Perturbation analysis of fluxon dynamics

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A two-stage scheme is presented to study structural perturbations of the sine-Gordon equation. Although the method is based upon the inverse scattering transform, detailed knowledge of this technical apparatus is not necessary in order to effect the calculations. In the first stage, slow modulations of speeds and positions for the soliton components are computed. The radiation resulting from the acceleration of the solitons is then calculated as a first-order correction through an easily constructed radiative Green's function. The method is exemplified by using it to study several outstanding problems that arise in applications of the Josephson transmission line. In particular we consider: (i) the pinning of flux quanta by microshorts, (ii) the quantum flux shuttle, (iii) annihilation conditions for fluxon-antifluxon collisions, (iv) breather decay, and (v) radiation from a moving fluxon.

I. INTRODUCTION

The dynamics of magnetic-flux propagation on a Josephson-junction transmission line (JTL) is a subject of increasing practical interest. It is now possible to make such lines with reasonable mechanical and electrical stability,[1,2,4,6] and they appear to have applications for transmission, storage, and processing of information.[3-7] The basic JTL property that is useful for computing applications is quantification of magnetic flux which appears in units of the flux quantum or *fluxon*,

$$\Phi_0 = h/2e = 2.068 \times 10^{-15}\ \text{V s}.$$

In the "quantum flux shuttle" discussed by Fulton, Dynes, and Anderson,[3-5] each fluxon carries one bit of information through a shift register. The Josephson memory element proposed by Guéret[6] uses a small segment of JTL to store a single bit in the form of a fluxon, and the Josephson computing network designed by Nakajima, Onodera, and Ogawa,[7] employs interactions between JTL fluxons to realize the logic functions: OR, AND, and NOT. Also, employing means to be described in this paper, radiation from a fluxon as it passes a series of equally spaced microshorts may serve as the basis for an electromagnetic oscillator in the submillimeter range.

The essential structure of a JTL is sketched in Fig. 1. Two superconducting metal strips are separated by an insulating barrier which is thin enough (~25 Å) to permit "Josephson" current to tunnel the barrier at superconducting temperatures. Magnetic flux can penetrate along the insulating barrier, and this flux can propagate in the longitudinal (x) direction. There are two possible orientations for the flux. A quantum of flux in one direction is called a "fluxon" and in the other direction an "antifluxon." If an adjustable bias current is present, it will exert a Lorentz force on a fluxon in one direction and on an antifluxon in the other direction.

If all dissipative effects and imperfections in the transmission line are neglected, flux propagation is governed by the "sine-Gordon" equation[8]

$$\phi_{tt} = \phi_{xx} - \sin\phi\,, \tag{1.1}$$

where ϕ is magnetic flux measured in units of $\Phi_0/2\pi$, and x (and t) denote suitably normalized

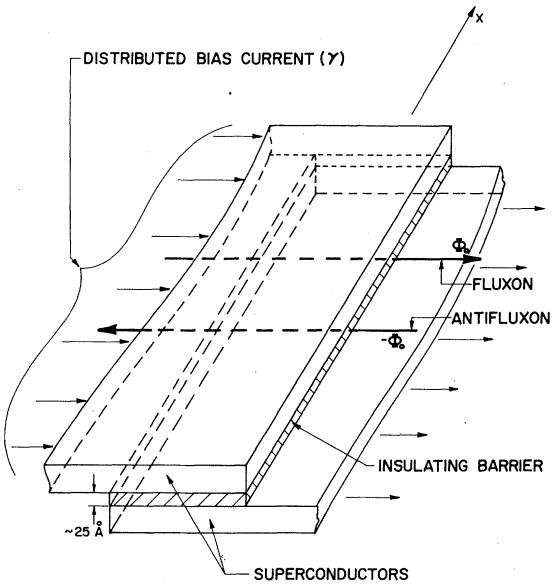

FIG. 1. Josephson junction transmission line (JTL) with bias current γ. Not drawn to scale.

space (and time) variables. (See Ref. 2 for a discussion of appropriate normalizations.) The sine-Gordon equation is a conservative, nonlinear, dispersive wave equation that supports special solutions called "solitons."[9] A sine-Gordon soliton is a localized wave [shown in Fig. 2(a) and 2(b)] that is analytically described by the two-parameter formula

$$\phi_{\pm}(x,t;u,x_0) \equiv 4\tan^{-1}\left[\exp\left(\pm\frac{x-ut-x_0}{(1-u^2)^{1/2}}\right)\right]. \tag{1.2}$$

The parameter u denotes the constant velocity of the soliton. This velocity u can take any value between -1 and $+1$, i.e., $-1<u<+1$. The other constant parameter, x_0, locates the initial center of the soliton. Notice that the $\pm$ signs imply that ϕ_+ increases monotonically by 2π as x goes from $-\infty$ to $+\infty$, while ϕ_- decreases by 2π. Thus, in this model, ϕ_+ represents a fluxon while ϕ_- represents an antifluxon. Finally, note that the factor $(1-u^2)^{1/2}$ fixes the width of the soliton. As $u\to\pm1$, the soliton "Lorentz contracts" to a step function of height 2π located at the "light cones" $(x=x_0\pm t)$.

The soliton of (1.2) is a special case of the general traveling-wave solution $\phi=\phi(\theta)$ (where $\theta \equiv \omega t-kx$) defined by the elliptic integral[2,8]

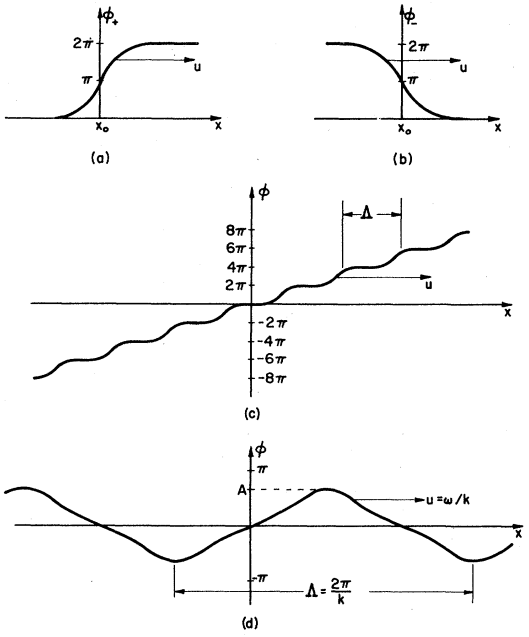

FIG. 2. Nonlinear solutions of the sine-Gordon equation (1.1); (a) fluxon, (b) antifluxon, (c) periodic array of fluxons, and (d) radiation component of wave number k.

$$\pm(\theta-\theta_0) = (k^2-\omega^2)^{1/2}\int_0^{\phi}\frac{d\phi'}{[2(E-\cos\phi')]^{1/2}}, \tag{1.3}$$

where E is a constant of the first integration. For the case $E=1$, (1.3) reduces to (1.2) with $u=\omega/k$. For $E>1$ and $u=\omega/k<1$, (1.3) indicates that ϕ is a monotone function of x; this can be considered as an infinite sequence of fluxons (or antifluxons) spaced apart by a distance $\Lambda=2\pi/k$ as shown in Fig. 2(c).

For $-1<E<+1$ and $u=\omega/k>1$, (1.3) gives ϕ as a periodic function of x with period $\Lambda=2\pi/k$ as shown in Fig. 2(d). The amplitude of this wave $A=\cos^{-1}E$. The frequency ω, wave number k, and amplitude A are related by the *nonlinear dispersion equation*

$$\omega^2-k^2=\tfrac{1}{4}\pi^2\{K[\tfrac{1}{2}(1-\cos A)]^{1/2}\}^{-2}, \tag{1.4}$$

where $K(\cdot)$ is the complete elliptic integral of the first kind. As A increases from 0 to π, the right-hand side of (1.4) falls from unity. Thus, for small-amplitude periodic waves, the lowest frequency is unity.

For each wave number k this periodic wave train has two free parameters [ω (or A) and θ_0] which fix its velocity (or amplitude) and phase. We shall refer to these periodic waves as "radiation in the transmission line at wave number k."

The general solution of the sine-Gordon equation which approaches zero (mod 2π) as $|x|\to\infty$, consists of a finite number of fluxons, a finite number of antifluxons, and a continuum of radiation. Before briefly describing this general solution, we discuss the special case of no radiation. Exact analytical solutions of the sine-Gordon equation are known which describe the interaction of N fluxons and M antifluxons.[10] These solutions are called "pure $N+M$-soliton states" (since no radiation is present) and are labeled by $2(N+M)$ parameters of which $N+M$ parameters fix the speeds of the $N+M$ solitons and the remaining $(N+M)$ parameters to fix their locations. This is exactly correct provided that only fluxons or only antifluxons are present in the wave; however, it is somewhat oversimplified if both fluxons and antifluxons are present. To see this, consider a pure two-soliton waveform composed of a fluxon-antifluxon pair. In this case, the general four-parameter waveform can represent two different physical states. In the simplest of these two situations, the fluxon separates from the antifluxon in both the distant past and the distant future; each then travels with its own velocity and has its own location parameter. The waveform in this situation is given by

$$\phi_D = -4\tan^{-1}\left(\frac{\sinh[u(t-t_0)/(1-u^2)^{1/2}]}{u\cosh[(x-x_0)/(1-u^2)^{1/2}]}\right), \tag{1.5}$$

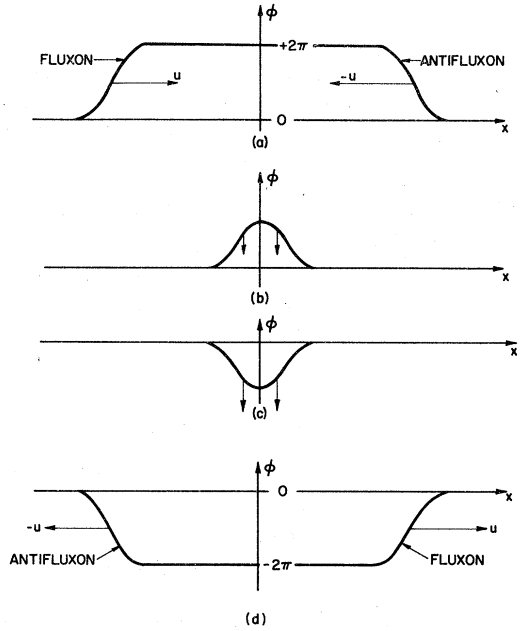

FIG. 3. Fluxon-antifluxon collision. Time increases from (a) to (d).

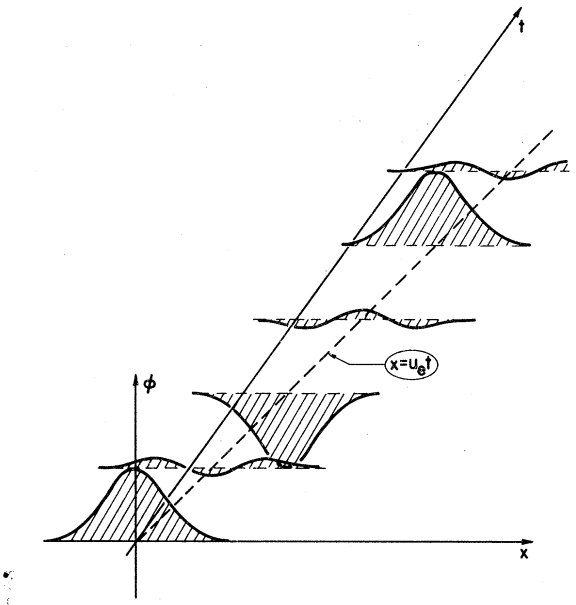

FIG. 4. Breather.

where ϕ_D depends upon three parameters instead of four because we have Lorentz transformed to a center-of-velocity coordinate frame. Note that as $t \to +\infty$ this "doublet" waveform becomes

$$\phi_D \approx 4\tan^{-1}\left[\exp\left(\frac{x - x_0 - \delta - ut}{(1-u^2)^{1/2}}\right)\right], \quad x > 0$$

$$\approx 4\tan^{-1}\left[\exp\left(\frac{-x + x_0 - \delta - ut}{(1-u^2)^{1/2}}\right)\right], \quad x < 0.$$

Thus, as time increases, ϕ_D separates into an antifluxon traveling to the left with velocity $-u$ and a fluxon traveling to the right with velocity $+u$. The reverse is true in the past as indicated in Fig. 3.

The other two-soliton solution consists of a fluxon-antifluxon pair bound together into an oscillatory state called a "breather" which is sketched in Fig. 4. It cannot decompose into a fluxon and an antifluxon as $t \to \pm\infty$ because its energy is less than the rest energy of the fluxon-antifluxon components. The four parameters which label this wave fix the velocity of the envelope, the internal "breathing" frequency, the initial location of the envelope, and an initial centering of the phase. Analytically, the breather waveform is given by

$$\phi_B = 4\tan^{-1}\left(\frac{\tan\nu \sin[(\cos\nu)(t - t_0)]}{\cosh[(\sin\nu)(x - x_0)]}\right), \qquad (1.6)$$

where we have assumed that the envelope is at rest in the (x, t) frame. The four parameters are ν, x_0, t_0 together with a velocity u_e which results after a Lorentz transformation $[x \to x' = (x - ut)/(1-u^2)^{1/2};\ t \to t' = (t - ux)/(1-u^2)^{1/2}]$ has boosted the envelope from rest to velocity u_e. Note that the frequency of the breather in its own frame is $\cos\nu$ and this frequency is always less than unity. Small-amplitude radiation, on the other hand, has a frequency greater than unity.

A general wave consists of a (nonlinear) superposition of N fluxons, M antifluxons, L breathers, and radiation wave trains at all wave numbers. Although these components would synthesize any wave ϕ [which approaches zero (mod 2π) as $x \to \pm\infty$], they are particularly natural and useful when the temporal evolution of ϕ is governed by the perfect sine-Gordon equation (1.1). In this case, the number of fluxons N, the number of antifluxons M, the number of breathers L, and the density of wave trains for each wave number k are all fixed by the initial data and do not vary with time. Moreover, the initial data selects the speeds of the fluxons, antifluxons, and breathers as well as the breather frequencies. All of these properties have been established analytically during the past five years through a new technique known as the "inverse scattering transform method" (ISTM).[11-13] This method can be viewed as a nonlinear generalization of the Fourier-transform method wherein the Fourier components are replaced by fluxons, antifluxons, breathers, and (nonlinear) radiation.

If the sine-Gordon equation were a perfect model

of the Josephson transmission line, these nonlinear modes would not interact even upon direct collision. But dissipation effects and junction irregularities are always present on a real JTL so real fluxons *do* interact when they collide. It is necessary to establish design control over such interactions if the above mentioned applications of the JTL are to be realized.

One analytical approach is to represent a more-realistic model of JTL as a structural perturbation of the sine-Gordon equation

$$\phi_{tt} - \phi_{xx} + \sin\phi = \epsilon f, \quad 0 \leq \epsilon \ll 1 . \tag{1.7}$$

Here the perturbation ϵf is a function of the wave ϕ, its spatial and temporal derivatives, and possibly of space and time. The perturbation will alter the speeds and shift the locations of any fluxons and antifluxons in the wave, and it will also modulate any radiation packets and breathers which are present. The perturbation may also cause fluxons and antifluxons to interact or to become bound into new breathers. Breathers might be forced to fission into fluxon-antifluxon pairs. In addition, the perturbation could open (or close) additional degrees of freedom which are not excited in the initial wave; it could create (or destroy) solitons as well as generate radiation.

The main purpose of this paper is to adapt a general perturbation method developed in Ref. 14 to the specific problem of a real JTL modeled as a structural perturbation of the perfect sine-Gordon equation. This perturbation scheme is constructed about the soliton components rather than an (old fashioned) linearization about Fourier modes. We try to present the perturbation scheme in sufficient generality that practicing scientists will find it easy to study particular structural perturbations; moreover, we apply the scheme to a perturbation that seems a realistic model of an actual JTL. Finally, we emphasize that it is not necessary to be familiar with the technical machinery of the inverse scattering transform in order to effectively undertake a perturbation analysis of soliton interactions.[15-18]

In Sec. II, we present the perturbation theory from the simplest possible perspective: an energy argument. This should be helpful to orient the physical intuition of the reader during the more detailed discussions to follow. Moreover, an energetic analysis is valuable in its own right because, when applicable, it is the most direct way to obtain useful results. Section III presents an abbreviated version of the perturbation scheme in a form which is appropriate for calculating interactions between fluxons, antifluxons, and breathers; and, in Sec. IV, this scheme is used to study the dynamics of a single fluxon in the presence of dissipation and microshorts across the Josephson junction. Section V considers the fluxon-antifluxon collision, derives conditions for subsequent decay into a breather, and offers some general comments on the propagation of fluxon bundles. Section VI develops the perturbation scheme with sufficient generality to permit calculation of the radiation from a modulated fluxon in Sec. VII. Finally, the Appendix uses inverse scattering theory to sketch a basic derivation of the Green's function upon which the calculation of radiation is based.

II. ENERGETIC ANALYSIS

In this section we indicate the physical origin of each term in the particular perturbation to be studied, describe qualitatively the response of the wave to each term, and present a simple way to make these considerations quantitative when studying the dynamics of a single fluxon. We take as a realistic model of the JTL a structural perturbation of the sine-Gordon equation

$$\phi_{tt} = \phi_{xx} - \sin\phi + \epsilon f, \quad 0 \leq \epsilon \ll 1 , \tag{2.1}$$

where ϵf is a generic term representing

$$\epsilon f = -\alpha\phi_t + \beta\phi_{xxt} - \gamma - \sum_i \mu_i \, \delta(x - a_i) \sin\phi . \tag{2.2}$$

Here, $\alpha\phi_t$ represents the dissipation due to tunneling of normal electrons across the barrier, $\beta\phi_{xxt}$ is the dissipation caused by flow of normal electrons parallel to the barrier, γ is a distributed bias current providing energy input, and the $\mu_i \, \delta(x - a_i) \sin\phi$ terms represent local regions of high Josephson current (microshorts, thin spots in the barrier, etc.). (Again, the reader should consult Ref. 2 for a discussion of the normalizations.)

The perfect sine-Gordon equation can be written as a Hamiltonian system for (ϕ, ϕ_t) with the Hamiltonian

$$H^{\mathrm{SG}} \equiv \int_{-\infty}^{\infty} (\tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi_x^2 + 1 - \cos\phi)\, dx ; \tag{2.3}$$

moreover, if we neglect the two dissipative terms (α and β), our more-realistic model can also be written as a Hamiltonian system with Hamiltonian

$$H = H^{\mathrm{SG}} + H^{\mathrm{P}} , \tag{2.4a}$$

$$H^{\mathrm{P}} \equiv \int_{-\infty}^{\infty} \left(\sum_i \mu_i \, \delta(x - a)(1 - \cos\phi) + \gamma\phi \right) dx . \tag{2.4b}$$

From these two Hamiltonians, it is easy to determine the nature of each term in the structural perturbation ϵf. First, let ϕ denote any solution of (2.1) with the parameters α and β set at zero. Then $H(\phi)$ is constant in time,

$$\frac{dH(\phi)}{dt}=0\,.$$

On the other hand, if ϕ denotes any solution of (2.1) with $\alpha,\beta\neq 0$, we compute

$$\frac{dH(\phi)}{dt}=-\int_{-\infty}^{\infty}(\alpha\phi_t^2+\beta\phi_{xt}^2)\,dx\,.$$

If α and β are positive, these terms extract energy from the wave; they are dissipative.

To see clearly the effect of the γ term (bias current) on the wave, set $\alpha=0$, $\beta=0$, and $\mu_i=0$. Then the energy density of the wave becomes

$$\mathcal{H}=(\tfrac{1}{2}\phi_t^2+\tfrac{1}{2}\phi_x^2+1-\cos\phi)+\gamma\phi=\mathcal{H}^{\mathrm{SG}}+\gamma\,\phi\,.$$

If γ is positive, the term $\gamma\phi$ in the energy density yields a negative force on the wave in ϕ space. As time increases, the wave is driven down in ϕ space. If ϕ is a single fluxon [see Fig. 2(a)], a force that causes ϕ to decrease drives the fluxon to the right. If, on the other hand, ϕ is a single antifluxon [see Fig. 2(b)] it is driven to the left. Thus, γ is an energy-injection term which accelerates fluxons (antifluxons) in the $+x$ ($-x$) direction.

To see the qualitative effect of a microshort on the wave, set $\alpha=0,\beta=0,\gamma=0$ and consider the energy density

$$\mathcal{H}=\mathcal{H}^{\mathrm{SG}}+\mu(1-\cos\phi)\,\delta(x-a)\,.$$

The factor $\mu(1-\cos\phi)$ represents the positive energy in a microshort located at $x=a$. This energy is zero unless $\phi(x,t)\neq 0 \pmod{2\pi}$ near $x=a$. Otherwise, energy is taken from H^{SG} and "stored in the short." Thus as fluxons and antifluxons approach the short, they must slow down.

In the case of a single fluxon (or antifluxon), these qualitative considerations can provide quantitative predictions of the effect of a perturbation. For example, set all $\mu_i=0$ and consider

$$\phi_{tt}=\phi_{xx}-\sin\phi-(\alpha\phi_t-\beta\phi_{xxt}+\gamma)\,, \tag{2.5}$$

where the parameters α,β,γ are small. First, we assume that the predominant effect of the perturbation on a single fluxon is to modulate its velocity. Next, we insert the soliton waveform (1.2) into H^{SG} to obtain

$$H^{\mathrm{SG}}(\phi_\pm)=8(1-u^2)^{-1/2}\,, \tag{2.6}$$

and take the time derivative

$$\frac{d}{dt}H^{\mathrm{SG}}(\phi_\pm)=8u(1-u^2)^{-3/2}\frac{du}{dt}\,. \tag{2.7}$$

On the other hand, if ϕ is any solution of (2.5) we compute

$$\frac{d}{dt}H^{\mathrm{SG}}(\phi)=-\int_{-\infty}^{\infty}(\gamma\phi_t+\alpha\phi_t^2+\beta\phi_{xt}^2)\,dx\,. \tag{2.8}$$

Since α,β,γ are small, we can approximate ϕ by a fluxon (antifluxon) with modulating velocity. Inserting the fluxon (antifluxon) wave (1.2) into the right-hand side of (2.8) and integrating gives

$$\frac{d}{dt}H^{\mathrm{SG}}(\phi_\pm)=\pm 2\pi\gamma u-8\alpha u^2(1-u^2)^{-1/2}-\tfrac{8}{3}\beta u^2(1-u^2)^{-3/2}\,. \tag{2.9}$$

Equating (2.7) and (2.9) yields a first-order ordinary differential equation for $u(t)$,

$$\frac{du}{dt}=\pm\tfrac{1}{4}\pi\gamma(1-u^2)^{3/2}-\alpha u(1-u^2)-\tfrac{1}{3}\beta u\,. \tag{2.10}$$

This simple equation describes the effect of the perturbation on the fluxon's velocity. Note that the α and β terms cause both the fluxon and the antifluxon to slow down, while the γ terms drives the fluxon to the right and the antifluxon to the left. Let u_∞ denote the equilibrium solutions of (2.10). These constant solutions represent "power balance" velocities at which the power input to the soliton is just balanced by its power loss to dissipation. For the case $\beta=0$, the "power-balance velocities" are

$$u_\infty=\pm\left[1+\left(\frac{4\alpha}{\pi\gamma}\right)^2\right]^{-1/2}\,. \tag{2.11}$$

The calculation just described uses one constant of the motion (total energy) to get one equation for the effect of the perturbation on the velocity of a single fluxon. The simplicity and directness of this calculation are its chief merits; however, the procedure has several disadvantages. First, it is difficult to check the accuracy of the result. Next, no method is provided to compute radiation generated by the perturbation, or for that matter any change in the wave other than a modulation of fluxon velocity. In a multisoliton wave, one must compute the modulations in several velocity parameters; thus additional constants of the motion are required. Although other constant are available for calculating additional dynamic properties of more complex solitary waves,[10] our experience indicates that they are not convenient for generating the extra equations associated with multisoliton interactions. In Sec. III we sketch a simple perturbation method that overcomes these difficulties. We find this method easier to use whenever the energy argument is insufficient. We

emphasize, however, that the energy argument is the method of choice whenever it is appropriate because it is so direct.

III. SUMMARY DESCRIPTION OF THE PERTURBATION SCHEME

In this section, we consider a wave which is initially a pure multisoliton state and show how to compute the response of the solitons to a generic structural perturbation ϵf. A more-detailed description of the perturbation scheme is given in Sec. VI and in Ref. 14. We begin by writing (1.6) as a first-order nonlinear system

$$\partial_t\begin{pmatrix}\phi\\ \phi_t\end{pmatrix}+\begin{pmatrix}0 & 1\\ -\partial_{xx}+\sin(\circ) & 0\end{pmatrix}\begin{pmatrix}\phi\\ \phi_t\end{pmatrix}=\epsilon\begin{pmatrix}0\\ f(\phi)\end{pmatrix},\qquad 0\le\epsilon\ll 1 \quad (3.1)$$

or, in vector notation for the wave $\vec{W}\equiv\mathrm{col}(\phi,\phi_t)$,

$$\frac{\partial}{\partial t}\vec{W}+N(\vec{W})=\epsilon\vec{f}, \quad (3.1')$$

where $\vec{f}\equiv\mathrm{col}[0,f(\phi)]$ and the nonlinear operator $N(\cdot)$ is given by

$$N(\vec{W})\equiv\begin{pmatrix}0 & 1\\ -\partial_{xx}+\sin(\circ) & 0\end{pmatrix}\begin{pmatrix}\phi\\ \phi_t\end{pmatrix}.$$

When $\epsilon=0$, a pure multisoliton wave is of the form

$$\vec{W}_{(N+M+2L)}=\vec{W}(x,u_1t+x_1,\ldots,u_{(N+M+2L)}t+x_{(N+M+2L)};\vec{u}), \quad (3.2)$$

where $\vec{u}$ is an $N+M+2L$-dimensional constant vector whose components fix the velocities of the N fluxons and M antifluxons in the wave, and the $2L$ frequencies and velocities of the breathers. The parameters $\vec{x}=(x_1,\ldots,x_{(N+M+2L)})$ fix the positions of the fluxons, antifluxons, and breathers and also the phases of the breathers. The waveform (3.2) has $2(N+M+2L)$ parameters. Frequently, we shall refer to these parameters collectively as a vector $\vec{p}=(\vec{u},\vec{x})$.

Note that time t enters the waveform $\vec{W}$ only through the combinations u_it $(i=1,\ldots,M+N+2L)$, although the velocity parameters u_i enter in other places as well. [For example, in (1.2) the velocity parameter u fixes both the speed and the width of the soliton.]

The first step in the perturbation scheme is to assume that the solution $\vec{W}$ of the full equation (3.1) is of the form

$$\vec{W}=\vec{W}_0+\epsilon\vec{w}+\cdots, \quad (3.3)$$

where $\vec{W}_0$ is the $N+M+2L$-soliton state represented by (3.2) except that all parameters in $\vec{p}$ are allowed to modulate as follows:

$$\vec{W}_0=\vec{W}_{(N+M+2L)}(x,X_1(t)+x_1(t),\ldots,X_{(N+M+2L)}(t)+x_{(N+M+2L)}(t);\vec{u}(t)), \quad (3.4)$$

where $dX/dt=u_i$. A few comments are needed to clarify this ansatz. First, the temporal modulation in the parameters $\vec{p}(t)$ is induced by the perturbation ϵf. Since the perturbation is $O(\epsilon)$, $d\vec{p}/dt=O(\epsilon)$. At this stage, the modulation is not known; it must be computed. Next, some care is needed to correctly identify the appropriate "velocity" parameters $(u_1,\ldots,u_{N+M+2L})$ before replacing u_it with $X_i(t)$. One approach to this identification begins with the asymptotic form of $\vec{W}_0$ for large t. In this limit, $\vec{W}_0$ will be decomposed into a collection of fluxons, antifluxons, and breathers that can be treated individually. The appropriate identification for the frequency of a breather is straightforward if one recognizes that the breather waveform (1.6) is an analytic continuation (in parameter space) of the doublet waveform for a fluxon-antifluxon pair (1.5).[19] As a concrete example of the zeroth-order ansatz, if a single fluxon is perturbed (i.e., $N=1, M=L=0$), (3.2) corresponds to (1.2) and (3.4) takes the form

$$\vec{W}_0=\begin{pmatrix}4\tan^{-1}\left[\exp\left(\pm\dfrac{x-X_0(t)-x_0(t)}{[1-u^2(t)]^{1/2}}\right)\right]\\ \dfrac{\mp 2u}{(1-u^2)^{1/2}}\operatorname{sech}\left(\dfrac{x-X_0(t)-x_0(t)}{[1-u^2(t)]^{1/2}}\right)\end{pmatrix}, \quad (3.4a)$$

with $X_0(t)=\int_0^t u(t')\,dt'$. We discuss the modulation of this wave in detail in Sec. IV.

The second step in the perturbation scheme is to find ordinary differential equations which govern the modulation of the parameters $\vec{p}(t)$. This calculation begins with the insertion of ansatz (3.3) into the nonlinear wave equation (3.1), Linearization about $\vec{W}_0$ then yields

$$L\vec{w}=\vec{\mathcal{F}}(\vec{W}_0),\quad \vec{w}(t=0)=\vec{0}. \quad (3.5)$$

where the linear operation L is given explicitly by

$$L\equiv\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\partial_t+\begin{pmatrix}0 & -1\\ -\partial_{xx}+\cos\phi_0 & 0\end{pmatrix}. \quad (3.6)$$

Thus, the first-order correction $\vec{w}$ satisfies a linear, variable-coefficient, nonhomogeneous initial-value problem. The choice of vanishing initial data is equivalent to the assumption that initially the wave is precisely the pure soliton state $\vec{W}_0$. In (3.5), the inhomogeneity or "effective source" $\vec{\mathcal{F}}$ is given by

$$\vec{\mathcal{F}}(\vec{W}_0)=\vec{f}(\vec{W}_0)-\frac{1}{\epsilon}\sum_{j=1}^{2(N+M+2L)}\frac{\partial p_j}{\partial t}\frac{\partial\vec{W}_0}{\partial p_j}, \quad (3.7)$$

where, in the calculation of the partial derivations, $\{X_1(t),\ldots,X_{N+M+2L}(t)\}$ are held fixed. Note that the

effective source includes a correction to the perturbation $\vec{f}$ because, since the parameters of $\vec{W}_0$ vary with time, $\vec{W}_0$ is not an exact solution of the unperturbed sine-Gordon equation. In the calculation of this effective source, $dp_j/dt = O(\epsilon)$, so the correction is $O(1)$, not $O(1/\epsilon)$ as the notation makes it appear.

Equation (3.5) for the first-order correction $\vec{w}$ is a linear initial-value problem that can be explicitly solved using a Green's function. This procedure is discussed in detail in Secs. VI and VII. In particular, this Green's function is useful for the computation of the radiation generated by an accelerating fluxon; however, if only the modulations of the fluxon itself are of interest, the calculation is much shorter and very easy. Generally the solution $\vec{w}(x,t)$ of Eq. (3.5) tends to grow with time t. When this growth occurs, the first-order correction $\epsilon\vec{w}$ becomes large, and approximation (3.3) becomes invalid. Mathematically, the origin of this growth is a resonance between the source $\vec{\mathcal{F}}$ and the Green's function for (3.3). We must choose the modulation of the parameters $\{p_j\}$ to eliminate this resonance; thus, we obtain a valid approximation. The worst secularity (linear growth in t) arises from a resonance between solitons in the effective source $\vec{\mathcal{F}}(\vec{W}_0)$ and the same solitons in the Green's function. This leading secularity will be eliminated if the effective source $\vec{\mathcal{F}}$ is orthogonal to a certain finite-dimensional subspace ("discrete subspace") of the null space of an operator $L^\dagger$ that is a formal adjoint of L,

$$L^\dagger \equiv -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\partial_t - \begin{pmatrix} 0 & \partial_{xx} - \cos\phi_0 \\ 1 & 0 \end{pmatrix}.$$

We denote the null spaces of L and $L^\dagger$ by $\mathfrak{N}(L)$ and $\mathfrak{N}(L^\dagger)$, and the special discrete subspaces by $\mathfrak{N}_d(L)$ and $\mathfrak{N}_d(L^\dagger)$. These subspaces have dimension $2(N+M+2L)$.

The idea is to select the modulation of the $2(N+M+2L)$ parameters $\{p_j(t)\}$ in a manner which forces the effective source $\vec{\mathcal{F}}$ to be orthogonal to $\mathfrak{N}_d(L^\dagger)$,

$$\vec{\mathcal{F}} \perp \mathfrak{N}_d(L^\dagger). \tag{3.8}$$

This orthogonality condition actually is an ordinary differential equation that determines the fluxon's response to the perturbation f. To obtain this differential equation, let $\{\vec{b}_j(x),\ j=1,2,\ldots,2(N+M+2L)\}$ denote a basis for the discrete subspace $\mathfrak{N}_d(L^\dagger)$. Then, using the definition (3.7) of the effect source $\vec{\mathcal{F}}$, we put the orthogonality condition into the form

$$\sum_{k=1}^{2(N+M+2L)} \left(\vec{b}_j, \frac{\partial \vec{W}_0}{\partial p_R}\right)\frac{dp_R}{dt_R} = \epsilon(\vec{b}_j, \vec{f}),$$

$$j = 1,2,\ldots,2(N+M+2L) \tag{3.9}$$

where we have used the "inner product" notation for the integral over x,

$$(\vec{F}, \vec{G}) \equiv \int_{-\infty}^{\infty} dx\, \vec{F}^T(x)\vec{G}(x)\,dx$$

Equation (3.9) provides a $2(N+M+2L)$-dimensional first-order system of ordinary differential equations for the slow temporal variations of the fluxons' parameters. We complete this section by showing that a basis for $\mathfrak{N}_d(L^\dagger)$ can be computed directly from the multifluxon waveform $\vec{W}_0$; thus, system (3.9) for the modulation of $\vec{p}$ depends only on the multifluxon waveform.

First, we notice from (3.9) that we need the basis $\{\vec{b}_j\}$ only through $O(1)$ in ϵ. This permits us to freeze the time dependence of $\vec{p}(t)$ in the computation of $\{\vec{b}_j(x)\}$. Thus, in the computation of the basis, we can replace (3.4) with (3.2).

Next, we observe that members of $\mathfrak{N}(L)$ can be generated by differentiating $\vec{W}_0$ with respect to the $2(N+M+2L)$ parameters $\vec{p}$. To see this, note that (3.1) has no explicit dependence upon these parameters; thus, differentiation of (3.1) with respect to a parameter p_j becomes simply $Ld\vec{W}_0/dp_j = \vec{0}$. Here the notation d/dp_j is to remind us that in this case, when computing $\partial/\partial u_i$, one does not hold $X_i = u_i t$ constant. In this way $2(N+M+2L)$ independent elements

$$\frac{d\vec{W}_0}{dp_j} \in \mathfrak{N}(L)$$

can be generated. But we seek elements of $\mathfrak{N}(L^\dagger)$. To find these note that if $\vec{V} \in \mathfrak{N}(L)$, then $J\vec{V} \in \mathfrak{N}(L^\dagger)$,

$$J \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(To verify this claim, begin with $\vec{V} \in \mathfrak{N}(L)$,

$$L\vec{V} = \vec{0} - \partial_t\vec{V} + \begin{pmatrix} 0 & -1 \\ -\partial_{xx} + \cos\phi_0 & 0 \end{pmatrix}\vec{V} = 0.$$

Multiplying by J gives

$$\partial_t J\vec{V} + J\begin{pmatrix} 0 & -1 \\ -\partial_{xx} + \cos\phi_0 & 0 \end{pmatrix}J^{-1}J\vec{V} = 0,$$

or $L^\dagger JV = 0$.) Thus, to leading order in ϵ,

$$\mathfrak{N}_d(L^\dagger) \equiv \operatorname{span}\left\{J\frac{d\vec{W}_0}{dp_j};\ j = 1,2,\ldots,2(N+M+2L)\right\}.$$

Finally, since

$$\vec{W}_0 \simeq \vec{W}_0(x, u_1 t + x_1, \ldots, u_{N+M+2L}t + x_{N+M+2L}; \vec{u}),$$

$$\frac{d\vec{W}_0}{du_j} \simeq t\frac{\partial \vec{W}_0}{\partial x_j} + \frac{\partial \vec{W}_0}{\partial u_j},$$

where on the right-hand side one holds $\{X_j\}$ fixed in computing the partial derivatives. Thus, for

fixed t as functions of x, $\text{span}\{Jd\vec{W}_0/dp_j\}$ $=\text{span}\{J\partial\vec{W}_0/\partial p_j\}$. In this basis, (3.9) takes the form

$$\sum_{k=1}^{2(N+M+2L)}\left(J\frac{\partial\vec{W}_0}{\partial p_j},\frac{\partial\vec{W}_0}{\partial p_k}\right)\frac{dp_k}{dt}=\epsilon\left(J\frac{\partial\vec{W}_0}{\partial p_j},\vec{f}(\vec{W}_0)\right),$$
$$j=1,2,\ldots,2(N+M+2L). \quad (3.9')$$

These equations are the main result of this section. They are the basic working tool of soliton perturbation theory. Equations (3.9′) for the velocity and location parameters permit the response of the fluxons to a structural perturbation f to be computed directly from the multisoliton waveform $\vec{W}_0$. No information is needed beyond an analytical formula for the multifluxon waveform. To compute the acceleration of the solitons caused by an external perturbation f, we take system (3.9′) [or, equivalently, system (3.9) in the arbitrary basis] as the starting point for the calculations.

From a broader perspective, our perturbation theory has reduced an infinite-dimensional system (the original partial differential equation) to a finite-dimensional system. The reduced system is a $2(N+M+2L)$-component system for the salient features of the solution, namely, the soliton positions and velocities.

In the Secs. IV and V we illustrate such calculations and then in later sections we check the accuracy of the approximations by computing the radiation generated by the fluxons as they respond to the perturbation f.

IV. PERTURBATION ANALYSIS FOR A SINGLE FLUXON

In this section, we use the perturbation scheme summarized in (3.9′) to study the dynamics of a single fluxon under the influence of an arbitrary structural perturbation. The results are then applied to several specific perturbations that are of practical interest.

A. General dynamic equations

Suppose a single fluxon (or antifluxon) experiences a generic perturbation ϵf. In this case, the modulated waveform $\vec{W}_0$ is given by (3.4a) with velocity parameter u and location parameter x_0. System (3.9′) or, equivalently, (3.9), is a coupled system for these two parameters which can be written in the form

$$\frac{du}{dt}=\mp\epsilon\tfrac{1}{4}(1-u^2)\int_{-\infty}^{\infty}f[\phi_0(\Theta,x,t)]\,\text{sech}\Theta\,dx, \quad (4.1a)$$

$$\frac{dx_0}{dt}=-\epsilon\tfrac{1}{4}u(1-u^2)^{1/2}\times\int_{-\infty}^{\infty}f[\phi_0(\Theta,x,t)]\Theta\,\text{sech}\Theta\,dx, \quad (4.1b)$$

where

$$\Theta=\Theta(x,t)\equiv\pm\left(x-\int_0^t u(t')\,dt'-x_0(t)\right)\Big/\,[1-u^2(t)]^{1/2}.$$

Equations (4.1) govern the response of a single soliton to a generic perturbation. In the remainder of Sec. IV A, we show in some detail the actual reduction of (3.9) to (4.1); and, in addition, we make several observations about the structure of system (4.1). We present this reduction process in full detail, without introducing any general notation that might tend to hide algebraic complications, in order to emphasize that only a few rather easy analytical manipulations are involved. Then, in Secs. IV B and IV C, we study some particular choices for the perturbation f.

We begin from system (3.9) for the single fluxon $\vec{W}_0$ with parameters $(p_1,p_2)=(u,x_0)$ and consider the null space

$$\mathfrak{N}_d(L)\equiv\text{span}\left\{\frac{\partial}{\partial u}\vec{W}_0,\frac{\partial}{\partial x_0}\vec{W}_0\right\}.$$

The calculations will be easier if we use a slight modification of this basis. Note (i) in (3.9) we only require the basis elements to $O(1)$ in ϵ, and (ii) to this leading order,

$$\vec{W}_0\simeq\begin{pmatrix}\phi_0\\ \partial_t\phi_0\end{pmatrix},$$
$$\frac{\partial\vec{W}_0}{\partial x_0}\simeq\frac{1}{u}\frac{\partial\vec{W}_0}{\partial t}.$$

Thus, to leading order, the null space $\mathfrak{N}_d(L)$ is spanned by $\{\partial\vec{W}_0/\partial u,\partial\vec{W}_0/\partial t\}$, and

$$\mathfrak{N}_d(L^\dagger)=\text{span}\left\{J\frac{\partial\vec{W}_0}{\partial u},J\frac{\partial\vec{W}_0}{\partial t}\right\}.$$

More explicitly, a basis for $\mathfrak{N}_d(L^\dagger)$ is given by

$$\vec{b}_1\cong\begin{pmatrix}\phi_{0,tt}\\ -\phi_{0,t}\end{pmatrix},\quad\vec{b}_2\cong\begin{pmatrix}\phi_{0,tu}\\ -\phi_{0,u}\end{pmatrix},$$

where the notation "$\cong$" reminds us that we only need the $O(1)$ part of the right-hand side. With this choice of parameters and basis, (3.9) takes the explicit form

$$\left(\int_{-\infty}^{\infty}(\phi_{ut}\phi_t-\phi_u\phi_{tt})\,dx\right)\dot{u}+\left(\int_{-\infty}^{\infty}(\phi_x\phi_{tt}-\phi_{xt}\phi_t)\,dx\right)\dot{x}_0=\epsilon\int_{-\infty}^{\infty}f(\phi)\phi_t\,dx, \quad (4.2a)$$

$$\left(\int_{-\infty}^{\infty}(\phi_{ut}\phi_u-\phi_u\phi_{ut})\,dx\right)\dot{u}+\left(\int_{-\infty}^{\infty}(\phi_x\phi_{ut}-\phi_{xt}\phi_u)\,dx\right)\dot{x}_0=\epsilon\int_{-\infty}^{\infty}f(\phi)\phi_u\,dx, \quad (4.2b)$$

where we have written $\phi = \phi_0$ for typographical convenience. Concentrate upon the first equation, (4.2a). Since ϕ_0 is given by

$$\phi_0 = 4\tan^{-1}\left[\exp\left(\pm\frac{x - x_0 - \int_0^t u(t')\,dt'}{(1-u^2)^{1/2}}\right)\right],$$

$\phi_{0,t}$ and $\phi_{0,x}$ are even functions of x, while their t derivatives $\phi_{0,tt}$ and $\phi_{0,xt}$ are odd. By these symmetry considerations, the coefficient of $\dot{x}_0$ vanishes and (4.2a) reduces to

$$\left(\int_{-\infty}^{\infty}(\phi_{ut}\phi_t - \phi_u\phi_{xx} + \phi_u\sin\phi)\,dx\right)\dot{u} = \epsilon\int_{-\infty}^{\infty} f(\phi)\phi_t\,dx,$$

where we have made the substitution $\phi_{tt} = \phi_{xx} - \sin\phi$. One integration by parts then yields

$$\left(\int_{-\infty}^{\infty}\frac{d}{du}\left(\tfrac{1}{2}\phi_t^2 + \tfrac{1}{2}\phi_x^2 + 1 - \cos\phi\right)dx\right)\dot{u} = \epsilon\int_{-\infty}^{\infty} f(\phi)\phi_t\,dx.$$

This is just the energy equation

$$\frac{dH(\phi)}{du}\,\dot{u} = \epsilon\int_{-\infty}^{\infty} f(\phi)\phi_t\,dx$$

which is discussed in Sec. II. Finally, for a single soliton $H(\phi) = 8/(1-u^2)^{1/2}$ and (4.1a) follows directly.

Now consider (4.2b). The coefficient of $\dot{u}$ vanishes identically. Integrating the second term by parts yields

$$\left(\frac{d}{du}\int_{-\infty}^{\infty}\phi_x\phi_t\,dx\right)\dot{x}_0 = \epsilon\int_{-\infty}^{\infty} f(\phi)\phi_u\,dx.$$

Writing $\phi_0 = 4\tan^{-1}(\exp\Theta)$ completes our derivation of (4.1b).

We conclude this subsection with two comments concerning the general structure of (4.1). First consider (4.1a) for the case in which f is independent of ϕ_0 and depends only upon x. This situation describes a soliton under the influence of an external force, and (4.1a) can be put in the form[17,20]

$$\frac{du}{dt} = -\frac{\partial}{\partial X}U_{\text{eff}},$$

where the effective potential is given by

$$U_{\text{eff}} \equiv \epsilon\tfrac{1}{8}(1-u^2)^{3/2}\int_{-\infty}^{\infty} f[\pm(1-u^2)^{1/2}\Theta - X] \times \phi_0(\Theta)\,d\Theta,$$

and where $X = X(t) \equiv \int_0^t u(t')\,dt' + x_0$ defines the center of the fluxon as a function of time. This form of equation (4.1a) emphasizes the particle nature of the fluxon. Its acceleration (caused by the perturbation f) is governed by a form of Newton's equation with an effective potential given by the external perturbation averaged over the fluxon itself.

Second, consider (4.1b). Note that the right-hand side vanishes if the perturbation $f[\phi_0(x,t),x,t]$ is an even function of Θ. In this case, $\dot{x}_0 = 0$ and the energetic analysis of Sec. II is entirely equivalent to the first stage of the perturbation analysis. However, with the general perturbation presented in (2.2), the terms

$$\sum_i \mu_i\delta(x - a_i)\sin\phi$$

(which represent the presence of microshorts of magnitude μ_i located at $x = a_i$) destroy the even symmetry of f. The effects of these shorts will now be discussed.

B. Pinning by microshorts

Consider first a single microshort located at the origin ($x = 0$) with maximum Josephson current μ. Then the perturbation becomes

$$\epsilon f = -\alpha\phi_t + \beta\phi_{xxt} - \gamma - \mu\delta(x)\sin\phi.$$

Assuming this perturbation to act on a single fluxon, we define the location of the center of the fluxon as

$$X \equiv \int_0^t u(t')\,dt' + x_0.$$

Thus,

$$\dot{X} = u + \dot{x}_0,$$

and, after performing the integrations over this perturbation, (4.1) becomes an autonomous set of ordinary differential equations in the variables $u(t)$ and $X(t)$:

$$\frac{du}{dt} = \pm\tfrac{1}{4}\pi\gamma(1-u^2)^{3/2} - \alpha u(1-u^2) - \tfrac{1}{3}\beta u + \tfrac{1}{2}\mu(1-u^2) \times \operatorname{sech}^2\left(\frac{X}{(1-u^2)^{1/2}}\right)\tanh\left(\frac{X}{(1-u^2)^{1/2}}\right), \tag{4.3a}$$

$$\frac{dX}{dt} = u - \tfrac{1}{2}\mu X u \times \operatorname{sech}^2\left(\frac{X}{(1-u^2)^{1/2}}\right)\tanh\left(\frac{X}{(1-u^2)^{1/2}}\right). \tag{4.3b}$$

These equations are conveniently analyzed in the (u,X) phase plane. Singular points (where $\dot{u} = 0$ and $\dot{X} = 0$) are found at

$$u = 0 \quad \text{and} \quad X = X_0,$$

where X_0 is a root of

$$\frac{\pi\gamma}{2\mu} + \operatorname{sech}^2(X_0)\ \tanh(X_0) = 0. \tag{4.4}$$

Consider some representative numerical inte-

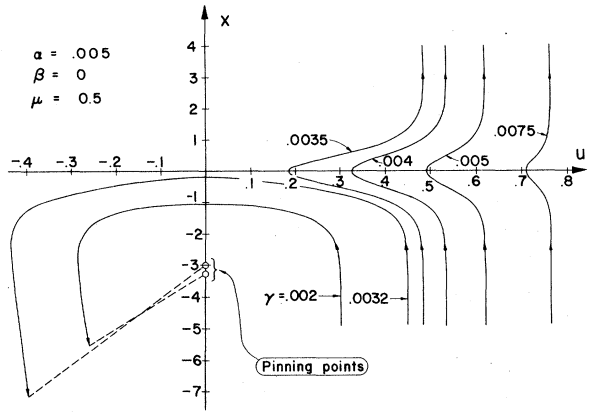

FIG. 5. Fluxon trajectories in the (u,X) phase plane from integration of (4.5) with $\alpha=0.005$, $\beta=0$, $\mu=5$.

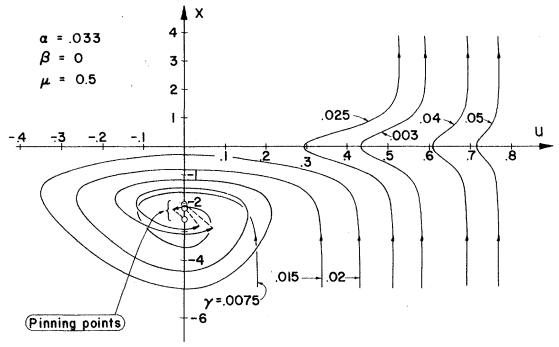

FIG. 6. Same as Fig. 5 but $\alpha=0.033$.

grations of (4.3). As an initial condition, we assume the fluxon to be approaching the microshort from large negative values of X. Then $\mathrm{sech}^2[X/(1-u^2)^{1/2}]\approx 0$, and the fluxon dynamics is well described by the energetic discussion of Sec. II and, in particular, (2.10). Thus, it is reasonable to assume that the fluxon approaches the microshort at the power-balance velocity u_∞. In Fig. 5 we take $\alpha=0.005$ and $\beta=0$, which corresponds to the dissipation in the JTL's studied experimentally in Ref. 2. The maximum Josephson current of the microshort μ is rather arbitrarily chosen to be 0.5, and the bias current γ is assumed to be under the adjustable control of an experimenter. If γ (therefore u_∞) is sufficiently large, the fluxon slows in the vicinity of the microshort but does not stop. If γ (or u_∞) is small enough, the fluxon will lose all its kinetic energy before passing the short; in this case it is "reflected" and eventually oscillates inward to the "pinning point" or singular point at $(u,X)=(0,X_0)$. Assuming that the energy dissipated (by the α and β terms) in the collision with the microshort is a negligible fraction of the incoming kinetic energy of the fluxon, the maximum value of γ for pinning is readily calculated using the energetic considerations of Sec. II. From (2.6), the rest energy of a fluxon is 8; thus, the kinetic energy E_{f1} of a fluxon incident on the microshort with velocity u_∞ is

$$E_{f1}=8\,\frac{1}{(1-u_\infty^2)^{1/2}}-1\ .$$

From (2.4b), the maximum energy absorbed by a microshort is 2μ. Equating these two energies and using (2.11) for u_∞ gives the maximum bias current for pinning as

$$\gamma_0=\frac{\alpha}{\pi}(8\mu+\mu^2)^{1/2}. \tag{4.5}$$

For the values of α and μ in Fig. 5, this indicates $\gamma_0=0.003\,28$ which is in good agreement with the observed value.

For the computations plotted in Fig. 6, the dissipation factor α has been increased to 0.033. The critical value of γ for pinning, calculated from (4.5), is now 0.0217, again in substantial agreement with the value observed. Thus it is interesting to compare the numerical trajectories in Fig. 6 with an analytical solution obtained by assuming $\alpha,\beta,\gamma=0$. In this case, (4.3) implies

$$\frac{d}{dt}\left(\frac{X}{(1-u^2)^{1/2}}\right)=\frac{u}{(1-u^2)^{1/2}},$$

which with (4.3a) can be integrated to obtain

$$X=\pm(1-u^2)^{1/2}\,\mathrm{sech}^{-1}\left[\frac{4}{\mu}\left(\frac{1}{(1-u_\infty^2)^{1/2}}-\frac{1}{(1-u^2)^{1/2}}\right)\right]^{1/2} \tag{4.6}$$

as the corresponding solution trajectory in the (u,X) phase plane. If the dissipative terms α,β,γ are not all zero, (4.6) remains a useful approximation as long as the fractional gain or loss of fluxon energy is not too large. For example, in Fig. 7 we compare a trajectory obtained by integrating (4.5) with $\alpha=0.033$, $\beta=0$, $\gamma=0.222$, and $\mu=0.5$ (solid line) with the approximate trajectory computed from (4.6) (dashed line).

From the computations plotted in Fig. 6, it is evident that fluxons reflected by the microshort undergo damped oscillations about the pinning points where they eventually come to rest. To study this oscillation analytically, we linearize (4.3) about the pinning point $(0,X_0)$. Assuming linear oscillation of the form.

$$u,(X-X_0)\sim \mathrm{Re}[\exp(st)],$$

the characteristic frequency is

$$s\cong -\tfrac{1}{2}(\alpha+\tfrac{1}{3}\beta)\pm i[\Omega^2-\tfrac{1}{4}(\alpha+\tfrac{1}{3}\beta)^2]^{1/2}, \tag{4.7}$$

where

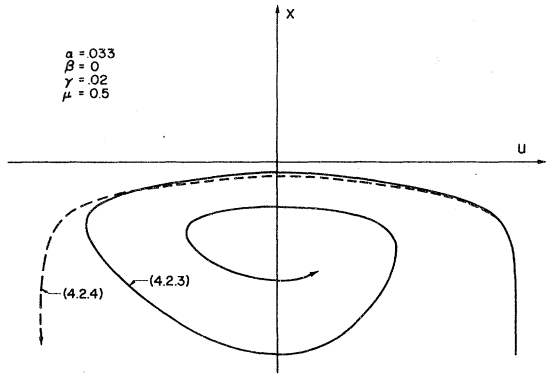

FIG. 7. Comparison of a pinning trajectory obtained by numerical integration of (4.5) (solid line) with the approximate trajectory computed from (4.6) (dashed line).

$$\Omega^2 \equiv \tfrac{1}{2}\mu(1+\tfrac{1}{4}\pi\gamma X_0)[2\,\mathrm{sech}^2(X_0)\tanh^2(X_0) - \mathrm{sech}^4(X_0)].$$

Since X_0, through (4.4), is a function only of μ and γ, the same is true of Ω. Figure 8 is presented to clarify this function. As $\gamma \to 0$, $\Omega^2 \sim \frac{1}{2}\pi\gamma$; and, as γ is increased, a critical value ($\gamma_c = 4\mu/\pi 3\sqrt{3}$) is reached at which Ω^2 falls to zero. From a different perspective, γ_c is the value of bias current above which (4.4) has no roots so pinning cannot occur under any conditions. From Fig. 8 it is evident that the oscillation frequency remains less than unity with the value of μ (0.5) chosen for the computations in Figs. 5 and 6. This is an important point for the applicability of the simple perturbation theory. Since the frequency of small-amplitude radiation, from (1.4), is greater than unity, no radiative coupling to the pinning oscillation will occur. Thus the "Q" of this oscillation will be $\Omega/(\alpha+\beta/3)$ as is indicated in (4.7).

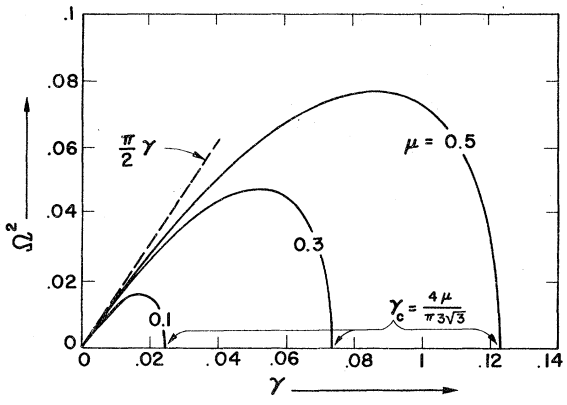

FIG. 8. Pinning oscillation frequency (squared) vs γ.

C. Quantum flux shuttle

We are now in a position to consider the "quantum flux shuttle" that has been proposed by Fulton, Dynes, and Anderson[3] as a Josephson-junction version of a shift register. This device consists of a normal (hyperbolic) transmission line that is periodically loaded with discrete Josephson junctions. Fluxons can remain stationary ("pinned") at locations between the discrete junctions. When a pulse of current bias is applied with the appropriate amplitude and duration, each fluxon is translated one period along the line.

The qualitative features of this device can be modeled by considering the case $\alpha \neq 0, \beta \neq 0, \gamma = \gamma(t)$, and taking the microshorts to be discrete Josephson junctions of equal spacing and equal strength. Then (4.3) is augmented to

$$\frac{du}{dt} = \tfrac{1}{4}\pi\gamma(1-u^2)^{3/2} - \alpha u(1-u^2) - \tfrac{1}{3}\beta u + \tfrac{1}{2}\mu(1-u^2)\sum_n \mathrm{sech}^2\left(\frac{X-na}{(1-u^2)^{1/2}}\right) \times \tanh\left(\frac{X-na}{(1-u^2)^{1/2}}\right), \qquad (4.8a)$$

$$\frac{dX}{dt} = u - \tfrac{1}{2}\mu u \sum_n (X-na)\,\mathrm{sech}^2\left(\frac{X-na}{(1-u^2)^{1/2}}\right) \times \tanh\left(\frac{X-na}{(1-u^2)^{1/2}}\right), \qquad (4.8b)$$

where, as before, X locates the center of the fluxon, and $n = 0, \pm 1, \pm 2, \ldots$. With bias current γ equal to zero, the equilibrium position for a fluxon is just half-way between the concentrations of Josephson current, i.e., at

$$X_n = (n - \tfrac{1}{2})a .$$

Thus (4.8) has a periodic array of singular points located at $(u, X) = (0, X_n)$.

Suppose a fluxon is resting at one of these singular points (say $X_0 = -\frac{1}{2}a$) when a bias-current pulse

$$\gamma(t) = A\delta(t)$$

is applied. Then at $t = 0^+$, u jumps from zero to an initial value

$$u_0 = \left[1 + \left(\frac{4}{A}\right)^2\right]^{-1/2},$$

after which (4.8) evolves with $\gamma = 0$. Note that the systems cannot operate properly as a quantum flux shuttle in the dissipationless limit; $\alpha, \beta = 0$. The reason for this failure is that if u_0 is large enough for the fluxon to get by the first junction (at $x = 0$), then the fluxon has enough energy to pass them all. In a shift register, however, it is

required that $\gamma(t)$ move the fluxon just one step along the periodically loaded JTL.

If α or β is not zero, the area A of the stepping pulse $\gamma(t)$ can be increased so the fluxon has enough energy to get by the first junction but dissipative losses prevent it from passing the next. In the "nonrelativistic" limit ($A \ll \frac{1}{4}\pi$), $\dot{u} \approx -(\alpha + \frac{1}{3}\beta)u$, and the fluxon spends a time of the order a/u_0 in moving a step. Thus $\Delta u \approx -(\alpha + \frac{1}{3}\beta)a$ and the allowable tolerance on A will be of the order

$$\Delta A \sim \frac{4}{\pi}(\alpha + \tfrac{1}{3}\beta)a . \tag{4.9}$$

This tolerance is relaxed as the spacing a between discrete Josephson junctions is increased. But a cannot be made too large or it will become energetically possible for *two* fluxons to remain stationary at one location of the shift register. To investigate the design requirement that only one fluxon be able to rest between any two discrete Josephson junctions, we consider the static case ($\phi_t = 0$). Then setting $\gamma(t) = 0$ yields the ordinary differential equation

$$\frac{d^2\phi}{dx^2} = \sin\phi \left(1 + \mu \sum_n \delta(x - na)\right) . \tag{4.10}$$

More generally, the quantum-flux-shuttle designer would be interested in solutions of $\phi'' = F(x)\sin\phi$, where $F(x) = F(x + a)$ is an accurate representation of the Josephson current. From this point of view, a further simplification of (4.10) to

$$\frac{d^2\phi}{dx^2} = \mu \sin\phi \sum_n \delta(x - na) \tag{4.10′}$$

is also a more-realistic model of an actual quantum flux shuttle.[3] Solutions of (4.10′) are readily constructed as "piecewise linear" functions which change slope by $\mu \sin\phi$ at the points $x = na$. Such a solution for which ϕ changes by 3π as x increases from 0 to a corresponds to the trapping of two fluxons and requires $\mu a = \frac{5}{2}\pi$. Thus,

$$\mu a < \tfrac{5}{2}\pi$$

is a design condition to avoid the resting of two fluxons at one location of the shift register.

Thus we see that $a \sim 1$ is a reasonable choice. If $a \ll 1$, the energy minimum between discrete Josephson junctions is not deep enough; if $a \gg 1$, two or more fluxons can be stored between junctions without energetic interaction. Thus, the tolerance on the area of a pulse $\gamma(t)$ will be of the order of the dissipation factor $\alpha + \frac{1}{3}\beta$. From (4.7), we see that setting $\alpha + \frac{1}{3}\beta \sim 0.3$ (say) will also ensure that the stepped fluxon settles rapidly to its stationary position.

V. FLUXON INTERACTIONS

A. Fluxon-antifluxon collisions

In their design of a Josephson computer, Nakajima, Onodera, and Ogawa[7] use two types of JTL: (i) *destrutive line* on which a fluxon and an antifluxon will annihilate each other upon collision, and (ii) *nondestructive line* on which a fluxon and an antifluxon will pass through each other after collision. In a similar way, the line terminations are divided into (i) *reflectionless*, and (ii) *reflective* varieties according to whether an incidnet fluxon disappears at the termination or is reflected as an antifluxon. Since the boundary condition at a termination ($\phi_x = 0$) can be satisfied by assuming collision with a virtual antifluxon, the critical values of line parameters that distinguish between destructive and nondestructive lines are identical to those for reflectionless and reflective terminations. Analytical control of the fluxon-antifluxon annihilation is therefore of fundamental importance for the design of JTL computer components.

To begin the perturbation analysis of fluxon-antifluxon annihilation, note that an exact two-soliton solution of (1.1) is given by the doublet solution of (1.5) in the special case of constant u, x_0, and t_0. This represents a nondestructive fluxon-antifluxon collision.

Suppose now that $\alpha \neq 0$ and $\gamma \neq 0$ so the nonlinear partial differential equation becomes

$$\phi_{tt} - \phi_{xx} + \sin\phi = -\alpha\phi_t - \gamma . \tag{5.1}$$

For the perturbation expansion $\vec{W} = \vec{W}_0 + \alpha\vec{W}_1 + \cdots$, an appropriate form for the slow modulation of the doublet is

$$\phi_0 = 4\tan^{-1}\frac{\sinh\{[\int_0^t u(t')\,dt' + x_0(t)]/(1-u^2)^{1/2}\}}{u\cosh\{[x - x_1(t)]/(1-u^2)^{1/2}\}}$$

together with a corresponding formula for the second component of $\vec{W}_0$.

This expression has only three independent parameters (u, x_1, and x_0), rather than the four which might be expected for a two-soliton solution, because it contains the assumption that the magnitudes of the fluxon and antifluxon velocities are always equal. (This is necessarily true at a termination where the antifluxon is "virtual.") If the magnitudes of the fluxon and antifluxon velocities are initially equal, they remain so since that symmetry is preserved by this perturbation.

The first-order perturbation, $\vec{W}_1 = \mathrm{col}(\phi_1, \phi_{1,t})$ is governed by the effective source (where again we write $\phi = \phi_0$ for typographical convenience)

$$\vec{\mathfrak{F}}(\vec{W}_0)=\begin{pmatrix} -\frac{1}{\alpha}\phi_u\dot{u}+\frac{1}{\alpha}\phi_x\dot{x}_1-\frac{1}{\alpha}\phi_{x_0}\dot{x}_0 \\ -\phi_t-\frac{\gamma}{\alpha}-\frac{1}{\alpha}\phi_{ut}\dot{u}+\frac{1}{\alpha}\phi_{xt}\dot{x}_1-\frac{1}{\alpha}\phi_{x_0t}\dot{x}_0 \end{pmatrix}$$

Three independent elements in $\mathfrak{N}(L^\dagger)$ are found by differentiating ϕ_0 with respect to x, t, and u. These are

$$\vec{b}_1=\begin{pmatrix}\phi_{0,xt}\\ -\phi_{0,x}\end{pmatrix}, \quad \vec{b}_2=\begin{pmatrix}\phi_{0,tt}\\ -\phi_{0,t}\end{pmatrix},$$

and

$$\vec{b}_3=\begin{pmatrix}\phi_{0,ut}\\ \phi_{0,u}\end{pmatrix}.$$

If the parameter x_1 is chosen to be zero, the orthogonality condition $(\vec{\mathfrak{F}},\vec{b}_1)$ is satisfied immediately. After eliminating terms that are zero, the other two orthogonality conditions can be written

$$\left(\int_{-\infty}^{\infty}(\phi_{tt}\phi_u-\phi_{ut}\phi_t)\,dx\right)\dot{u} = \int_{-\infty}^{\infty}(\alpha\phi_t^2+\gamma\phi_t)\,dx, \quad (5.2a)$$

and

$$\left(\int_{-\infty}^{\infty}(\phi_{ut}\phi_{x_0}-\phi_{x_0t}\phi_u)\,dx\right)\dot{x}_0 = \int_{-\infty}^{\infty}(\alpha\phi_u\phi_t+\gamma\phi_u)\,dx, \quad (5.2b)$$

with $\phi=\phi_0$. Just as in Sec. IV A, (5.2a) can be written in the form of a power-balance condition

$$\dot{u}\frac{dH(\phi_0)}{du}=-\int_{-\infty}^{\infty}(\alpha\phi_{0,t}^2+\gamma\phi_{0,t})\,dx.$$

Here

$$H(\phi_0)=16(1-u^2)^{-1/2}$$

since the asymptotic form of ϕ_0 is a free fluxon and antifluxon each with energies $8(1-u^2)^{-1/2}$.

Thus, $\dot{u}$ can be evaluated to be

$$\frac{du}{dT}=F(\gamma,\alpha,u,T), \quad (5.3)$$

where

$$F\equiv\gamma\frac{\pi(1-u^2)^{3/2}\cosh T}{4(\sinh^2T+u^2)^{1/2}}-\alpha\frac{u^3(1-u^2)\cosh^2T}{\sinh^2T+u^2} \times\left(\frac{1}{u^2}-\frac{\ln|u^{-1}\ [(\sinh^2T+u^2)^{1/2}-\sinh T]|}{\sinh T(\sinh^2T+u^2)^{1/2}}\right)$$

and

$$T\equiv\frac{\int_0^t u(t')\,dt'+x_0(t)}{(1-u^2)^{1/2}}.$$

From this definition of T,

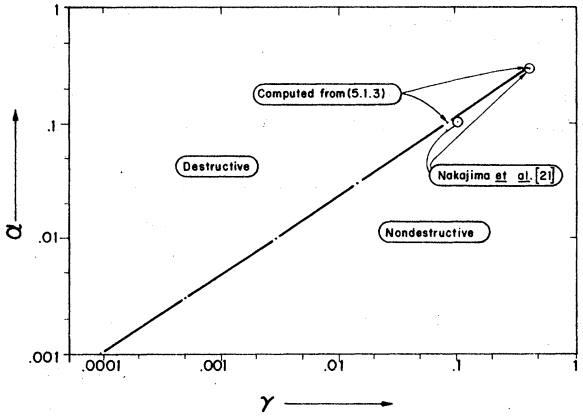

FIG. 9. The critical values of γ and α which separate nondestructive collisions from destructive ones (fluxon-antifluxon annihilation).

$$\frac{dT}{dt}=\frac{u+\dot{x}_0}{(1-u^2)^{1/2}}$$

and analysis of (5.2b) shows that, as $u\to 0$, $\dot{x}_0 \ll \alpha u$. Thus to zeroth order in α, $\dot{T}=u/(1-u^2)^{1/2}$ and to first order in α

$$\frac{du}{dT}=\frac{(1-u^2)^{1/2}}{u}F.$$

This equation has been numerically integrated from the initial condition $u=u_\infty$ at $T=-\infty$, where u_∞ is the power-balance velocity of an isolated fluxon given in (2.11). Of particular interest are the critical values of γ and α which separate nondestructive from destructive collisions. This critical situation is determined, for a fixed value of α, as the maximum value of γ for which the velocity parameter u falls to zero. For all larger values of γ, the velocity parameter will remain positive and eventually rise again to the power-balance velocity, u_∞. The critical locus is plotted on the γ-α plane in Fig. 9. Comparison is made with the approximate values obtained by Nakajima *et al.*[21] from direct integration of the full partial differential equation (5.1). Evidently the agreement between these two calculations is good, even for rather large values of α, where the perturbation approach could be expected to become inaccurate.

B. Breather decay

In our description of a nondestructuve fluxon-antifluxon collision, the velocity parameter u falls to some minimum value and then rises again to u_∞. If $u^2<0$, however, the fluxon and antifluxon become bound together in a "breather" state described by (1.6). Note in particular that the energy of a breather is

$$H_B=16\sin\nu$$

and that (1.5) is obtained from (1.6) by substituting

$$\nu = \sin^{-1}\left(\frac{1}{(1-u^2)^{1/2}}\right). \tag{5.4}$$

Thus when $u^2 = 0$ and $\nu = \frac{1}{2}\pi$, the total energy is equal to the rest energy of a fluxon and an antifluxon (i.e., 16). For $u^2 < 0$, $\nu < \frac{1}{2}\pi$ and $H_B < 16$.

Under the perturbations indicated in (5.1), ν will relax from $\frac{1}{2}\pi$ to zero as $t \to \infty$. The first-order ordinary differential equation governing $\nu(t)$ is obtained from the perturbation analysis, or from substitution of (5.4) into (5.3):

$$\frac{d\nu}{dt} = \pi\gamma \frac{\tan\nu \cos[(\cos\nu)t]}{\{\tan^2\nu \sin^2[(\cos\nu)t] + 1\}^{1/2}} - \alpha \frac{\tan\nu \cos^2[(\cos\nu)t]}{\tan^2\nu \sin^2[(\cos\nu)t] + 1} \times\left(1 - \frac{\ln|\{\tan^2\nu \sin^2[(\cos\nu)t] + 1\}^{1/2} - \tan\nu \sin[(\cos\nu)t]|}{\tan\nu \sin[(\cos\nu)t]\{\tan^2\nu \sin^2[(\cos\nu)t] + 1\}^{1/2}}\right). \tag{5.5}$$

The dynamics of breather decay is important for the design of a JTL computing system because the residual energy of breathers may be an important source of computer noise.

As $\nu \to 0$, (5.5) becomes approximately

$$\frac{d\nu}{dt} \cong \tfrac{1}{4}\pi\gamma \cos t - 2\alpha\nu \cos^2 t .$$

Averaging over a period of the breather oscillation then gives

$$\frac{d\langle\nu\rangle}{dt} \cong -\alpha\langle\nu\rangle .$$

Thus, the mean free time of a breather is approximately α^{-1}. If R breathers form per unit time and per unit length of the JTL, the resulting noise-energy density will be approximately $16R/\alpha$.

This estimate has ignored loss of breather energy due to radiation into the continuum of periodic waves, indicated in Fig. 2(d). Since the lowest frequency of small-amplitude radiation, from (1.4), is $\omega = 1$, and the frequency of a stationary breather ($\cos\nu$) is always less than unity, we expect such radiative interaction to become important only as $\nu \to 0$. But in this limit, the radiation is governed by the linear partial differential equation $\nu_{xx} - \nu_{tt} - \nu = \alpha\nu_t$ which is damped at the same rate as the breather. Thus, although some of the breather energy may be transformed into radiation field energy, the total background energy (i.e., noise) should not be affected.

C. Flux bundles

Consider the perturbed sine-Gordon equation

$$\phi_{tt} - \phi_{xx} + \sin\phi = \beta\phi_{xxt} - \alpha\phi_t - \gamma . \tag{5.6}$$

It has been known for some time that this equation supports traveling-wave solutions that consist of "congealed" states in which an arbitrary number of fluxons can participate. The stability of some of these solutions has been established by direct integration of the partial differential equation and also from measurements on a mechanical analog.[21]

In the context of the present discussion, it is interesting to inquire what a solition perturbation analysis can add to our quantitative knowledge of the properties of these states. The simplest case involves two fluxons for which a corresponding unperturbed solution of (1.1) can be written,

$$\phi_2 = 4\tan^{-1}\left\{\left[1 - \left(\frac{1 - u_1u_2 - [(1-u_1^2)(1-u_2^2)]^{1/2}}{1 - u_1u_2 + [(1-u_1^2)(1-u_2^2)]^{1/2}}\right)\exp\left(-\frac{x - x_1 - u_1t}{(1-u_1^2)^{1/2}} - \frac{x - x_2 - u_2t}{(1-u_2^2)^{1/2}}\right)\right] \Big/ \left[\exp\left(-\frac{x - x_1 - u_1t}{(1-u_1^2)^{1/2}}\right) + \exp\left(-\frac{x - x_2 - u_2t}{(1-u_2^2)^{1/2}}\right)\right]\right\} .$$

There are now four independent components of the null space of $L^{\dagger}$ which are obtained by differentiating ϕ_2 with respect to the four parameters u_1, u_2, x_1, x_2. The corresponding four orthogonality conditions can then be written as the system

$$A\begin{pmatrix}\dot{u}_1\\ \dot{u}_2\\ \dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}\left(\left[\phi_{0,t}-\frac{\beta}{\alpha}\phi_{0,xxt}-\frac{\gamma}{\alpha}\right],\phi_{0,u_1}\right)\\ \left(\left[\phi_{0,t}-\frac{\beta}{\alpha}\phi_{0,xxt}-\frac{\gamma}{\alpha}\right],\phi_{0,u_2}\right)\\ \left(\left[\phi_{0,t}-\frac{\beta}{\alpha}\phi_{0,xxt}-\frac{\gamma}{\alpha}\right],\phi_{0,x_1}\right)\\ \left(\left[\phi_{0,t}-\frac{\beta}{\alpha}\phi_{0,xxt}-\frac{\gamma}{\alpha}\right],\phi_{0,x_2}\right)\end{pmatrix}, \tag{5.7}$$

where $\underline{A}$ is a 4×4 matrix of appropriate inner products of the null-space components.

The congealed state of two fluxons owes its existence to the β term; if this β term were absent, no congealed state would exist. For fixed values of (α,β,γ), the congealed state of two fluxons travels with a fixed speed $u_c=u_c(\alpha,\beta,\gamma)$, with its two fluxons at fixed separation $x_1-x_2=x_c(\alpha,\beta,\gamma)$. The entire state possesses a translational degree of freedom. In this perturbation analysis, the special values for the speed and location presumably arise as equilibrium solutions of (5.7), that is, as those special values of $u_1=u_2=u_c$ and of x_1-x_2 which make the right-hand side of (5.7) vanish. Clearly, these equilibrium solutions are functions of the parameters (α,β,γ). Presumably no equilibrium solution exists if $\beta=0$.

As a computational procedure to establish the existence and properties of the congealed state, this perturbation approach must be compared to a direct traveling-wave analysis of (5.6) in which it is assumed that $\phi=\phi_T(x-ut)$. Then the partial differential equation (5.6) becomes the third-order ordinary differential equation

$$-\beta u\phi_T'''+\phi_T''(1-u^2)+\alpha u\phi_T'-\sin\phi_T-\gamma=0. \tag{5.8}$$

Since the phase-space dimension for (5.8) is 3 when $\beta\neq0$ and only 2 when $\beta=0$, elementary topological (i.e., continuity or "shooting") arguments confirm that $\beta\neq0$ is a necessary condition for fluxons to congeal into traveling wave bundles.[22,23] Numerically, it appears more attractive to deal with (5.8) than the right-hand side of (5.7) for several reasons: (i) (5.8) is exact while (5.7) is an order-α approximation. (ii) Four parameters must be adjusted in (5.7) while only one (u) is needed in (5.8). (iii) More fluxons in the congealed state add no difficulty to (5.8), but the right-hand side of (5.7) has $2N$ components for N fluxons.

Although the traveling-wave analysis of (5.8) is superior to the perturbation theory for establishing the existence and shape of the congealed states, perturbation methods should be useful to understand the stability of congealed states and, in particular, to describe time-dependent transitions from unstable to stable congealed states. Numerical analysis of (5.8) shows that for an N-fluxon congealed state $(N>1)$ there is a critical value of γ below which the state does not exist. As γ is increased above this critical value, two traveling-wave solutions for (5.8) appear. Numerical study of the partial differential equation (5.6) indicates that the faster branch (A in Fig. 10) is stable and the slower branch is unstable. This observation is to be expected from a simple energy argument. From (2.8) it is clear that along the traveling-wave locus indicated in Fig. 10,

$$2\pi N\gamma=u\int_{-\infty}^{\infty}(\alpha\phi_x^2+\beta\phi_{xx}^2)\,dx. \tag{5.9}$$

If γ is increased above this value, $\dot{H}>0$ and u will increase. This carries a fluxon bundle away from branch B and toward branch A. If γ is decreased below the value in (5.9), $\dot{H}<0$. Then u will decrease again moving the bundle away from branch B and toward branch A. Thus branch B should be unstable and branch A, stable. The perturbation theory provides a method to investigate the time-dependent, nonequilibrium transition from an unstable to stable branch of this bifurcation diagram. At the time of this writing we have not

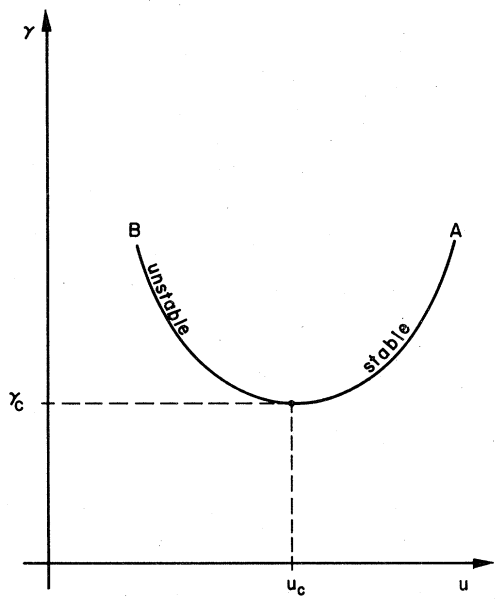

FIG. 10. Traveling-wave locus near a critical point for a congealed fluxon state.

done this calculation, but it should proceed from (5.2) using methods similar to those described recently by Newell.[24]

VI. A GREEN'S FUNCTION FOR FLUXON RADIATION

Our next task is the computation of the radiation generated by a multifluxon waveform as it responds to the perturbation. The primary tool in this calculation is a Green's function $\mathcal{G}$ for the linear perturbation equation (3.5). In this section, we present explicit constructions of $\mathcal{G}$; in Sec. VII, we describe a typical application of this Green's function to compute the radiation emitted by a fluxon. In addition to such analyses of the structure of the radiation field, this Green's function provides a systematic derivation of the orthogonality conditions (3.9) that we have already used to compute the fluxon's response to a perturbation, and permits us to check the accuracy of these computations.

A few comments about the organization of this section seem appropriate. The most systematic construction of $\mathcal{G}$ employs direct and inverse scattering theory. This theoretical method provides a complete set of functions (together with orthogonality relations) that is a natural basis for the construction of $\mathcal{G}$. This representation of $\mathcal{G}$ is valid for any zeroth-order solution, although it is not very explicit unless the zeroth-order wave is a pure multifluxon waveform. In this special case, which is our problem, an alternative construction of $\mathcal{G}$ can be given using Bäcklund transformations. This derivation minimizes the use of scattering theory and has other advantages that are described in the text. Thus, in this section we construct $\mathcal{G}$ with the Bäcklund transformations, and present its systematic construction through scattering theory in the Appendix.

Many of the formulas of direct scattering theory that we present in the Appendix are new, although their derivations are natural extensions of work of Kaup and Newell[15-17] for the "Zakharov-Shabat eigenvalue problem" to an eigenvalue problem of Takhtadjian and Fadeev.[12] The theoretical representation of $\mathcal{G}$ derived in the Appendix extends work of Keener and McLaughlin[14,25] to the sine-Gordon equation. The use of Bäcklund transformations to construct the Green's function is new.

A. Definition of the Green's function

The solution of the linear inhomogeneous equation (3.5) for the first-order correction $\vec{\mathcal{W}}$ can be represented in terms of a matrix Green's function:

$$\vec{\mathcal{W}} = \int_0^t \int_{-\infty}^{\infty} \mathcal{G}(x,t|x',t')\vec{\mathcal{F}}(x',t')\,dx'dt', \tag{6.1}$$

where the matrix kernel $\mathcal{G}(x,t|x',t')$ is defined by the homogeneous initial-value problem in (x,t) coordinates,

$$\begin{pmatrix} \partial_t & -1 \\ -\partial_{xx} + \cos\phi_0 & \partial_t \end{pmatrix} \mathcal{G}(x,t|x',t') = 0, \qquad t > t' \geq 0 \tag{6.2a}$$

$$\lim_{t\to t'} \mathcal{G}(x,t|x',t') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x-x'). \tag{6.2b}$$

Equivalently, $\mathcal{G}^T$ satisfies an adjoint problem in (x',t') coordinates,

$$-\begin{pmatrix} \partial_{t'} & \partial_{x'x'} - \cos\phi_0 \\ 1 & \partial_{t'} \end{pmatrix} \mathcal{G}^T(x,t|x',t') = 0, \quad t > t' \geq 0$$

$$\lim_{t'\to t} \mathcal{G}^T(x,t|x',t') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(x-x').$$

Thus, the columns of the matrix $\mathcal{G}$, as functions of (x,t), belong to the null space of L, $\mathfrak{N}(L)$, while the rows of $\mathcal{G}$, as functions of (x',t'), belong to the null space of $L^\dagger$, $\mathfrak{N}(L^\dagger)$. To find a representation of the Green's function $\mathcal{G}$ we must study these two null spaces.

In our application, the zeroth-order wave $\vec{\phi} = \vec{W}_0$ describes a pure $N+M+2L$-fluxon waveform with speed and location parameters $\{p_j,\ j=1, 2, \ldots 2(N+M+2L)\}$ that modulate in response to the perturbation ϵf. The first-order correction $\vec{\mathcal{W}}$ describes the radiation field that is created by these modulating fluxons. Since the linear operator L arises from linearization of the nonlinear equation about the multifluxon waveform $\vec{W}_0$, some members of the null spaces $\mathfrak{N}(L)$ and $\mathfrak{N}(L^\dagger)$ can be generated by differentiating $\vec{W}_0$ with respect to each of its velocity and location parameters $\{p_j,\ j=1,2,\ldots,2(N+M+2L)\}$. Clearly, this finite collection of parameters cannot generate all members of these infinite-dimensional null spaces; we must enlarge this collection of parameters into an infinite set that generates a basis for each null space. With these bases, an explicit representation of the Green's function $\mathcal{G}$ can be constructed.[14,25]

B. Representation of $\mathcal{G}$

In general, $\vec{\phi}$ contains $N+M+2L$ fluxons together with a continuum of radiation at all wavelengths. Just as we can differentiate $\vec{\phi}$ with respect to its $2(N+M+2L)$ fluxon parameters, we can differentiate it with respect to the amplitude of its radiation components. In this manner we

generate an infinite number of members of $\mathfrak{N}(L)$,

$$S=\left(\frac{\partial\vec{\phi}(x,t)}{\partial p_j},\ j=1,2,\ldots,2(N+M+2L);\right.$$
$$\left.\frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,0)},\ -\infty<\lambda<+\infty\right).$$

Here $\rho_+(\lambda,0)$ is the initial value (at $t=0$) of a certain "forward-reflection coefficient" that characterizes the radiation density at wave number $k(\lambda)=2\lambda-1/8\lambda$. ρ_+ is precisely defined in (A4) of the Appendix.

In our application, $\vec{\phi}=\vec{W}_0$, a pure multisoliton waveform with no radiation, and $\rho_+(\lambda,0)$ is not explicitly available in $\vec{W}_0$ for differentiation. Thus the derivative $\delta\vec{\phi}/\delta\rho_+$ must be computed before ρ_+ is set at zero. One procedure for this calculation is presented in Sec. VI C, and an alternative method in the Appendix.

The set S includes the discrete components previously computed plus an infinite number of radiative components that depend upon the parameter λ. This set is complete (for functions of x at fixed t); it spans $\mathfrak{N}(L)$ and can be used to construct the Green's function $\mathcal{G}$. The discrete components span the $2(N+M+2L)$-dimensional subspace of $\mathfrak{N}(L)$ that we have earlier called the "discrete subspace." The "continuous subspace" of $\mathfrak{N}(L)$, $\mathfrak{N}_c(L)$, is defined as the span of $\{\delta\phi/\delta\rho_+(\lambda,0),\ -\infty<\lambda<+\infty\}$.

Since the set s spans $\mathfrak{N}(L)$, we seek a representation of the Green's function $\mathcal{G}$ in the form

$$\mathcal{G}(x,t|x',t')=\mathcal{G}_c(x,t|x',t')+\mathcal{G}_d(x,t|x',t'),$$

where

$$\mathcal{G}_c(x,t|x',t')=\int_{-\infty}^{\infty}\frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,0)}\vec{A}^T(x',t';\lambda)\,d\lambda$$

and

$$\mathcal{G}_d(x,t|x',t')=\sum_{j=1}^{2(N+M+2L)}\frac{\partial\vec{\phi}(x,t)}{\partial p_j}\vec{A}_j^T(x',t').$$

Here the expansion coefficients $\{\vec{A}(x',t';\lambda),\ \vec{A}_j(x',t')\}$ are to be found. Note that the expansion is guaranteed to satisfy the differential equation (6.2a) since the columns (as functions of x and t) belong to $\mathfrak{N}(L)$. The expansion coefficients must be selected so the initial data is satisfied. This selection can be made since the set $\{\partial\vec{\phi}/\partial p_j,\ \delta\vec{\phi}/\delta\rho_+(\lambda,0)\}$ is complete.

In the Appendix, we derive formulas for these expansion coefficients. In particular, the continuous component of $\mathcal{G}$ is given by[26]

$$\mathcal{G}_c(x,t|x',t')=-\tfrac{1}{4}\pi i\int_{-\infty}^{\infty}\frac{\lambda}{a^2(\lambda)}\left(\frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,0)}\right)\times\left(\frac{\delta\vec{\phi}(x',t')}{\delta\rho_-(\lambda,0)}\right)^T J\,d\lambda. \quad (6.3)$$

In this formula, $\rho_-(\lambda)$ denotes a "backward-reflection coefficient" which is precisely defined in (A4) of the Appendix, and should be viewed as a different characterization of the radiation density in $\vec{\phi}$. The integration density $[a(\lambda)]^{-2}$ is the square of a transmission coefficient which is defined in the Appendix. Its magnitude is related to $\rho_+(\lambda)$ by the formula

$$|a(\lambda)|^{-2}=1+|\rho_+(\lambda)|^2.$$

In the case when $\vec{\phi}$ is a pure $N+M+2L$-fluxon state, $a(\lambda)$ is given explicitly by

$$a(\lambda)=\prod_{j=1}^{N+M+2L}\left(\frac{\lambda-\zeta_j}{\lambda-\zeta_j^*}\right), \quad (6.4)$$

where the complex parameters $\{\zeta_j, j=1,2,\ldots,N+M+2L\}$ are equivalent to the "velocity parameters" $\{p_j,\ j=1,2,\ldots,N+M+2L\}$. The ζ_j must be pure imaginary or must lie in pairs (ζ_j,ζ_{j+1}) for which $\zeta_j=-\zeta_{j+1}^*$. A ζ_j on the imaginary axis corresponds to a *fluxon* with asymptotic velocity

$$u_j=\frac{16\zeta_j^2+1}{16\zeta_j^2-1}$$

A pair $(\zeta_j,\ -\zeta_j^*)$ implies a *breather* with envelope velocity

$$u_{ej}=\frac{\zeta_j+\frac{1}{16}\zeta_j^{-1}-\zeta_j^*-\frac{1}{16}\zeta_j^{*-1}}{\zeta_j-\frac{1}{16}\zeta_j^{-1}-\zeta_j^*+\frac{1}{16}\zeta_j^{*-1}}.$$

With this representation of $\mathcal{G}_c$, the structure of the first-order correction becomes apparent. Using this $\mathcal{G}_c$ in (6.1) with the evaluation of $\vec{\mathcal{F}}$ including the modulating parameters, we obtain

$$\vec{w}(x,t)=\int_{-\infty}^{\infty}\frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,0)}\hat{w}(\lambda,t)\,d\lambda, \quad (6.5)$$

where

$$\hat{w}(\lambda,t)\equiv\frac{-\pi i\lambda}{4a^2(\lambda)}\int_0^t\int_{-\infty}^{\infty}\left(\frac{\delta\vec{\phi}(x',t')}{\delta\rho_-(\lambda,0)}\right)^T J\times\vec{\mathcal{F}}(x',t')\,dx'\,dt'.$$

Recalling that $\delta\vec{\phi}(x,t)/\delta\rho_+(\lambda,0)$ denotes the component of the radiation field at wave number $k=(2\lambda-1/8\lambda)$, we note that $\hat{w}(\lambda,t)$ is the amplitude of radiation at wave number k that has been created by the effective source $\vec{\mathcal{F}}$. Formula (6.5) then gives the total radiation emitted by N fluxons, M antifluxons, and L breathers as they respond to the perturbation ϵf.

C. A more-useful form of $\mathcal{G}_c$

In order that the representations (6.3) of $\mathcal{G}_c$ and (6.5) of $\vec{w}$ be practical, we need explicit formulas for the variations $\delta\vec{\phi}/\delta\rho_\pm$ at least in the case $\vec{\phi}=\vec{W}_0$, a pure multifluxon waveform. That is, we must compute

$$\left.\frac{\delta\vec{\phi}(x,t)}{\delta\rho_{\pm}(\lambda,0)}\right|_{\rho_{\pm}(\lambda,0)=0},$$

which we denote by $\dot{\vec{\phi}}_{\pm}(x,t;\lambda)$. In this notation, $\mathcal{G}_c^T$ takes the form

$$\mathcal{G}_c(x,t|x',t') = -\tfrac{1}{4}\pi i \int_{-\infty}^{\infty} \frac{\lambda}{a^2(\lambda)} \dot{\vec{\phi}}_{+}(x,t;\lambda) \times \dot{\vec{\phi}}_{-}^{T}(x',t';\lambda) J\, d\lambda . \tag{6.6}$$

There are several methods for the computation of $\dot{\vec{\phi}}_{\pm}(x,t;\lambda)$. One approach that uses "squared eigenfunctions" from the inverse scattering method is described in the Appendix. Here we introduce an alternative approach through Bäcklund transformations which we believe has two advantages. First, it uses the idea that is behind our construction of $\mathcal{G}$—that members of the null spaces $[\mathcal{N}(L), \mathcal{N}(L^{\dagger})]$ can be generated by differentiating the zeroth-order solution with respect to free parameters. Second, the Bäcklund-transform approach requires less background information than inverse scattering theory.

Clearly, $\dot{\vec{\phi}}_{\pm}$ belongs to $\mathcal{N}(L)$; that is, it satisfies $L\dot{\vec{\phi}}_{\pm}=0$. In components, this equation takes the form $\dot{\vec{\phi}}_{\pm} \equiv \mathrm{col}(\dot{\phi}_{\pm}, \partial_t \dot{\phi}_{\pm})$ and

$$[\partial_{tt} - \partial_{xx} + \cos\phi]\dot{\phi}_{\pm} = 0 . \tag{6.7}$$

As can be seen from (A2), (A8), and (A9) of the Appendix, $\dot{\phi}_{\pm}$ must satisfy the boundary conditions

$$\dot{\phi}_{\pm}(x,t;\lambda) \simeq \frac{1}{\pi\lambda} \exp\{\mp i[k(\lambda)x + \omega(\lambda)t]\}, \quad \text{as } x \to \mp\infty \tag{6.8}$$

where $k(\lambda) = 2\lambda - 1/8\lambda$ and $\omega(\lambda) = 2\lambda + 1/8\lambda$. Once we solve (6.7) together with boundary conditions (6.8) for $\dot{\phi}_{\pm}$ and then compute $\partial_t\dot{\phi}_{\pm}$, we will have all the ingredients necessary to construct $\mathcal{G}_c$.

Note that $\omega^2(\lambda) = k^2(\lambda) + 1$. Thus, boundary conditions (6.8) are indeed consistent with Eqs. (6.7) since $\phi \to 0 (\mathrm{mod} 2\pi)$ as $x \to \pm\infty$. The normalization factor $1/\pi\lambda$ will be seen to be correct as we begin, for orientation, with the simplest situation.

1. Zero-soliton case

Here $\phi = 0$; (6.2) reduces to the Klein-Gordon equation $(\partial_{tt} - \partial_{xx} + 1)\dot{\phi}_{\pm} = 0$ and the solutions that satisfy boundary condition (6.8) are the Fourier modes

$$\dot{\phi}_{\pm}(x,t;\lambda) = \frac{1}{\pi\lambda} \exp\{\mp i[k(\lambda)x + \omega(\lambda)t]\} .$$

Since the scattering potentials (ϕ_t and ϕ_x; see the Appendix) are zero, the inverse transmission coefficient $a(\lambda)$ is equal to 1. Putting all these ingredients into (6.6) yields the representation

$$\mathcal{G}_c(x,t|x',t') = \frac{1}{4\pi i} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda} \begin{pmatrix} i\omega & -1 \\ \omega^2 & i\omega \end{pmatrix} \times \exp\{-i[k(\lambda)(x-x') + \omega(\lambda)(t-t')]\} .$$

To put this into the usual form, we change the variable of integration from λ to k using the definitions $k(\lambda) = 2\lambda - 1/8\lambda$ and $\omega(\lambda) = 2\lambda + 1/8\lambda$. Thus,

$$\frac{d\lambda}{\lambda} = \frac{dk}{\omega} .$$

Furthermore, as λ runs from $-\infty$ to 0, k goes from $-\infty$ to $+\infty$; and, as λ proceeds from 0 to $+\infty$, k goes again from $-\infty$ to $+\infty$. Also, ω is negative (positive) when λ is negative (positive). Thus, we have the Fourier representation

$$\mathcal{G}_c(x,t|x',t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \begin{pmatrix} \cos\omega(t-t') & \frac{1}{\omega}\sin\omega(t-t') \\ -\omega\sin\omega(t-t') & \cos\omega(t-t') \end{pmatrix} \times \exp[-ik(x-x')] ,$$

where $\omega^2 = k^2 + 1$.

2. Single-fluxon case

Next, we construct $\mathcal{G}_c$ for $\phi(x,t)$ a single fluxon. Thus, we seek solutions of (6.7) with

$$\phi = \phi_s = 4\left[\tan^{-1}\exp\left(\frac{x-ut-x_0}{(1-u^2)^{1/2}}\right)\right] .$$

One approach[20] is to Lorentz-transform the independent variables from (x,t) to $\xi = (x-ut-x_0)/(1-u^2)^{1/2}$ and $\tau = (t-ux)/(1-u^2)^{1/2}$. Then (6.7) takes the form $[\partial_{\xi\xi} - \partial_{\tau\tau} - (1 - 2\mathrm{sech}^2\xi)]\dot{\phi}_{\pm} = 0$. Upon separation of variables, the eigenfunctions can be found on pages 734–5 of Ref. 27. Although we favor the use of old tools whenever appropriate, in this case a more-general approach will help prepare the reader to deal with multisoliton cases. To this end we direct attention to the Bäcklund transform. Recall our difficulty. Although the pure multisoliton waveform is defined by the complete absence of radiation, we must differentiate this waveform with respect to the amplitude of radiation at wavelength $2\pi/k$ in order to generate solutions of (6.7). We need a generalization of the pure multisoliton wave that consists of a multisoliton wave "riding over"

radiation. The Bäcklund transformation can be used to generate such a solution.

The Bäcklund transform takes a given solution of the sine-Gordon equation and "adds" a soliton; that is, it transforms a solution of the sine-Gordon equation into another solution which can be interpreted as the first solution with one additional soliton.[28-30] The Bäcklund transform is a "soliton creation operator." The usual application of the Bäcklund transform begins with the trivial solution $\phi = 0$. The Bäcklund transform applied to this "vacuum state" generates a pure fluxon. Next, the Bäcklund transform of the fluxon generates a pure two-soliton waveform, either a doublet or a breather depending upon a choice of parameter. In this manner, the general $N+M+2L$-pure-multifluxon waveform can be generated from the vacuum by a sequence of $N+M+2L$ Bäcklund transformations. Here, rather than starting from the vacuum, we begin with a pure radiation state at wave number k. In this manner, we generate a multifluxon waveform riding with radiation at wave number k. Such a wave can be differentiated with respect to the amplitude of this radiation, after which this amplitude can be set at zero.[31]

Analytically, the Bäcklund transformation is most conveniently described after introducing a coordinate change

$$x \to z \equiv \tfrac{1}{2}(x-t) ,$$

$$t \to \tau \equiv \tfrac{1}{2}(x+t) ,$$

whereupon the sine-Gordon equation becomes

$$\phi_{z\tau} = \sin\phi .$$

It is easily demonstrated that if ϕ_r is a solution of the sine-Gordon equation then ϕ, generated from integration of the pair

$$\begin{aligned} &\frac{\partial}{\partial z}\tfrac{1}{2}(\phi-\phi_r) = 4i\zeta \sin\tfrac{1}{2}(\phi+\phi_r) , \\ &\frac{\partial}{\partial \tau}\tfrac{1}{2}(\phi+\phi_r) = \frac{1}{4i\zeta}\sin\tfrac{1}{2}(\phi-\phi_r) , \end{aligned} \tag{6.9}$$

is also a solution. The integration of these first-order equations effects a Bäcklund transformation from an "old" (i.e., known) solution ϕ_r to a "new" solution ϕ. The new solution has one additional fluxon component with velocity

$$u = (16\zeta^2+1)/(16\zeta^2-1).$$

The location of this fluxon is determined by the integration constant.

We let $\phi_r(x,t;\lambda,a)$ denote a pure radiation state with wave number $k(\lambda)$ and amplitude fixed by constant a. An explicit formula such as (1.3) could be given for ϕ_r. Since ϕ_r depends upon the parameter a, so does its Bäcklund transform ϕ which satisfies (6.9). Thus, we can differentiate both ϕ_r and ϕ with respect to the amplitude parameter a,

$$\mathring{\phi}_r(x,t;\lambda) \equiv \frac{\partial}{\partial a}\phi_r(x,t;\lambda,a)\Big|_{a=0} ,$$

$$\mathring{\phi}(x,t;\lambda) \equiv \frac{\partial}{\partial a}\phi(x,t;\lambda,a)\Big]_{a=0} .$$

Clearly from the sine-Gordon equation, $\mathring{\phi}_R$ satisfies

$$\partial_{z\tau}\mathring{\phi}_r = \mathring{\phi}_r , \tag{6.10a}$$

while $\mathring{\phi}$ solves

$$\partial_{z\tau}\mathring{\phi} + (\cos\phi_s)\mathring{\phi} = 0 , \tag{6.10b}$$

where ϕ_s denotes a single-soliton state. On the other hand, we can differentiate the Bäcklund transformation (6.9) with respect to a and find $\mathring{\phi}$ satisfies

$$\begin{aligned} &\partial_z(\mathring{\phi}-\mathring{\phi}_r) = 4i\zeta\cos(\tfrac{1}{2}\phi_s)[\mathring{\phi}+\mathring{\phi}_r] , \\ &\partial_\tau(\mathring{\phi}+\mathring{\phi}_r) = \frac{1}{4i\zeta}\cos(\tfrac{1}{2}\phi_s)[\mathring{\phi}+\mathring{\phi}_r] . \end{aligned} \tag{6.11}$$

This linear first-order pair of equations (linear Bäcklund transformation) is easier to solve for $\mathring{\phi}$ than is (6.10b). Moreover, given that $\mathring{\phi}_r$ solves (6.10a), any solution $\mathring{\phi}$ of (6.11) will also satisfy (6.10b). Thus, in the single-soliton case, in order to compute the ingredients of $\mathcal{G}_c$ (i.e., $\mathring{\phi}_\pm$) we must choose for $\mathring{\phi}_r$ a solution of (6.10a) that, when Bäcklund transformation by (6.11), will generate functions $\mathring{\phi}_\pm$ that satisfy the appropriate boundary data (6.8).

Let's concentrate on $\mathring{\phi}_+$. In this case, we must find $\mathring{\phi}_+$ satisfying

$$\begin{aligned} &\partial_z(\mathring{\phi}_+ - \mathring{\phi}_r) = -4i\zeta(\tanh\tilde{x})(\mathring{\phi}_+ + \mathring{\phi}_r) , \\ &\partial_\tau(\mathring{\phi}_+ + \mathring{\phi}_r) = \frac{-1}{4i\zeta}(\tanh\tilde{x})(\mathring{\phi}_+ - \mathring{\phi}_r) , \\ &\mathring{\phi}_+(z,\tau;\lambda) \simeq \frac{1}{\pi\lambda}\exp\{-i[k-\omega)z + (k+\omega)\tau]\} , \end{aligned}$$

as $z, \tau \to -\infty$

where

$$\begin{aligned} \tilde{x} &\equiv 4i\zeta(z+z_0) - \tau/16\zeta^2 \\ &= 2i\zeta\left(1-\frac{1}{16\zeta^2}\right)\left[x - x_0 - \left(\frac{16\zeta^2+1}{16\zeta^2-1}\right)t\right], \end{aligned}$$

and we have used the fact that for a single soliton ϕ_s, $\cos\tfrac{1}{2}\phi_s = -\tanh\tilde{x}$. If we choose

$$\mathring{\phi}_r^+(x,t;\lambda) = \frac{1}{\pi\lambda}\left(\frac{\zeta-\lambda}{\zeta+\lambda}\right)\exp\{-i[k(\lambda)x + \omega(\lambda)t]\} ,$$

we find

$$\dot{\phi}_+(x,t;\lambda)=\frac{1}{\pi\lambda}\left(\frac{\zeta^2+\lambda^2+2\zeta\lambda\tanh\tilde{x}}{(\zeta+\lambda)^2}\right)$$

$$\times\exp\{-i[k(\lambda)x+\omega(\lambda)t]\}\ . \qquad (6.12)$$

Note that this satisfies the boundary condition (6.8) since (for ζ pure imaginary) $\tilde{x}\to+\infty$ as $x\to-\infty$. Similarly, the choice

$$\dot{\phi}_r^-(x,t;\lambda)=\frac{1}{\pi\lambda}\left(\frac{\zeta-\lambda}{\zeta+\lambda}\right)\exp\{+i[k(\lambda)x+\omega(\lambda)t]\}$$

generates

$$\phi_-(x,t;\lambda)=\frac{1}{\pi\lambda}\left(\frac{\zeta^2+\lambda^2-2\zeta\lambda\tanh\tilde{x}}{(\zeta+\lambda)^2}\right)$$

$$\times\exp\{+i[k(\lambda)x+\omega(\lambda)t]\}\ .$$

With these ingredients we construct the Green's function $\mathcal{G}_c$ from (6.6) for the single-fluxon case. Explicitly, we find

$$\mathcal{G}_c(x,t\,|\,x',t')$$

$$=\frac{1}{4\pi i}\int_{-\infty}^{\infty}d\lambda\begin{pmatrix}g_{11}&g_{12}\\g_{21}&g_{22}\end{pmatrix}\times\frac{\exp\{-i[k(\lambda)(x-x')+\omega(\lambda)(t-t')]\}}{\lambda(\zeta^2-\lambda^2)^2}\ , \qquad (6.13)$$

where

$$g_{11}\equiv(\zeta^2+\lambda^2+2\zeta\lambda\tanh\tilde{x})\times[i\omega(\zeta^2+\lambda^2-2\zeta\lambda\tanh\tilde{x}')-2i\zeta\lambda\omega(\zeta)\operatorname{sech}^2\tilde{x}\,]$$

$$g_{12}\equiv-(\zeta^2+\lambda^2+2\zeta\lambda\tanh\tilde{x})(\zeta^2+\lambda^2-2\zeta\lambda\tanh\tilde{x}')$$

$$g_{21}\equiv[-i\omega(\zeta^2+\lambda^2+2\zeta\lambda\tanh\tilde{x})+2i\zeta\lambda\omega(\zeta)\operatorname{sech}^2\tilde{x}']\times[(\zeta^2+\lambda^2-2\zeta\lambda\tanh\tilde{x}')-2i\zeta\lambda\omega(\zeta)\operatorname{sech}^2\tilde{x}']$$

$$g_{22}\equiv[+i\omega(\zeta^2+\lambda^2+2\zeta\lambda\tanh\tilde{x})-2i\zeta\lambda\omega(\zeta)\operatorname{sech}^2\tilde{x}]\times(\zeta^2+\lambda^2-2\zeta\lambda\tanh\tilde{x}')\ .$$

In these formulas,

$$\tilde{x}'\equiv\frac{x'-\int^{t'}u(t'')\,dt''-x_0(t')}{[1-u^2(t')]^{1/2}}$$

and $\tilde{x}$ is given by the same equation with the single primes dropped.

3. *Two-fluxon case*

In order to construct the radiative Green's function about a two-fluxon solution, we apply a Bäcklund transform once again, this time using a single soliton riding over a periodic wave as the known solution. Fortunately, at higher levels of the soliton hierarchy, explicit integrals can be obtained algebraically from commutability of the Bäcklund transformation. Thus it is readily shown that[28,29] a two-fluxon wave riding over ϕ_r is given by ϕ_2, where

$$\tan\tfrac{1}{4}(\phi_2-\phi_r)=\left(\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right)\tan\tfrac{1}{4}(\phi_A-\phi_B)\ . \qquad (6.14)$$

Here, ϕ_A (ϕ_B) denotes a single soliton with speed fixed by the parameter ζ_A (ζ_B) riding over the background radiation ϕ_r, and ϕ_2 is a Bäcklund transform of ϕ_A through ζ_B and simultaneously a transform of ϕ_B through ζ_A. Differentiating formula (6.14) with respect to the amplitude of radiation and then setting the amplitude at zero yields

$$\dot{\phi}_2=\dot{\phi}_r+\left[\left(\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right)\frac{\cos^2\tfrac{1}{4}\phi_D}{\cos^2\tfrac{1}{4}(\phi_{SA}-\phi_{SB})}\right](\dot{\phi}_A-\dot{\phi}_B)\ ,$$

where ϕ_D denotes a pure double-fluxon wave with speed parameters ζ_A and ζ_B, ϕ_{SA} (ϕ_{SB}) denotes a pure single fluxon-state with speed fixed by ζ_A (ζ_B), and $\dot{\phi}_A$ ($\dot{\phi}_B$) denotes the derivative of ϕ_A (ϕ_B) with respect to the amplitude of radiation. We can compute $\dot{\phi}_A$ ($\dot{\phi}_B$) as in Sec. VI C 2. Analytically, ϕ_D is given by (6.14) with ϕ_r set at zero.

$$\tan\tfrac{1}{4}\phi_D=\left(\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right)\tan\tfrac{1}{4}(\phi_{SA}-\phi_{SB})\ . \qquad (6.15)$$

Clearly, $\dot{\phi}_2$ satisfies the linear equation (6.7). Next, we choose $\dot{\phi}_r$ such that the boundary condition (6.8) is satisfied. In particular, the choices

$$\dot{\phi}_r^{\pm}=\exp\{\mp i[k(\lambda)x+\omega(\lambda)t]\}\times\left\{\pi\lambda\left[1+\left(\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right)\left(\frac{\zeta_A+\lambda}{\zeta_A-\lambda}\right)\left(\frac{\zeta_B+\lambda}{\zeta_B-\lambda}\right)\right]\right\}^{-1} \qquad (6.16)$$

yield

$$\dot{\phi}_2^{\pm}=\dot{\phi}_r^{\pm}\left[1+\left(\frac{\zeta_A+\zeta_B}{\zeta_A-\zeta_B}\right)\left(\frac{\cos\tfrac{1}{4}\phi_D}{\cos\tfrac{1}{4}(\phi_{SA}-\phi_{SB})}\right)\times\left(\frac{\zeta_A^2+\lambda^2\pm2\zeta_A\lambda\tanh x_A}{(\zeta_A^2-\lambda^2)}\right)\times\left(\frac{\zeta_B^2+\lambda^2\pm2\zeta_B\lambda\tanh x_B}{(\zeta_B^2-\lambda^2)}\right)\right]\ ,$$

where

$$x_A\equiv2i\zeta_A\left(1-\frac{1}{16\zeta_A^2}\right)\left[x-\left(\frac{16\zeta_A^2+1}{16\zeta_A^2-1}\right)t-x_{0A}\right]$$

and similarly for x_B.

Substitution into (6.6) gives the radiative Green's function for a two-fluxon solution.

4. Breather case

Here the formulas are the same as in the two-fluxon case but the parameters ζ_A and ζ_B are complex and satisfy $\zeta_A = -\zeta_B^*$.

5. Multisoliton case

The same techniques can be used at each level of the soliton hierarchy. Differentiating with respect to the amplitude of the radiation component and then setting the amplitude at zero gives

$$\dot{\phi}_N = \dot{\phi}_{N-2} + \left[\left(\frac{\zeta_p + \zeta_q}{\zeta_p - \zeta_q}\right) \frac{\cos^2\frac{1}{4}(\phi_{SN} - \phi_{S(N-2)})}{\cos^2\frac{1}{4}(\phi_{Sp} - \phi_{Sq})}\right] \times (\dot{\phi}_p - \dot{\phi}_q) ,$$

where ϕ_{SN} is a pure N-soliton wave, $\phi_{S(N-2)}$ is the pure $N-2$-soliton wave with solitons of velocities related to ζ_p and ζ_q missing, and ϕ_{Sp} (ϕ_{Sq}) is a pure $(N-1)$ soliton wave with the soliton of velocity related to ζ_q (ζ_p) missing. Just as before, $\dot{\phi}_N$, $\dot{\phi}_{N-2}$, $\dot{\phi}_p$, and $\dot{\phi}_q$ are the corresponding radiative solutions of (6.7). With these formulas, all ingredients of the radiative Green's function $\mathcal{G}_c$ are explicitly available.

VII. RADIATION FROM A FLUXON

Here we present a brief recapitulation of our perturbation scheme in the concrete context of analyzing a fluxon generator for electromagnetic radiation. Although results equivalent to those presented here have been obtained from an analysis based upon a less elaborate theoretical perspective,[20,32] we discuss this work for the following reasons: (i) It provides a simple example of our perturbation scheme in action. (ii) The oscillation structure to be considered may be of technical importance at electromagnetic wavelengths of 100 μm or less, and the equations we develop are directly applicable to the design of this oscillator. (iii) Once understood, our perturbation scheme is readily extended to calculations of radiation from multifluxon collisions and breathers and to estimates of radiation reaction on fluxon motion.

The structure we will analyze is the quantum flux shuttle discussed in Sec. IVC, but in practice it may be modified into the annular shape of Fig. 11(a) for technical convenience. We assume that the bias current γ is large enough so the fluxon does not become pinned between microshorts but executes a "wobbling" motion of frequency ω_0 and mean velocity u_m. Our aim is to estimate the power radiated by the fluxon.

A. Find the nonradiative motion of the fluxon

The motion of a fluxon is described by the formula

$$\phi_0 = 4\tan^{-1}\left[\exp\left(\frac{x-X}{(1-u^2)^{1/2}}\right)\right] ,$$

where

$$X \equiv \int_0^t u(\tau)\,d\tau + x_0(t) .$$

From Sec. IV C, the appropriate ordinary differential equations obtained from the orthogonality conditions (3.9) on the discrete subspace of $\mathfrak{N}(L^\dagger)$ are

$$\dot{u} = \tfrac{1}{4}\pi\gamma(1-u^2)^{3/2} - \alpha u(1-u^2) - \tfrac{1}{8}\beta u + \tfrac{1}{2}\mu(1-u^2) \times \sum_n \operatorname{sech}^2\left(\frac{X-na}{(1-u^2)^{1/2}}\right) \tanh\left(\frac{X-na}{(1-u^2)^{1/2}}\right) ,$$

$$\dot{x}_0 = -\tfrac{1}{2}\mu u \sum_n (X-na)\operatorname{sech}^2\left(\frac{X-na}{(1-u^2)^{1/2}}\right) \times \tanh\left(\frac{X-na}{(1-u^2)^{1/2}}\right) .$$

Integrating these equation (as in Figs. 5 and 6) to obtain the periodic motion

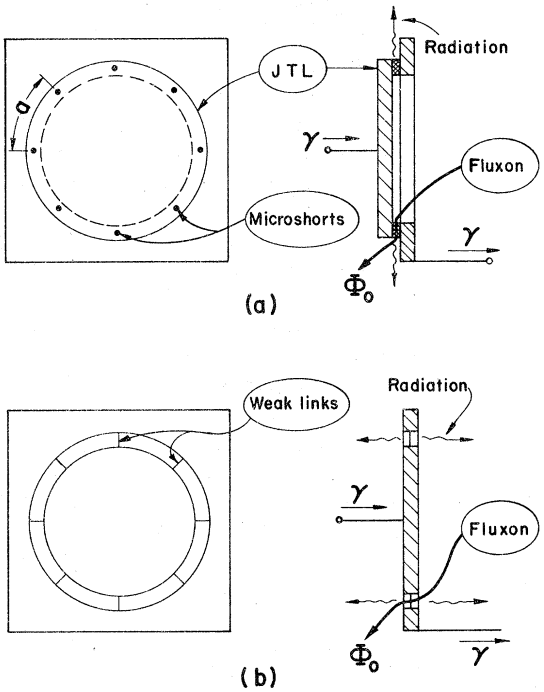

FIG. 11. (a) JTL fluxon oscillator. (b) Modification of the structure in (a) to permit increased lateral radiation.

$$u(t) = u(t + 2\pi/\omega_0)$$

then determines the wobble frequency ω_0 and the mean fluxon velocity u_m. To zeroth order, these will be related by the condition

$$\omega_0 = 2\pi u_m / a \ ,$$

where a is the spacing between microshorts. At this point, the dc power input to the oscillator for nonradiative dissipation P_{nr} can be calculated as the product of the Lorentz force on the fluxon γ times the mean fluxon velocity. Thus

$$P_{nr} = \gamma u_m \ . \tag{7.1}$$

B. Calculate the effective source

From the periodic modulations in $\dot{u}$ and $\dot{x}_0$, the effective source

$$\vec{\mathcal{F}} = \begin{pmatrix} -\frac{1}{\alpha}\phi_{0,u}\dot{u} + \frac{1}{\alpha}\phi_{0,x}\dot{x}_0 \\ f - \frac{1}{\alpha}\phi_{0,ut}\dot{u} + \frac{1}{\alpha}\phi_{0,x't}\dot{x}_0 \end{pmatrix},$$

where

$$f \equiv -\phi_{0,t} + \frac{\beta}{\alpha}\phi_{0,xxt} - \frac{\gamma}{\alpha} - \frac{\mu}{\alpha}\sum_n \delta(x - na)\sin\phi_0 \ ,$$

will have a corresponding periodicity. To zeroth order this can be evaluated by approximating

$$X \cong u_m t + x_0$$

in evaluating ϕ_0, f, $\dot{u}$, and $\dot{x}_0$. Then $\vec{\mathcal{F}}$ can be written as a Fourier series in ω_0:

$$\vec{\mathcal{F}}(x', t') = \vec{F}_0(x' - u_m t') + \vec{F}_1(x' - u_m t')e^{i\omega_0 t'} + \vec{F}_1^*(x' - u_m t')e^{-i\omega_0 t'} + (\text{higher harmonics})$$

C. Calculate the first-order corrections

Here it is convenient to consider separately each harmonic in the Fourier series beginning with the average term

$$\vec{w}_0(x, t) = \frac{1}{4\pi i}\int_{-\infty}^{\infty} d\lambda \int_0^t dt' \int_{-\infty}^{\infty} dx' \, \frac{\exp\{-i[k(\lambda)(x - x') + \omega(\lambda)(t - t')]\}}{\lambda(\zeta^2 - \lambda^2)^2}[g_{ij}]\vec{F}_0(x' - u_m t') \ . \tag{7.2}$$

To zeroth order, the components of $[g_{ij}]$ depend upon x' and t' only as $(x' - u_m t')$. Thus, integration over x' leaves a t' dependence in the integrand as the factor $\exp\{i[\omega(\lambda) + k(\lambda)u_m]t'\}$. Integration over t' from 0 to t in (7.2) gives the transient response caused by turning $\vec{F}_0$ on at time $t = 0$. Instead, since we are interested in the steady-state behavior, we integrate over t' from $-t$ to $+t$ and then let $t \to +\infty$. Then

$$\lim_{t\to\infty}\int_{-t}^{t} \exp[i(\omega + ku_m)t']\,dt' = 2\pi\delta(\omega + k u_m) \ .$$

Final integration over λ then gives

$$\vec{w}_0 = \vec{w}_0(x - u_m t) \ . \tag{7.3}$$

Thus the steady-state perturbation caused by $\vec{F}_0$ rides with the fluxon. It is a "meson dressing" of the zeroth fluxon that makes a first-order correction to the shape of ϕ_0.[20] This first-order correction can be neglected when ϕ_0 is used for other first-order calculations.

Consider next the effect of the fundamental harmonic in $\vec{\mathcal{F}}$ on the sinusoidal steady-state response defined by

$$\vec{w}_{1ss}(x, t) = \frac{1}{4\pi i}\lim_{t\to\infty}\int_{-\infty}^{\infty} d\lambda \int_{-t}^{t} dt' \int_{-\infty}^{\infty} dx' \, \frac{\exp\{-i[k(\lambda)(x - x') + \omega(\lambda)(t - t')]\}}{\lambda(\zeta^2 - \lambda^2)^2} \times [g_{ij}][\vec{F}_1(x' - u_m t')e^{i\omega_0 t'} + \vec{F}_1^*(x' - u_m t')e^{-i\omega_0 t'}] \ .$$

Again integration over x' introduces a factor $\exp(iku_m t')$ so the t' integral involves only the factor

$$\lim_{t\to\infty}\int_{-t}^{t} \exp\{i[\omega(\lambda) + k(\lambda)u_m \pm \omega_0]t'\}\,dt' = 2\pi\delta(\omega + ku_m \pm \omega_0) \ .$$

This drives radiation at the Doppler-shifted wobble frequencies $-(ku_m \pm \omega_0)$. For oscillator applications the higher frequency (ω_+) is of primary interest. Here

$$\omega_+ = -(\omega_0 + k_+ u_m)$$

and

$$k_+^2 = \omega_+^2 - 1$$

so

$$k_+ = \frac{u_m\omega_0 + (\omega_0^2 + u_m^2 - 1)^{1/2}}{1 - u_m^2}$$

and

$$\omega_+ = -\omega_0\left(\frac{1 + u_m[1 - (1 - u_m^2)/\omega_0^2]^{1/2}}{1 - u_m^2}\right) . \tag{7.4}$$

Note that as $u_m \to 1$, $\omega_+ \to -\omega_0/(1 - u_m)$ so a substantial increase in the radiation frequency is obtained through the Doppler shift.

The final integration over λ is effected by introducing a variable change from λ to ω through $d\lambda/\lambda = d\omega/k$. The final result is

$$\vec{\mathfrak{W}}^+_{1SS} = 128\frac{\exp[-i(k_+x + \omega_+t + \frac{1}{2}\pi)]}{k_+[(1 + u_m)/(1 - u_m) + (k_+ + \omega_+)^2]^2}$$

$$\times \int_{-\infty}^{\infty} [g_{ij}(x - u_m t\,|\,x')]\exp(ik_+x')$$

$$\times \vec{F}_1(x')\,dx' + \text{c.c.} ,$$

where g_{ij} [see (6.13)] is evaluated with $\zeta = i[(1 + u_m)/16(1 - u_m)]^{1/2}$ and $\lambda = \frac{1}{4}(k_+ + \omega_+)$.

D. Determine the radiated power

We begin by writing the amplitude of the higher-frequency Doppler radiation as

$$\vec{\mathfrak{W}}^+_{1SS} \equiv \text{col}(w, w_t) .$$

Away from the fluxon this radiation is governed by the linear Klein-Gordon equation $w_{xx} - w_{tt} = w$ for which the corresponding energy (Hamiltonian) density is

$$\mathcal{H} = \tfrac{1}{2}(w_x^2 + w_t^2 + w^2) .$$

Using the dispersion equation ($w^2 = k^2 + 1$) and the plane-wave relations, $w_x^2 = -k_+^2 w^2$ and $w_t^2 = -\omega_+^2 w^2$, this can be written

$$\mathcal{H} = w_t^2 = \omega_+^2 w^2 .$$

This energy propagates at the group velocity $d\omega/dk = -k_+/\omega_+$. Thus, the radiated power is

$$P_r = \frac{[u_m\omega_0 + (\omega_0^2 + u_m^2 - 1)^{1/2}][\omega_0 + u_m(\omega_0^2 + u_m^2 - 1)^{1/2}]w^2}{(1 - u_m^2)^2} . \tag{7.5}$$

E. Determine output power and efficiency

The radiated power will be absorbed by the losses α, β of the JTL and by a useful load. Let us assume $\beta = 0$ and suppose α is composed of two components: (i) α' represents internal losses and (ii) α'' represents power absorbed by a shunt load. Then $\alpha = \alpha' + \alpha''$ and the power absorbed by the load would be

$$P_0 = \frac{\alpha''}{\alpha} P_r \tag{7.6}$$

with an efficiency of $100\,P_0/(P_r + P_{nr})$. It should be possible to increase the number of fluxons up to the number of discrete Josephson junctions (8 in Fig. 11) with a corresponding increase in output power and somewhat lower efficiency.

F. Final comments

The above discussion has been presented entirely in the normalized units discussed in Ref. 2. In laboratory units, the limiting velocity of a fluxon $\tilde{c}$ is about 1/15 of the velocity of light or 2×10^7 m/s. Setting $a = 1$ means that the microshorts are separated by a distance equal to the "Josephson length" $\lambda_J = 5\times10^{-4}/\sqrt{J_c}$ μm, where J_c is the Josephson current density measured in A/m^2. With a current density of 10^8 A/m^2, $\lambda_J = 5$ μm and the wobble frequency $f_0 = u_m/\lambda_J$ would be $4\times10^{12}(u_m/\tilde{c})$ Hz. As previously noted, the corresponding Doppler-shifted radiation frequency would then approach

$$f_+ \to 4\times10^{12}\,\frac{u_m}{c - u_m}\ \text{Hz}$$

as $u_m \to c$, the relativistic limit. For $u_m/c = \frac{1}{2}$, this corresponds to a free-space wavelength of 0.075 mm.

If the oscillator structure of Fig. 11(a) is to be useful, the radiation field must be effectively coupled to an external load. In terms of (7.6), α'' must be a significant fraction of α. This requirement is somewhat difficult to meet for the JTL structure because the fields are concentrated between the shielding effect of the two superconductors. To alleviate this problem, the junction width should be decreased; and in the limit of zero width we arrive at the structure indicated in Fig. 11(b). Here the radiation is into an open wave guide with rather favorable geometry for lateral emission, and the Josephson effect is entirely concentrated at "weak link" junctions. Our perturbation theory will not be quantitatively correct for this structure, but we expect the performance to be qualitatively similar to that of Fig. 11(a).[33]

APPENDIX: BASIC DERIVATION OF THE GREEN'S FUNCTION

Here we sketch our derivation of the Green's function $\mathcal{G}$ from the viewpoint of inverse-scattering-transform theory. Takhtadjian and Faddeev[12] have completely integrated the sine-Gordon equation (1.1) using the associated linear problem

$$\left[\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\frac{d}{dx}+\tfrac{1}{4}iw\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}+\frac{1}{16\lambda}\begin{pmatrix}e^{i\phi} & 0\\ 0 & e^{-i\phi}\end{pmatrix}\right]\times\begin{pmatrix}f_1\\ f_2\end{pmatrix}=\lambda\begin{pmatrix}f_1\\ f_2\end{pmatrix}. \quad \text{(A1)}$$

Here, $\phi(x,t)$ is a solution of the sine-Gordon equation (1.1) that vanishes (mod2π) as $x\to\pm\infty$, $w\equiv\phi_x+\phi_t$, and λ is an eigenvalue parameter. Consider two linearly independent sets of solutions of (A1), $\{g,\bar{g}\}$ and $\{f,\bar{f}\}$ that are defined for real λ by the following asymptotic conditions:

$$g(x;\lambda)\simeq\begin{pmatrix}1\\ -i\end{pmatrix}e^{-[ik(\lambda)/2]x}, \quad \text{as } x\to-\infty; \quad \text{(A2a)}$$

$$\bar{g}(x;\lambda)\simeq\begin{pmatrix}1\\ i\end{pmatrix}e^{[ik(\lambda)/2]x} \quad \text{as } x\to-\infty \quad \text{(A2b)}$$

and

$$f(x;\lambda)\simeq\begin{pmatrix}1\\ i\end{pmatrix}e^{[ik(\lambda)/2]x}, \quad \text{as } x\to+\infty; \quad \text{(A2c)}$$

$$\bar{f}(x;\lambda)\simeq\begin{pmatrix}1\\ -i\end{pmatrix}e^{[-ik(\lambda)/2]x} \quad \text{as } x\to+\infty \quad \text{(A2d)}$$

Here $k(\lambda)\equiv(2\lambda-1/8\lambda)$. The solution f can be expanded in the basis $\{g,\bar{g}\}$,

$$f(x;\lambda)=a(\lambda)\bar{g}(x;\lambda)+b(\lambda)\bar{g}(x;\lambda), \quad \text{(A3a)}$$

similarly,

$$g(x;\lambda)=a(\lambda)\bar{f}(x;\lambda)+b^*(\lambda)f(x;\lambda). \quad \text{(A3b)}$$

From (A3a) and (A2),

$$f(x;\lambda)\simeq\begin{pmatrix}1\\ i\end{pmatrix}e^{[ik(\lambda)/2]x} \quad \text{as } x\to+\infty$$

$$\simeq a(\lambda)\begin{pmatrix}1\\ i\end{pmatrix}e^{[ik(\lambda)/2]x}+b(\lambda)\begin{pmatrix}1\\ -i\end{pmatrix}\times e^{[-ik(\lambda)/2]x} \quad \text{as } x\to-\infty.$$

Thus, $[a(\lambda)]^{-1}$ is a transmission coefficient and $b(\lambda)/a(\lambda)$ is the reflection coefficient which we called $\rho_+(\lambda)$ in Sec. VI. In a similar way, the asymptotic behavior of g define the reflection coefficient $\rho_-(\lambda)$. So we record the definitions

$$\rho_+(\lambda)\equiv b(\lambda)/a(\lambda), \quad \text{(A4a)}$$

$$\rho_-(\lambda)\equiv b^*(\lambda)/a(\lambda). \quad \text{(A4b)}$$

Next, we label the bound-state eigenvalues as $\{\lambda_j\}$ for (A1) and their corresponding normalization. [These parameters fix the solitons in $\phi(x,t)$.] At an (upper half plane) eigenvalue, g and f are proportional so $f(x;\lambda_j)\equiv b_j g(x;\lambda_j)$. In addition to the b_j, we define

$$m_j^-\equiv(i\dot{a}_j b_j)^{-1}, \quad \text{(A5a)}$$

$$m_j^+\equiv b_j/i\dot{a}_j, \quad \text{(A5b)}$$

where $\dot{a}_j\equiv da(\lambda)/d\lambda\,|_{\lambda=\lambda_j}$.

From (A4) and (A5) we can define two equivalent sets of "scattering data":

$$\mathcal{S}_\pm\equiv\{\rho_\pm(\lambda) \text{ for all real } \lambda;\ \lambda_j, m_j^\pm \text{ for } j=1,2,\ldots,N\}. \quad \text{(A6)}$$

Takhtadjian and Faddeev have shown that there is a one-to-one, invertible map between the potentials $\{\phi,w\}$ and the scattering data $\mathcal{S}_+$ (or $\mathcal{S}_-$). If the potentials $\{\phi,w\}$ evolve according to the sine-Gordon equation, the corresponding temporal evolution of the scattering data is trivial:

$$\mathcal{S}_\pm=\{\rho_\pm(\lambda;t)=\exp[\mp i\omega(\lambda)t]\rho(\lambda;0);\ \lambda_k(t)=\lambda_k(0);m_k^\pm(t)=\exp[\mp i\omega(\lambda_k)t]m_k^\pm(0)\}, \quad \text{(A7)}$$

where $\omega(\lambda)\equiv 2\lambda+1/8\lambda$.

As discussed in Sec. VI, components of the Green's function can be obtained by varying solutions of the sine-Gordon equation (ϕ,ϕ_t) with respect to a family of parameters. Here we use the initial values of the scattering data for these parameters. The variations are computed using a form of the Marchenko equation that compares the potentials $(\phi+\delta\phi,w+\delta w)$ to the reference potentials (ϕ,w).[12] Thus,

$$\delta\phi(x)=\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda}\right)\delta\rho_+(\lambda)\,d\lambda+2i\sum_{k=1}^{N}\left[\frac{g_1(x;\lambda_k)g_2(x;\lambda_k)}{\lambda_k}dm_k^+ +m_k^+\frac{d}{d\lambda}\left(\frac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda}\right)_{\lambda_k}d\lambda_k\right],$$

$$\delta w(x)=\frac{2i}{\pi}\int_{-\infty}^{\infty}[g_2^2(x;\lambda)-g_1^2(x;\lambda)]\delta\rho_+(\lambda)+4i\sum_{k=1}^{N}\left([g_2^2(x;\lambda_k)-g_1^2(x;\lambda_k)]dm_k^+ +m_k^+\frac{d}{d\lambda}[g_2^2(x;\lambda)-g_1^2(x;\lambda)]_{\lambda_k}d\lambda_k\right),$$

where all functions are to be taken at time t. From these variations, it is clear that

$$\frac{\delta\phi(x,t)}{\delta\rho_+(\lambda,t)}=\frac{i}{\pi}\frac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda},$$

$$\frac{\delta\phi(x,t)}{\delta m_k^+(t)}=2i\frac{g_1(x;\lambda_k)g_2(x;\lambda_k)}{\lambda_k},$$

$$\frac{\delta\phi(x,t)}{\delta\lambda_k^\mp(t)}=2im_k^+(t)\frac{d}{d\lambda}\left(\frac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda}\right)_{\lambda_k},$$

$$\frac{\delta w(x,t)}{\delta\rho_+(\lambda,t)}=\frac{2i}{\pi}\left(g_2^2(x;\lambda)-g_1^2(x;\lambda)\right),$$

$$\frac{\delta w(x,t)}{\delta m_k^+(t)} = 4i\left(g_2^2(x;\lambda_k) - g_1^2(x;\lambda_k)\right),$$

$$\frac{\delta w(x,t)}{\delta \lambda_k^+(t)} = 4im_k^+(t)\frac{d}{d\lambda}\left(g_2^2(x;\lambda) - g_1^2(x;\lambda)\right)_{\lambda_k},$$

where $\delta\phi/\delta\lambda_k^+$ denotes the variation of ϕ with respect to λ_k *while holding the rest of the scattering data* $\mathcal{S}_+$ *constant*. Note that these variations are naturally expressed in terms of the quadratic products of the components of the eigenfunction g; hence, we define

$$\Phi(x;\lambda) \equiv \begin{pmatrix} \dfrac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda} \\ 2[g_2^2(x;\lambda) - g_1^2(x;\lambda)] - \dfrac{\partial}{\partial x}\left(\dfrac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda}\right) \end{pmatrix}. \tag{A8}$$

Then, after using the definitions $w = \phi_t + \phi_x$, $\vec{\phi} = \mathrm{col}(\phi, \phi_t)$, the variations take the compact form

$$\begin{aligned} \frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,t)} &= \frac{i}{\pi}\Phi(x;\lambda), \\ \frac{\delta\vec{\phi}(x,t)}{\delta m_k^+(t)} &= 2i\Phi(x;\lambda_k), \\ \frac{\delta\vec{\phi}(x,t)}{\delta\lambda_k^+(t)} &= 2im_j^+(t)\dot{\Phi}(x). \end{aligned} \tag{A9}$$

After defining the "adjoint state"

$$\Phi^A(x;\lambda) \equiv \begin{pmatrix} 2[f_2^2(x;\lambda) - f_1^2(x;\lambda)] - \dfrac{\partial}{\partial x}\left(\dfrac{f_1(x,\lambda)f_2(x,\lambda)}{\lambda}\right) \\ -\dfrac{f_1(x;\lambda)f_2(x;\lambda)}{\lambda} \end{pmatrix}, \tag{A8a}$$

similar computations yield

$$\begin{aligned} \frac{\delta\vec{\phi}(x,t)}{\delta\rho_-(\lambda,t)} &= \frac{i}{\pi}J\Phi^A(x;\lambda), \\ \frac{\delta\vec{\phi}(x,t)}{\delta m_k^-(t)} &= 2iJ\Phi^A(x;\lambda_k), \\ \frac{\delta\vec{\phi}(x,t)}{\delta\lambda_k^-(t)} &= 2im_k^-(t)J\dot{\Phi}^A(x), \end{aligned} \tag{A9a}$$

where, as before, $\delta\vec{\phi}/\delta\lambda_k^-$ means the variation of $\vec{\phi}$ with respect to λ_k *while holding the rest of the scattering data* $\mathcal{S}_-$ *constant*. These variations with respect to scattering data at time t can be converted to variations with respect to the initial values of the scattering data using temporal evolution (A7); for example

$$\frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,0)} = \frac{1}{\pi}\exp[-i\omega(\lambda)t]\Phi(x;\lambda), \tag{A10}$$

with similar formulas for the other variations of (A9) and (A9a).

Note from (A10) that $\exp(-i\omega t)\Phi$ is a derivative of $\vec{\phi}$ with respect to a parameter; hence $\exp(-i\omega t)\Phi$ belongs to $\mathfrak{N}(L)$ and satisfies

$$L(e^{-i\omega t}\Phi) = \left[\begin{pmatrix}1&0\\0&1\end{pmatrix}\partial_t + \begin{pmatrix}0&-1\\-\partial_{xx}+\cos\phi&0\end{pmatrix}\right] \times \begin{pmatrix}e^{-i\omega t}\Phi_1\\ e^{-i\omega t}\Phi_2\end{pmatrix} = 0.$$

Thus we see that the "squared eigenfunctions" provide a natural set for the expansion of $\mathcal{S}$. In particular, the first component satisfies

$$[\partial_{tt} - \partial_{xx} + \cos\phi](e^{-i\omega t}\Phi_1) = 0,$$

while Φ_2 is obtained from $\partial_t(e^{-i\omega t}\Phi_1) = e^{-i\omega t}\Phi_2$. To find Φ, we must solve this second-order linear equation subject to the correct boundary data. This fact is the basis of the "Bäcklund transformation method" to generate $\mathcal{S}_c$ which we discussed in Sec. VI.

The orthogonality and completeness relations for this "basis of squared eigenfunctions" are summarized in Table I.

Table I needs some explanation. The properties summarized in this table permit one to expand two-vector valued *functions of* x on the basis of squared eigenfunctions. For example, it follows immediately from the "completeness relation" that

$$\vec{h}^T(x) = (h_1(x), h_2(x)) = \int_\Gamma \hat{h}(\lambda)\Phi^T(x;\lambda)\,d\lambda,$$

where

$$\begin{aligned} \hat{h}(\lambda) &= \frac{1}{4\pi i}\frac{\lambda}{a^2(\lambda)}\langle h(\cdot), \Phi^A(\cdot;\lambda)\rangle\rangle \\ &= \frac{1}{4\pi i}\frac{\lambda}{a^2(\lambda)}\int_{-\infty}^{\infty}[h_1(x)\Phi_1^A(x;\lambda) + h_2(x) \times \Phi_2^A(x;\lambda)]\,dx, \end{aligned}$$

and where we have used the definition of the "inner product" $\langle\ ,\ \rangle$,

$$\langle\vec{h}(\cdot), \vec{g}(\cdot)\rangle \equiv \int_{-\infty}^{\infty}[h_1(x)g_1(x) + h_2(x)g_2(x)]dx.$$

Note, in particular, that no complex conjugates are present in this definition. In these formulas (and in Table I) Γ denotes a contour in the complex Γ plane running from $\lambda = -\infty + i0^+$ to 0^- and then from 0^+ to $\lambda = +\infty + i0^+$ and chosen to lie above all zeros of $a(\lambda)$ in the upper half λ plane.[34]

For an analogous eigenvalue problem (the Zakharov-Shabat system), Kaup[15] has shown that the squared-eigenfunction basis is complete. His proof uses a set of Marchenko equations to show

TABLE I. Properties of the basis of squared eigenfunctions.

Basis:

$$\left\{\Phi(\cdot;\lambda) \text{ for all real } \lambda;\ \Phi(\cdot;\lambda_k), \frac{d}{d\lambda}\Phi(\cdot;\lambda)\Big|_{\lambda_k} \text{ for } k=1,2,\ldots,N\right\}$$

Adjoint basis:

$$\left\{\Phi^A(\cdot;\lambda) \text{ for all real } \lambda;\ \Phi^A(\cdot;\lambda_k), \frac{d}{d\lambda}\Phi^A(\cdot;\lambda)\Big|_{\lambda_k} \text{ for } k=1,2,\ldots,N\right\}$$

Orthogonality relations:

$$\langle \Phi^A(\cdot;\mu), \Phi(\cdot;\lambda)\rangle = \frac{4\pi i}{\lambda} a^2(\lambda)\delta(\lambda-\mu), \quad \lambda, \mu \text{ both real}$$

$$= 0, \quad \text{all other cases.}$$

$$\langle \Phi^A(\cdot;\mu), \dot{\Phi}_j(\cdot)\rangle = \langle \dot{\Phi}_j^A(\cdot), \Phi(\cdot;\mu)\rangle$$

$$= \begin{cases} 0, & \mu \text{ real} \\ -\dfrac{2}{\lambda_j}\dot{a}_j^2\delta_{jk}, & \mu=\lambda_k \end{cases}$$

$$\langle \dot{\Phi}_k^A(\cdot), \dot{\Phi}_j(\cdot)\rangle = \left\{-\frac{d}{d\lambda}\left[\frac{1}{\lambda}\left(\frac{d}{d\lambda}a(\lambda)\right)^2\right]\right\}_{\lambda_j}\delta_{jk}$$

Completeness relation:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta(x-x') = \frac{1}{4\pi i}\int_\Gamma \frac{\lambda}{a^2(\lambda)}\Phi^A(x';\lambda)\Phi^T(x;\lambda)\,d\lambda$$

$$= \frac{1}{4\pi i}\int \frac{\lambda}{a^2(\lambda)}\Phi^A(x';\lambda)\Phi^T(x;\lambda)\,d\lambda + \sum_{k=1}^{N} \text{"discrete state" or residues}$$

that the squared-eigenfunction basis is equivalent to a Fourier basis. We are certain that Kaup's verification can be extended to (A1) although we have not carried it through.

Note from (A9) that our basis of squared eigenfunctions $\Phi(\cdot,\lambda)$ actually consists of $\delta\vec{\phi}(x,t)/\delta\mathcal{S}_+(t)$. Since this basis is used to expand functions of x, it is equivalent to $\delta\vec{\phi}/\delta\mathcal{S}_+(t=0)$ by such formulas as (A10). Finally, the members of the basis $\Phi(\cdot,\lambda)$ are the eigenfunctions for an operator $\mathcal{L}$ which is not self-adjoint; $\Phi^A(\cdot;\lambda)$ are the eigenfunctions of the adjoint of $\mathcal{L}$. This lack of self-adjointness is the reason the "orthogonality relations" are biorthogonality relations involving the pairs (Φ,Φ^A).

The verification of these orthogonality relations is a painful but straightforward manipulation involving Wronskian identities and the asymptotic behavior of the eigenfunctions f and g. Our techniques follow those of Zakharov and Manakov[35] for the Zakharov-Shabat eigenvalue problem. For example, consider

$$\langle \Phi^A(\cdot;\mu), \Phi(\cdot;\lambda)\rangle = \lim_{L\to+\infty}\int_{-L}^{L}[\Phi_1^A(x;\mu)\Phi_1(x;\mu) + \Phi_2^A(x,\mu)\Phi_2(x,\lambda)\,dx$$

$$= \lim_{L\to+\infty}\int_{-L}^{L}\left\{\left[2[f_2^2(x;\mu) - f_1^2(x;\mu)] - \frac{d}{dx}\left(\frac{f_1(x;\mu)f_2(x;\mu)}{\mu}\right)\right]\left(\frac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda}\right)\right.$$

$$-\left(\frac{f_1(x;\mu)f_2(x;\mu)}{\mu}\right)\left[2[g_2^2(x;\lambda) - g_1^2(x;\lambda)]\right.$$

$$\left.\left. - \frac{d}{dx}\left(\frac{g_1(x;\lambda)g_2(x;\lambda)}{\lambda}\right)\right]\right\}dx. \tag{A11}$$

The following Wronskian formula follows directly from the eigenvalue problem (A1):

$$\frac{d}{dx}\left[\left(\frac{g_1(x;\lambda)f_2(x;\mu)}{\lambda}-\frac{g_2(x;\lambda)f_1(x;\mu)}{\mu}\right)\left(\frac{g_1(x;\lambda)f_2(x;\mu)}{\mu}-\frac{g_2(x;\lambda)f_1(x;\mu)}{\lambda}\right)\right]$$
$$=\left(\frac{\lambda}{\mu}-\frac{\mu}{\lambda}\right)\left[\left(\frac{g_1(x;\lambda)f_2(x;\mu)}{\lambda}-\frac{g_2(x;\lambda)f_1(x;\mu)}{\mu}\right)\left(g_2(x;\lambda)f_2(x;\mu)+\frac{1}{16\mu\lambda}e^{iu}g_1(x;\lambda)f_1(x;\mu)\right)\right]$$
$$+\left(\frac{g_1(x;\lambda)f_2(x;\mu)}{\mu}-\frac{g_2(x;\lambda)f_1(x;\mu)}{\lambda}\right)\left[\left(g_1(x;\lambda)f_1(x;\mu)+\frac{1}{16\mu\lambda}e^{-iu}g_2(x;\lambda)f_2(x;\mu)\right)\right].$$

Using the eigenvalue problem (A1), we see that the integrand of (A11) is proportional to the right-hand side of this Wronskian identity; hence the integrand is a perfect x derivative. Evaluation of this integral places (A11) in the form

$$\langle\Phi^A(\cdot;\mu),\Phi(\cdot;\lambda)\rangle=\left\{\frac{\lambda\mu}{(\lambda-\mu)(\lambda+\mu)}\lim_{L\to+\infty}\left[\left(\frac{g_1^2(x;\lambda)f_2^2(x;\mu)+g_2^2(x;\lambda)f_1^2(x;\mu)}{\lambda\mu}\right)\right.\right.$$
$$\left.\left.-\left[\left(\frac{1}{\mu^2}+\frac{1}{\lambda^2}\right)[g_1(x;\lambda)g_2(x;\lambda)f_1(x;\mu)f_2(x;\mu)]\right]_{-L}^{L}\right\}.$$

Next, we use formulas (A2) and (A3) to express the large-L behavior of $g(\pm L;\lambda)$, $f(\pm L;\lambda)$ in terms of the scattering coefficients a and b. The result is

$$\langle\Phi^A(\cdot;\mu),\Phi(\cdot;\lambda)\rangle=2\left(1+\frac{1}{16\lambda\mu}\right)\lim_{L\to+\infty}\left[\frac{\lambda-\mu}{\lambda\mu}[b^{*2}(\lambda)-b^2(\mu)]\frac{\exp\{2i[k(\lambda)+k(\mu)]L\}}{2[k(\lambda)+k(\mu)]}\right.$$
$$\left.+\frac{\lambda+\mu}{\lambda\mu}\left(a^2(\mu)\frac{\exp\{2i[k(\lambda)-k(\mu)]L\}}{2[k(\lambda)-k(\mu)]}-\frac{a^2(\lambda)\exp\{-2i[k(\lambda)-k(\mu)]\}}{2[k(\lambda)-k(\mu)]}\right)\right].$$

This limit must be evaluated in the sense of distributions. We use the identity

$$\lim_{L\to\infty}\frac{e^{ikL}}{k}=\pi i\delta(k)$$

and the symmetry of the scattering coefficients

$$b(-\lambda)=-b^*(\lambda),\quad \lambda \text{ real},$$

to obtain

$$\langle\Phi^A(\cdot,\mu),\Phi(\cdot,\lambda)\rangle=\frac{4\pi i}{\lambda}a^2(\lambda)\delta(\lambda-\mu).$$

The other orthogonality relations are evaluated with similar manipulations.

Since the basis is assumed to be complete, these orthogonality relations quickly yield the "completeness relation." For, consider the vector $\mathrm{col}[\delta(x-x'),0]$ and seek an expansion of the form

$$\begin{pmatrix}\delta(x-x')\\0\end{pmatrix}=\int_{-\infty}^{\infty}\hat{\delta}(\lambda;x')\Phi(x;\lambda)\,d\lambda$$
$$\sum_{k=1}^{N}[\hat{\delta}_k^1(x')\Phi(x;\lambda_k)+\hat{\delta}_k^2(x')\dot{\Phi}(x;\lambda_k)].$$

Taking the inner product with $\Phi^A(x;\lambda')$ yields

$$\Phi_1^A(x';\lambda')=\int_{-\infty}^{\infty}\hat{\delta}(\lambda;x')\langle\Phi^A(\cdot,\lambda'),\Phi(\cdot,\lambda)\rangle\,d\lambda$$
$$=\int_{-\infty}^{\infty}\hat{\delta}(\lambda;x')\left(\frac{4\pi i}{\lambda}a^2(\lambda)\delta(\lambda-\lambda')\right)d\lambda$$
$$=\hat{\delta}(\lambda';x')\left(\frac{4\pi i}{\lambda'}a^2(\lambda')\right),$$

where we have used the orthogonality relations. Thus, the expansion coefficient $\hat{\delta}$ is given by

$$\hat{\delta}(\lambda;x')=\frac{1}{4\pi i}\frac{\lambda}{a^2(\lambda)}\Phi_1^A(x';\lambda).$$

In this manner, the completeness relation is obtained.

This completes our discussion of the properties summarized in Table I. With these properties, the computation of the Green's function $\mathcal{G}$ is straightforward. Recall that $\mathcal{G}$ is defined as the (matrix) solution of

$$L\mathcal{G}=\begin{pmatrix}\partial_t & -1\\ -\partial_{xx}+\cos\phi_0 & \partial_t\end{pmatrix}\mathcal{G}(x,t\,|\,x',t')=0$$

$$\lim_{t\to t'}\mathcal{G}(x,t\,|\,x',t')=\begin{pmatrix}1&0\\0&1\end{pmatrix}\delta(x-x').$$

Thus, the columns of $\mathcal{G}$, as functions of (x,t), belong to the null space of L, and we seek an expansion of the form

$$\mathcal{G}(x,t\,|\,x',t')=\int_\Gamma\frac{\delta\vec{\phi}(x,t)}{\delta\rho_+(\lambda,t=0)}\vec{A}^T(x',t';\lambda)\,d\lambda.$$

Using (A10) places this representation in the form

$$\mathcal{G}(x,t\,|\,x',t')=\frac{i}{\pi}\int_\Gamma\exp[-i\omega(\lambda)t]\Phi(x,t;\lambda)$$
$$\times\vec{A}^T(x',t';\lambda)\,d\lambda.$$

Forcing this solution to satisfy the initial data yields the expansion coefficient $\vec{A}(x',t';\lambda)$:

$$\lim_{t\to t'} \mathcal{G}(x,t\,|\,x',t') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\delta(x-x')$$
$$= \frac{i}{\pi}\int_\Gamma \exp[-i\omega(\lambda)t']\Phi(x,t';\lambda) \times \vec{A}^T(x',t';\lambda)\,d\lambda\,.$$

From the completeness relation we see that $\vec{A}(x',t';\lambda)$ must be given by

$$\vec{A}(x',t';\lambda) = -\tfrac{1}{4}\exp[i\omega(\lambda)t']\frac{\lambda}{a^2(\lambda)} \times \Phi^A(x',t';\lambda)\,.$$

In this way, the Green's function is represented,

$$\mathcal{G}(x,t\,|\,x',t') = \frac{1}{4\pi i}\int_\Gamma \frac{\lambda\,d\lambda}{a^2(\lambda)}\exp[-i\omega(\lambda)(t-t')]\Phi(x,t;\lambda) \times [\Phi^A(x',t';\lambda)]^T\,.$$

Using (A10), the squared eigenfunctions in this formula can be replaced by variations of $\vec{\phi}$ with respect to the scattering data. This leads directly to the representation of $\mathcal{G}_c$ in (6.3).

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$$\int_\infty^\infty \mathcal{G}_d(x,t\,|\,x',t')\,\vec{\mathcal{F}}(x',t')\,dx' = 0,$$
and it is sufficient for our purposes to concentrate upon the continuous component $\mathcal{G}_c$.
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