec{\partial}_1 \vec{1} \cdot \vec{\partial}_2 \frac{V_l c(l)}{|\epsilon_{\vec{1}}(\vec{1} \cdot \vec{v}_1)|^2}$$

$$\times \left[\pi\delta(\vec{1} \cdot \vec{v}_1 - \vec{1} \cdot \vec{v}_2) \varphi(\vec{p}_1) \varphi(\vec{p}_2) - \delta(\vec{p}_1 - \vec{p}_2) \varphi(\vec{p}_1) \int d\vec{p}_3 \pi\delta(\vec{1} \cdot \vec{v}_1 - \vec{1} \cdot \vec{v}_3) \varphi(\vec{p}_3) \right],$$

(4.7)

where the dielectric constant $\epsilon_{\vec{k}}^+(z)$ is defined as

$$\epsilon_{\vec{k}}^+(z) = 1 - \frac{c(k)}{\beta} \int d\vec{p} \, \frac{\vec{k} \cdot \vec{\delta\varphi}(\vec{p})}{z - \vec{k} \cdot \vec{v}} \, . \qquad (4.8)$$

If we now also approximate the *static* properties by their weak-coupling limiting value, we have to replace the direct correlation function $c(k)$ by its weak-coupling Debye-Huckel expression $c(k) \simeq c_{\mathrm{DH}}(k) = -n\beta V_k$. Substituting this result into Eqs. (4.7) and (4.8), the collision operator of Eq. (4.7) is seen to reduce exactly to the linearized BGL operator. Hence it is justified to call Eq. (3.5) a generalization of the BGL theory to finite values of $\vec{k}, z$ as well as to strongly coupled situations.

Before closing this section let us indicate an interesting property of Eq. (4.7). As stated above, in the BGL theory one takes $c(l) \simeq -\beta n V_l$, and hence the l integral of Eq. (4.7) is seen to diverge for large l values. From the computer results of Hansen et al.[2(a)] we know, however, that

$$c(r = 0) = \int \frac{d\vec{l}}{8\pi^3} c(l)$$

is finite and hence $c(l)$ has to decrease at least as $l^{-3-\alpha}$ ($\alpha > 0$) for large l, in which case the l integral of Eq. (4.7) is seen to be convergent, repairing one of the major defects of the BGL theory.[1]

V. SHEAR VISCOSITY

The approximate collision operator given in Eq. (3.5) will now be used to evaluate the shear viscosity defined by Eq. (2.6). Very fortunately, the third term in the right-hand side of Eq. (3.5) involving the equilibrium triple correlations does not contribute to the viscosity for symmetry reasons. Let us first introduce a dimensionless viscosity $\eta^* = \eta/\eta_0$ with $\eta_0 = nm\omega_p k_0^{-2}$, where k_0 is a characteristic wave vector which we will take equal to the Debye wave vector k_D when using $\lambda = k_D^3/n$ as coupling constant, whereas it is more convenient to put k_0 equal to $a^{-1} = (\frac{4}{3}\pi n)^{1/3}$ when comparing with the results of Hansen et al.[2] who use $\Gamma = e^2\beta a^{-1}$ as coupling parameter. We recall that for a OCP of number density n, charge density en, mass density mn, and temperature $T = (\beta k_B)^{-1}$ we have $\omega_p^2 = 4\pi n e^2/m$ and $k_D^2 = 4\pi e^2 n\beta$, whereas the relation between the two coupling parameters λ and Γ reads

$$\Gamma = \frac{1}{3}\left(\frac{3\lambda}{4\pi}\right)^{2/3}, \quad \lambda = \frac{4}{3}\pi(3\Gamma)^{3/2}. \qquad (5.1)$$

Notice also that $ak_D = (3\Gamma)^{1/2}$. Substituting now Eq. (3.5) into the right-hand side of Eq. (2.6a) we obtain for the direct part of η^*

$$\eta_{\mathrm{dir}}^* \equiv \eta_{\mathrm{dir}}/\eta_0 = \lambda I_1(\lambda), \qquad (5.2)$$

with the dimensionless integral I_1 defined as

$$I_1 = \left(\frac{k_0}{k_D}\right)^3 \frac{\omega_p}{k_0^3} \int \frac{d\vec{l}}{8\pi^3} \int_0^\infty dt \, c(l) G_{nn}(l,t)(\vec{\epsilon} \cdot \vec{1})^2 \left[\frac{1}{2} G_{nn}(l,t)\left(\hat{k} \cdot \frac{\partial}{\partial \vec{1}}\right)^2 \bar{V}_l + \left(\hat{k} \cdot \frac{\partial}{\partial \vec{1}}\bar{V}_l\right)\left(\hat{k} \cdot \frac{\partial}{\partial \vec{1}}G_{nn}(l,t)\right)\right], \qquad (5.3)$$

where $\hat{k} = \vec{k}/|\vec{k}|$, while $\vec{\epsilon}$ denotes a unit vector $\vec{\epsilon}^2 = 1$ orthogonal to $\vec{k}$. In Eq. (5.3), we have put moreover $\bar{V}_l = k_0^2/l^2$ and $G_{nn}(l,t) = (1/n) \int d\vec{p} \int d\vec{p}' S(\vec{l}, \vec{t}; \vec{p}, \vec{p}')$ which denotes the dimensionless density-density correlation function. Notice that in Eq. (5.3) and this section we will leave the scale factor k_0 undetermined, a procedure which allows an easy translation between the $\lambda - \Gamma$ and $k_D - a^{-1}$ languages.

In order to compute the indirect part of the shear viscosity η_{ind}, defined by Eq. (2.6b), we have to invert $\Sigma^c(\vec{0}, 0)$ in the subspace orthogonal to the hydrodynamical one. If we expand $\Sigma^c(\vec{0}, 0; \vec{p}, \vec{p}')$ into a Hermite polynomial basis with respect to $\vec{p}$ and $\vec{p}'$, the first nonzero contribution to $\bar{Q}\Sigma^c(\vec{0}; 0)\bar{Q}$ will come from the pressure-tensor states ($\sim p_i p_j$). If we stop the expansion of $\bar{Q}\Sigma^c\bar{Q}$ at this first nonvanishing contribution, which is equivalent to a one-Sonine-polynomial expansion, one can easily invert $\bar{Q}\Sigma^c\bar{Q}$ and compute Eq. (2.6b) to be

$$\eta_{\mathrm{ind}} = nm \lim_{k \to 0} k^{-2}\langle \perp|\Sigma^0 + \Sigma^c(\vec{k}, 0)|l\perp\rangle$$
$$\times \langle l\perp|i\Sigma^c(\vec{0}, 0)|\perp l\rangle^{-1}$$
$$\times \langle \perp l|\Sigma^0 + \Sigma^c(\vec{k}, 0)|\perp\rangle, \qquad (5.4)$$

where $|l\perp\rangle \equiv |\perp l\rangle = \hat{k} \cdot \vec{p}\,\vec{\epsilon} \cdot \vec{p}(m\beta^{-1}n^{1/2})^{-1}$ is the only surviving component of the pressure states. We proceed now by evaluating the right-hand side of Eq. (5.4) with the aid of the approximate collision operator (3.5) and obtain for the indirect part of η^*

$$\eta_{\mathrm{ind}}^* \equiv \frac{\eta_{\mathrm{ind}}}{\eta_0} = \frac{[1 + \lambda I_2(\lambda)]^2}{\lambda I_3(\lambda)}, \qquad (5.5)$$

where we have introduced two more dimensionless integrals

$$I_2 = \frac{k_0}{k_D} k_0^{-3} \int \frac{d\vec{l}}{8\pi^3} \int_0^\infty dt \, (\hat{l} \cdot \hat{k})^2(\hat{l} \cdot \vec{\epsilon})^2 G_{nn}(l,t)$$
$$\times [\partial_t G_{nn}(l,t)]l\frac{\partial}{\partial l}[\bar{V}_l c(l)], \qquad (5.6)$$

$$I_3 = -2 \frac{k_0}{k_D} k_0^{-3} \omega_p \int \frac{d\vec{l}}{8\pi^3} \int_0^\infty dt\, c(l)(\hat{l} \cdot \hat{k})^2$$

$$\times G_{nn}(l,t) G_\perp(l,t). \quad (5.7)$$

In Eq. (5.6), $G_\perp(l,t)$ denotes the dimensionless transverse momentum correlation function

$$G_\perp(l,t) = \beta(nm)^{-1} \int d\vec{p} \int d\vec{p}'\, \vec{\epsilon} \cdot \vec{p}\, \vec{\epsilon} \cdot \vec{p}'\, S(\vec{l},t;\vec{p},\vec{p}').$$

The evaluation of the total (reduced) viscosity $\eta^* = \eta^*_{\text{dir}} + \eta^*_{\text{ind}}$ is now seen to depend on the value of the three integrals I_1, I_2, and I_3 defined, respectively, by Eqs. (5.3), (5.6), and (5.7). Each of these integrals contains information about the static and dynamic properties of the system. In order to separate them completely, we put $G_{nn}(l,t) = S(l)\overline{G}_{nn}(l,t)$ with $G_{nn}(l,t=0) = S(l) = 1 + h(l) = [1 - c(l)]^{-1}$ denoting the static structure factor, whereas $G_\perp(l,t) = \overline{G}_\perp(l,t)$, because $G_\perp(l,t=0) = 1$. Using these forms in the expressions of I_1, I_2, and I_3, we see that all the information about the statics can be expressed in terms of the static structure factor $S(l)$, whereas all the information about the dynamics is contained now in the two correlation functions $\overline{G}_{nn}(l,t)$ and $\overline{G}_\perp(l,t)$. Both functions $\overline{G}_{nn}(l,t)$ and $\overline{G}_\perp(l,t)$ have the same limiting values: $\overline{G}_{nn}(l,t=0) = \overline{G}_\perp(l,t=0) = 1$ and $\overline{G}_{nn}(l,t=\infty) = \overline{G}_\perp(l,t=\infty) = 0$. With the help of this information we can perform the time integration in I_2, Eq. (5.6), with the exact result

$$I_2 = \frac{1}{60\pi^2} \frac{k_0}{k_D} \int_0^\infty dx\, \{2[S^2(x) - 1] + [1 - S(x)]\}, \quad (5.8)$$

where we have put $x = l/k_0$. Hence, from Eq. (5.8), I_2 is seen to be independent of the dynamics. For the evaluation of I_1 and I_3, we need, on the contrary, to specify the dynamics through the time integrals of $\overline{G}^2_{nn}(l,t)$ and $G_{nn}(l,t)\overline{G}_\perp(l,t)$. The precise way in which these integrands drop from one (at $t=0$) to zero (as $t \to \infty$) will depend on the value of the coupling constant λ. The only available expressions for $\overline{G}_{nn}(l,t)$ and $\overline{G}_\perp(l,t)$ concern their (a) free-particle limit, (b) random-phase approximation, and (c) hydrodynamical limit.[5(a)] A closer inspection of I_1 and I_3 reveals that the domain of the l integrals corresponding to small and very large values of l contributes little. Hence the major contribution to I_1 and I_3 comes from the intermediate l small t region because for later times the integrands start to oscillate rapidly. Therefore our basic approximation for I_1 and I_3 will consist in taking the simple free-particle (FP) approximation for the dynamics ($\omega_p = k_D v_0$)

$$\overline{G}^{\text{FP}}_{nn}(l,t) = e^{-(1/2)(lv_0 t)^2}, \quad (5.9a)$$

$$\overline{G}^{\text{FP}}_\perp(l,t) = e^{-(1/2)(lv_0 t)^2}, \quad (5.9b)$$

while keeping the statics $S(l)$ exact.

Because of the long range of the Coulomb potential, the static properties become of overwhelming importance and, hence, we are forced to use the numerical values of $S(l)$ as input. As far as the dynamics are concerned, our approximation treats them correctly for short times especially because as a result of the sum rule (4.2) we start from the exact initial condition. Hence, the major uncertainty of our calculation resulting from the use of Eq. (5.9) stems from the neglect of the oscillatory tails in the time integrals of Eqs. (5.3) and (5.7). Note, however, that in the intermediate l region, which contribution dominates, these oscillations are strongly damped, as are the plasma oscillations which give rise to them. This remains true for all values of λ, the coupling constant. Hence, we expect our approximation to reproduce, at least qualitatively, the behavior of η as a function of λ over the whole range of λ values belonging to the fluid phase.

If we now proceed to compute I_1 and I_3 with the help of Eq. (5.9) we obtain from Eq. (5.3)

$$I_1^{\text{FP}} = \frac{1}{60\pi^{3/2}} \left(\frac{k_0}{k_D}\right)^2, \quad (5.10)$$

and from Eq. (5.7)

$$I_3^{\text{FP}} = \frac{1}{10\pi^{3/2}} \int_0^\infty dx\, x\, [1 - S(x)]. \quad (5.11)$$

To obtain Eq. (5.10), we have used the general properties $S(l=0) = 0$ and $S(l=\infty) = 1$. Most remarkably, I_1^{FP} is independent of the statics. To proceed with I_2 and I_3^{FP} we have to use a numerical integration, the results of which are presented in Sec. VI.

VI. NUMERICAL RESULTS

In order to compare our results with those of Hansen et al.,[2] we take the scale factor k_0 equal to $a^{-1} = (\frac{4}{3}\pi n)^{1/3}$ and measure the coupling with the aid of Γ of Eq. (5.1). To proceed with the evaluation of I_2, Eq. (5.8) and I_3^{FP}, Eq. (5.11), and hence of η^* through Eq. (5.2) and Eq. (5.5), we need the values of $S(k)$ which we will take from the literature.

A. Large Γ: $2 \leqslant \Gamma \leqslant 160$

For the Γ values in the range $2 \leqslant \Gamma \leqslant 160$ we have used the Monte Carlo results of Hansen et al.[2(a)2(f)] for $S(x)$ with $x = ka \geqslant 1$. For $x \leqslant 1$ we have used the analytic result[2(e),5(b)]

$$S(x) = \left(\frac{3\Gamma}{x^2} + \frac{\chi^0_T}{\chi_T}\right)^{-1}, \quad \text{for } x \leqslant 1, \quad (6.1)$$

where χ^0_T/χ_T is the ratio of the isothermal com-

pressibilities of, respectively, the perfect gas and the OCP. We have computed χ_T from the equation of state given by Galam and Hansen.[2(f)] The right-hand side of Eqs. (5.8) and (5.11) have then been computed analytically in the range $0 \leqslant x \leqslant 1$ and numerically in the range $1 \leqslant x \leqslant \infty$. To test the precision of the numerical integration, we have compared our results for the integral $\int_0^\infty dx\,[1 -S(x)]$ which appears in I_2, Eq. (5.8), with the Monte Carlo results for the correlation energy density[2(a)] to which this integral is proportional. We found the error to be less than 1% for all the values of Γ listed in Table I. We also found that the numerical values obtained in this way can be fitted to within a few percent to those obtained for η^* (but not for I_2 and I_3^{FP} separately) from the following power laws:

$$I_2 = (1/60\pi^2)(0.49-2.23\Gamma^{-1/3})\,, \qquad (6.2a)$$

$$I_3^{FP} = (1/10\pi^{3/2})(2.41\Gamma^{1/9})\,, \qquad (6.2b)$$

whereas Eqs. (5)–(10) yields exactly

$$I_1^{FP} = (1/60\pi^{3/2})(1/3\Gamma)\,. \qquad (6.3)$$

From Eqs. (6.2), (6.3), (5.1), (5.2), and (5.5) we see that both η_{dir}^* and η_{ind}^* are separately positive for all Γ values as they should be[6] and that, as expected,[5(d)] η_{dir}^* increases while η_{ind}^* decreases with increasing Γ. Hence $\eta^* = \eta_{dir}^* + \eta_{ind}^*$ is bound to exhibit a minimum as a function of Γ. The precise values of η^* are given in Table I, whereas the variation of η^* with Γ over the whole fluid phase $2 \leqslant \Gamma \leqslant 160$ is pictured in Fig. 1. We have also compared our results with those of Hansen et al.[2(e)] The overall agreement is rather good, whereas the existence of a pronounced minimum is also confirmed here. The position of the minimum is, however, rather sensitive to the various approximations. We have located the minimum at $\Gamma \simeq 8$ corresponding to $\eta_{min}^* = 0.070$, whereas

Hansen et al. found the minimum at $\Gamma \sim 20$ with a slightly higher value of η_{min}^*. For $\Gamma = 152.4$ an independent estimate of η^*, based on a molecular-dynamics[2(c)] result, did yield $\eta^* \simeq 0.25$, as indicated by the isolated point in Fig. 1.

B. Intermediate Γ: $0.1 \leqslant \Gamma \leqslant 2$

In this domain of Γ values no Monte Carlo results for $S(k)$ are available. Here, we have used the values obtained for $S(k)$ by Hansen[9] through numerical solution of the hypernetted-chain equations[10] for $0.1 \leqslant \Gamma \leqslant 1$. In this region, the fits provided by Eq. (6.2) are no longer valid while we do not possess as yet sufficient Γ values to present fits adequate for the intermediate Γ values.

It is interesting to note that in this region we are not yet allowed to use the Debye-Huckel approximation for the statics or the BGL approximation for the dynamics. Indeed, the use of the Debye-Huckel approximation for $S(k)$ leads, for instance, to correlation energies which are too large by a factor of 4 at, for example, $\Gamma = 2$. As far as the dynamics is concerned, let us recall[11] that η^*, as computed from the Landau kinetic equation [with cutoffs taken as usual at $k = k_D$ and $k = (e^2\beta)^{-1}$] would read

$$\eta_L^* = \frac{1}{3\Gamma}\,\frac{5\pi^{1/2}}{2}\,(\epsilon\,\ln\epsilon^{-1})^{-1}\,, \qquad (6.4)$$

where $\epsilon = \lambda/4\pi$ can be related to Γ by Eq. (5.1). Similarly, the value of η^* as computed from the
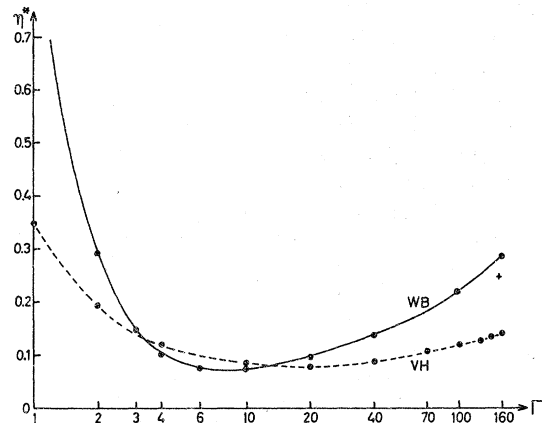
TABLE I. Reduced shear viscosity $\eta^* = \eta/nm\omega_p a^2$ for $2 \leqslant \Gamma \leqslant 160$ as calculated from the present kinetic theory. The results obtained by Vieillefosse and Hansen [Ref. 2(e)] for η^* are denoted by η_{VH}^* and compared to the kinetics-theoretic results.

Γ	η^*	η_{VH}^*	η^*/η_{VH}^*
2	0.29	0.19	1.5
3	0.15	···	···
4	0.10	0.12	0.8
6	0.074	···	···
10	0.072	0.083	0.9
20	0.097	0.078	1.2
40	0.14	0.088	1.6
100	0.22	0.12	1.8
160	0.29	0.14	2.1

FIG. 1. Reduced shear viscosity $\eta^* = \eta/nm\omega_p a^2$ as a function of Γ for $2 \leqslant \Gamma \leqslant 160$. The solid line (WB) denotes the present kinetic-theory results. The dashed line (VH) corresponds to the results of Vieillefosse and Hansen [Ref. 2(e)], while the isolated point (cross) at $\Gamma = 152.4$ is taken from Hansen, McDonald, and Pollock [Ref. 2(c)].

BGL equation [with a cutoff at $k = (e^2\beta)^{-1}$] is[11]

$$\eta^*_{\text{BGL}} = \eta^*_L\left(1 + \frac{0.346}{\ln\epsilon^{-1}}\right)^{-1} . \qquad (6.5)$$

These approximations of η^* are still inadequate for the intermediate Γ values as can be seen from Table II. At the lowest possible value $\Gamma = 0.1$, these weak-coupling theories lead to $\eta^*_L = 93$ and $\eta^*_{\text{BGL}} = 83$, whereas we obtain $\eta^* = 86$. The agreement here is already rather good in view of the cutoff uncertainty in Eqs. (6.4) and (6.5) and indicates that we are approaching the region where the weak-coupling theories become adequate. Our values of η^* for the intermediate Γ can be found in Table II, whereas Fig. 2 indicates the transition from weak to strong coupling. In the weak-coupling region $\Gamma \leqslant 0.1$, no values of $S(k)$ are available but the agreement found at $\Gamma = 0.1$ seems to indicate that for $\Gamma \leqslant 0.1$ Eqs. (6.4) and (6.5) should yield reliable values of η.

C. Importance of first sum rule

At various stages we have insisted on the importance of using an approximate collision operator $\Sigma^c(\vec{k}, t)$ which still starts from the exact initial condition $\Sigma^c(\vec{k}, t = 0)$ given in Eq. (4.3). The shear viscosity is determined by the zero-frequency collision operator $\Sigma^c(\vec{k}, z = 0)$ which is the time integral of $\Sigma^c(\vec{k}, t)$; $\Sigma^c(\vec{k}, z = 0) = \int_0^\infty dt\, \Sigma^c(\vec{k}, t)$. Because of the damped plasma oscillations which show up in $\Sigma^c(\vec{k}, t)$ at later times, the value of $\int_0^\infty dt\, \Sigma^c(\vec{k}, t)$ is very sensitive to the short-time behavior of $\Sigma^c(\vec{k}, t)$ and hence to its initial condition $\Sigma^c(\vec{k}, t = 0)$. As our approximate collision operator of Eq. (3.5) does satisfy the sum rule, Eqs. (4.2) and (4.3), its initial value is exact. As discussed in Sec. III this would not have been the case if we had used Mazenko's symmetric renormalization. To estimate the quantitative importance of the sum rule we have compared the kinetic shear viscosity $\eta^*_K = (\lambda I_3)^{-1}$ as computed from both theories. In the present theory, the kinetic shear viscosity η^*_K is determined by I_3^{FP} as given in Eq. (5.11). With Mazenko's renor-

malization Eq. (5.11) is replaced by

$$\hat{I}_3^{\text{FP}} = \frac{1}{10\pi^{3/2}}\left(\frac{k_0}{k_D}\right)^2 \int_0^\infty dx\, x^3 \frac{[1 - S(x)]^2}{S(x)} , \qquad (6.6)$$

where $x = k/k_0$. The ratio of $\hat{\eta}_K$, computed with the aid of Eq. (6.6), to η_K, computed with Eq. (5.11), is given in Table III. We see that $\hat{\eta}_K$ systematically exceeds η_K by a factor of about $\frac{3}{2}$. This ratio is of the same order of magnitude as the one between the values of the self-diffusion constant computed by Gould and Mazenko to the molecular-dynamics values. We conjecture that a modification of their calculation that retains the exact first sum rule for Σ^c might bring their calculated values closer to the molecular-dynamics ones. Note also that at the edge of the weak-coupling region $\Gamma = 0.1$, the ratio $\hat{\eta}_K/\eta_K = 1.35$ is still significant, whereas our value of η_K already compares favorably with the BGL value. In the weak-coupling region $\Gamma \ll 1$, all theories should nevertheless become equivalent.

VII. CONCLUSIONS

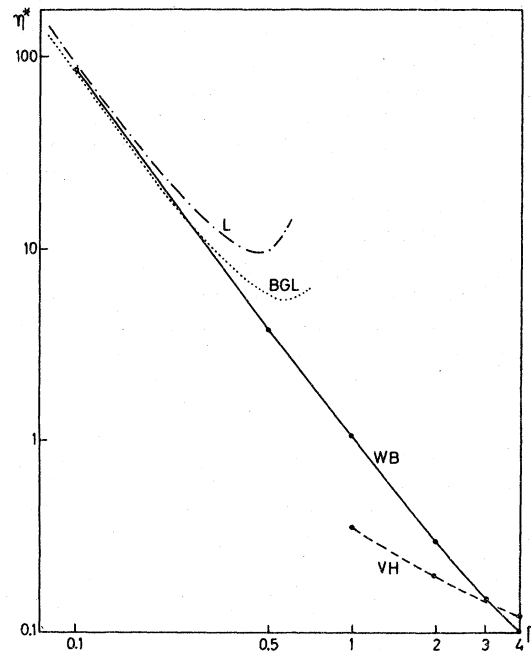We have set up a microscopic theory capable of studying the transport properties of strongly

FIG. 2. Reduced shear viscosity $\eta^* = \eta/nm\omega_p a^2$ as a function of Γ in the transition region from weak to strong coupling $0.1 \leqslant \Gamma \leqslant 4$. The solid line (WB) denotes the present kinetic-theory results. The dashed-dotted line (L) and the dotted line (BGL) represent the weak-coupling results given by the Landau and BGL expressions of, respectively, Eqs. (6.4) and (6.5). The dashed line (VH) corresponds to the strong-coupling results of Vieillefosse and Hansen [Ref. 2(e)].

TABLE II. Reduced shear viscosity $\eta^* = \eta/nm\omega_p a^2$ for $0.1 \leqslant \Gamma \leqslant 2$ as calculated from the present kinetic theory and compared with the results from the Landau (η^*_L) and BGL theory (η^*_{BGL}) and from Ref. 2(e) (η^*_{VH}).

Γ	η^*	η^*_L	η^*_{BGL}	η^*_{VH}
0.1	86	93	83	$\cdots$
0.5	3.7	9.8	5.8	$\cdots$
1	1.0	$\cdots$	$\cdots$	0.35
2	0.29	$\cdots$	$\cdots$	0.19

coupled plasmas when the equilibrium properties are used as input. To this end, we have used a general kinetic theoretical setup, summarized in Sec. II, which follows closely the previous studies of Forster[6] and Mazenko[7] as adapted by one of us[5(a)] to the case of Coulomb systems. It provides an elegant scheme for the study of the equilibrium fluctutations and transport properties of the classical OCP. In order to perform explicit calculations, an approximate expression of the collision operator $\Sigma^c(\vec{k}, z)$ is, however, needed. Such an approximate collision operator was derived in Sec. III on the basis of a slightly modified version of Mazenko's FRKT.[7] The result, Eq. (3.5), represents a rather straightforward generalization of the BGL theory. It does, however, satisfy a number of additional requirements, correcting thereby a number of well-known defects of the BGL theory. Amongst them we cite the fact that no divergent integrals appear, that both the wavelength and frequency dependence of the collision operator is retained, and that the collision process is reasonably well described even for finite coupling constants $\lambda = k_D^3/n$. For instance, at $t = 0$ the collision process is described exactly to all orders in λ.

To test our approximation, we have computed the shear viscosity of the OCP over a wide range of λ values covering the fluid phase. Here, besides the standard one-Sonine-polynomial approximation for computing the inverse of the collision operator, other approximations have been necessary. We took the point of view that the behavior of η as a function of λ could be rendered, at least qualitatively, for all λ, when keeping the static properties exact while using a rather naive approximation for the dynamics. This point of view is based on the observation that, in the presence of the long-ranged Coulomb forces, the static properties become, indeed, of overwhelming importance, while, on the other hand, the dynamics will always lead to compensating plasma oscillations which emphasize therefore the short-time dynamics which, in turn, is treated exactly within the present approximation.

From Table I and Fig. 1 it can be seen that the results of this rather simple kinetic theory, without adjustable parameters, compare at least qualitatively with the results obtained by Vieillefosse and Hansen[2(e)] from the formalism of generalized hydrodynamics with a Gaussian approximation for the memory function of the shear viscosity and fitting the parameters of the Gaussian with the aid of sum rules. These authors also used the superposition approximation for the triplet equilibrium correlation function which appeared in their second sum rule. For smaller

values of the coupling than those considered by the previous authors we have obtained, moreover, the transition from the strong- to the weak-coupling regime which in turn can be adequately described by the BGL theory as seen from Table II and Fig. 2. Finally the importance of keeping the description of the short-time collision process exact for all λ values is shown quantitatively in Table III.

As a by-product, we also obtained a simple explanation[5(d)] of the appearance of a minimum in η^* as a function of λ, one of the most striking features of the results of Vieillefosse and Hansen.[2(e)] Indeed, within the present kinetic theory, the shear viscosity η appears as the sum of two partial viscosities η_{dir} and η_{ind}, each of which is separately positive.[6] Note that our approximations do preserve this property. On dimensional grounds, one expects η_{dir}, which is proportional to the collision operator, to increase with λ, whereas η_{ind}, which is inversely proportional to the collision operator, to decrease with λ and hence their sum is bound to exhibit a minimum. This property, which is certainly true for small λ, is seen here to persist for large λ. Indeed, from Eqs. (5.2) and (6.3) we see that η_{dir}^* increases as $\Gamma^{1/2}$, whereas Eqs. (5.5) and (6.2) indicate that η_{ind}^* decreases roughly as $\Gamma^{-29/18}$ [here Γ is simply related to λ by Eq. (5.1)]. The behavior of η^* as a function of Γ reflects the behavior of η as a function of the inverse temperature T^{-1}. Note also that the behavior of η as a function of the density n is given by $\Gamma^{5/2}\eta^*(\Gamma)$ (which is seen to increase monotonically with $\Gamma \sim n^{1/3}$). The very position of the minimum of $\eta^*(\Gamma)$ is, however, seen to depend on the approximations as we locate it at $\Gamma \sim 8$, whereas Vieillefosse and Hansen did find $\Gamma \sim 20$. Hence, in order to select a better theory and test the approximations, a direct molecular-dynamics calculation of η would be of much interest.

In fact, very recently, after this work was completed [see Ref. 5(d)], we learned about a molecu-

TABLE III. Reduced kinetic shear viscosity $\eta_K^* = \eta_K/nm\omega_p a^2$ as computed from Eq. (5.11) compared with the results $\hat{\eta}_K^*$ obtained from Eq. (6.6) based on a symmetric vertex renormalization.

Γ	η_K^*	$\hat{\eta}_K^*$	$\hat{\eta}_K^*/\eta_K^*$
0.1	87	117	1.35
2	0.35	0.53	1.54
6	0.058	0.086	1.47
20	0.0089	0.012	1.38
100	0.00075	0.00097	1.30

lar dynamics (MD) calculation of the Green-Kubo formula for the shear viscosity.[12] The ratio, η_{WB}/η_{MD} of our result (η_{WB}) to the molecular-dynamics one (η_{MD}) reads, $\eta_{WB}/\eta_{MD} = 0.97$, 0.86, and 1.22 for, respectively, $\Gamma = 1.0$, 10.4, and 100.4, which are the three values of Γ considered in these MD experiments. The statistical errors in these experiments have been estimated[12] to be of the order of 15%. Hence, the agreement with our relatively simple kinetic theory, without adjustable parameters, is rather good.

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