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Effective electric field in an inhomogeneous medium

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This paper investigates the effective field acting in an infinite, nonmagnetic, inhomogeneous medium. A self-consistent, integrodifferential equation may be written for the effective field at a point, accounting for scattering at every other site in the medium. One may solve this equation by iteration to write the effective field as a power series of operators acting on the uniform-medium effective field. We rewrite this series so that all local fields are augmented by all orders of the self-field. This means that each order of scattering is represented by a single integral operator. The action of this scattering operator on an arbitrary field may be expressed in terms of integrals over reciprocal space, if the product of the fluctuating function and the field is Fourier transformed. One of the integrals contains a weighting function that discriminates in favor of Fourier wave numbers similar to the uniform medium wave number. This integral gives the Bragg-like contributions to the effective field. The other integral extends over all of reciprocal space and contributes equally to the effective field for all fluctuation wave vectors. This integral determines the background effective field. A general term of the complete multiple-scattering expansion for the effective field is given. If the fluctuation distribution has a single cosine Fourier wave vector, then the effective-field expansion simplifies considerably. The Bragg contributions are examined. Finally, the background effective field that ignores all Bragg-like contributions is given, where all orders of scattering are considered. This field is modulated by the fluctuations in the medium in a straightforward way and it obviously provides a better approximation of the local field than either the incident or uniform-medium fields. This background field is used to find the effective dielectric constant that is correct for media in which the inhomogeneities are uncorrelated over lengths comparable to a wavelength.

I. INTRODUCTION

In considering the propagation of an electromagnetic field in an infinite, inhomogeneous, nonmagnetic medium, the effective field that polarizes a macroscopically small region is of vital concern. In classical electromagnetic theory, one is in the habit of dealing with macroscopic fields which are the appropriate averages of the effective fields.[1] The macroscopic fields obey differential equations involving macroscopic parameters (e.g., the dielectric constant). One of the more significant problems of electromagnetic theory in dealing with media is the means by which these macroscopic parameters are related to microscopic properties. This forces consideration of the local effective field. We are familiar with the textbook procedure of finding the contributions to this field from the near and far regions in the medium. The near region is treated discretely and the far region, as if it were continuous. If there is no contribution from the discrete summation, then one is led to the familiar Clausius-Mossotti or Lorentz-Lorenz equation.[2]

In general, the effective field is of interest because: (i) in wave propagation, the coherent, forward propagating field is the average of the effective field and (ii) the scattered field at a point is the summation of fields scattered from every point at which the effective field has acted. Conventional treatments of wave propagation are based upon self-consistent integral equations that relate the effective field at a point to the scattered fields emitted from all other points. These equations are merely self-consistent because the scattered fields are radiated by dipoles induced by the effective field itself. In inhomogeneous media the polarizability fluctuates. But if the range of fluctuation is much smaller than a wavelength, then the entire medium may be considered to be continuous. The effective field can be solved for in terms of the incident field and is found to propagate with the wave number $k_{un}=(\epsilon_{un})^{1/2}k_0$, where ϵ_{un} is the uniform-medium dielectric constant and k_0 is the vacuum wave number, equal to $2\pi/\lambda$. The uniform-medium dielectric constant is of course related to the local polarizability by the Clausius-Mossotti equation. This approximation is appropriate for optical fields in molecular systems, for example.[3]

There are two complementary approaches for dealing with the self-consistent integral equations for the effective field, when one may not ignore the fluctuations in the polarizability. They are multiple scattering and perturbation theories. In a recent paper[4] the two approaches are discussed and the perturbative approach is further developed. There we followed the same procedure used in the uniform case by treating the inhomogeneities as perturbations from a uniform dielectric instead of from vacuum. This results in a perturbation expansion for the effective field in terms of the uniform-medium effective field. In the original form of the perturbation expansion,[5] each member of the expansion contained both a self-field term and a scattering term represented by an integral. In Sec. II of this paper, we rewrite the expansion so that the mth member is of the mth order of the same integral operator. A consequence of this is that at each order of the operator (i.e., for each order of scattering) the local field must be modified by all orders of the self-field.

Section III examines the action of this integral operator on an arbitrary vector field. The spatial dependence of the integrand is complicated by a curl-curl operator acting on the Green's function. Nevertheless, after Fourier transforming the portion of the integrand that fluctuates with the medium, all the spatial integrals may be evaluated. What remains are two types of integrals over reciprocal space. The first contains a weighting function that discriminates in favor of wave numbers similar to the uniform-medium number. This results in Bragg-like contributions. The second integral extends over all of reciprocal space and contains no weighting function. Needless to say, the field obtained after applying the integral operator to an arbitrary vector field contains both nondepolarized and depolarized fields.

Section IV then uses these results to write the first few terms of the expansion for the effective field. One may then generalize to an arbitrary term in an obvious way. In principle, this allows the complete specification of the effective field considering all orders of scattering if the Fourier transform of the fluctuating polarizability is known. In fact, the expression is not very tractable. Section V helps the situation by emphasizing regularity in the medium; we assume that the fluctuating polarizability has only a single cosine Fourier component. This results in a much more tractable series expansion for the effective field. The consequence of the exact Bragg condition is discussed. Finally, if all Bragg-like contributions are ignored, the expansion may be summed to give a closed-form expression for the background effective field. This field is modulated by the fluctuating medium and includes all orders of scattering. This field completely describes all wavelength-independent corrections to the uniform-medium field due to the inhomogeneities.

Section VI discusses the various approximations for the effective field in an inhomogeneous medium. Comparisons are made to results of other authors. The background field is used to derive an expression for the effective dielectric constant that is correct to all orders of scattering as long as the inhomogeneities are uncorrelated over lengths comparable to a wavelength. This result is of general interest and has often been misrepresented. Finally, we include a few comments about the use of the effective field in determining the scattered field and irradiance.

II. PERTURBATION EXPANSIONS FOR THE EFFECTIVE FIELD

In Ref. 4 we considered the propagation of an electromagnetic wave in an infinite, inhomogeneous, and nonmagnetic medium. To start the present discussion, we will write again the perturbation expansion for the effective field as[6]

$$\vec{E}' = (1 + L + LL + \cdots)\,\vec{E}_{un}\,. \tag{2.1}$$

In Eq. (2.1), the operator L acting on the arbitrary vector field $\vec{F}$ is

$$L\vec{F} = c_1\alpha_1(\vec{r})\vec{F}(\vec{r}) + c_2\nabla_r\times\nabla_r \times\int_\sigma G_{un}(|\vec{r}-\vec{r}_1|)\,\alpha_1(\vec{r}_1)\vec{F}(\vec{r}_1)\,d^3r_1\,, \tag{2.2}$$

where c_1 and c_2 are the constants

$$c_1 = -\tfrac{8}{3}\pi(1+\tfrac{8}{3}\pi\,\alpha_{un})^{-1}\,, \tag{2.3a}$$

$$c_2 = (1+\tfrac{8}{3}\pi\,\alpha_{un})^{-1}(1-\tfrac{4}{3}\pi\,\alpha_{un})^{-1}\,. \tag{2.3b}$$

The inhomogeneities of the medium are described in terms of the dimensionless polarization function $\alpha(\vec{r})$,

$$\alpha(\vec{r}) = \alpha_{un} + \alpha_1(\vec{r})\,. \tag{2.4}$$

We have taken the polarizability to be the total dipole moment per unit volume divided by the effective field. We have also assumed that the polarizability varies continuously over a macroscopically small range. Note that the average polarizability is α_{un} (i.e., $\langle\alpha_1\rangle = 0$), and if $\alpha = \alpha_{un}$ then the effective field equals the uniform-medium effective field,

$$\vec{E}'|_{\alpha=\alpha_{un}} = \vec{E}'_{un}\,. \tag{2.5}$$

This field propagates with the wave vector

$$\vec{k}_{un} = (\epsilon_{un})^{1/2}\vec{k}_0\,, \tag{2.6}$$

where $\vec{k}_0$ is the vacuum wave vector, and

$$\epsilon_{un} = (1+\tfrac{8}{3}\pi\,\alpha_{un})/(1-\tfrac{4}{3}\pi\,\alpha_{un})\,, \tag{2.7a}$$

or

$$\tfrac{4}{3}\pi\,\alpha_{un} = (\epsilon_{un}-1)/(\epsilon_{un}+2)\,, \tag{2.7b}$$

the usual Clausius-Mossotti equation. It is important to note that one of the consequences of writing a perturbation expansion in terms of the fluctuations in the polarizability about α_{un} is that the propagator in the integral operator of Eq. (2.2), is the uniform-medium Green's function

$$G_{\rm un}(|\vec{r}-\vec{r}_1|)=e^{(ik_{\rm un}|\vec{r}-\vec{r}_1|)}/|\vec{r}-\vec{r}_1|. \qquad (2.8)$$

Note also that the integral of $L\vec{F}$ excludes the small volume σ about the point of interest. This represents the volume of the infinitesimal region on which the effective field is acting. The limit in which this volume goes to zero is always implied. The first term of the vector field $L\vec{F}$ is the self-field, the second is the field of the summation of polarization potentials. In Eq. (2.1) the term of the nth order in L includes all combinations of m self-field terms and $n-m$ integral terms (where $m=0,1,\ldots,n$). The first task of the present paper is to rewrite Eq. (2.1) as an expansion involving another operator where the action of this new operator may be expressed as a single integral. To begin, we first take the curl curl inside the integral in Eq. (2.2), making use of the identity[7]

$$\nabla_r\times\nabla_r\times\int_\sigma G_{\rm un}(|\vec{r}-\vec{r}_1|)\vec{f}(\vec{r}_1)d^3r_1 = \frac{8\pi}{3}\vec{f}(\vec{r})+\int_\sigma \nabla_r\times\nabla_r\times G_{\rm un}(|\vec{r}-\vec{r}_1|)\vec{f}(\vec{r}_1)d^3r_1, \qquad (2.9)$$

so

$$L\vec{F}=c\alpha_1(\vec{r})\vec{F}(\vec{r}) + c_2\int_\sigma \nabla_r\times\nabla_r\times G_{\rm un}(|\vec{r}-\vec{r}_1|)\,\alpha_1(\vec{r}_1)\vec{F}(\vec{r}_1)\,d^3r_1, \qquad (2.10a)$$

where

$$c=c_1+\tfrac{8}{3}\pi c_2. \qquad (2.10b)$$

To facilitate our reordering of the perturbation expansion, we define

$$\mathcal{L}_{ij}\vec{f}\equiv\int_\sigma \nabla_{r_i}\times\nabla_{r_i}\times G_{\rm un}(|\vec{r}_i-\vec{r}_j|)\vec{f}(\vec{r}_j)\,d^3r_1. \qquad (2.11)$$

Note that the functions inside of $\mathcal{L}_{ij}$ are evaluated at $\vec{r}_j$. Any function appearing without a $\mathcal{L}$ operator to the left is evaluated at the point of interest for the effective field, taken to be $\vec{r}_0$. Then the contribution to $\vec{E}'(\vec{r}_0)$ involving n orders of L is

$$\begin{aligned}\vec{E}_n'(\vec{r}_0)=L^n\vec{E}_{\rm un}'=\{&c^n\alpha_1^n+c^{n-1}c_2[\alpha_1^{n-1}\mathcal{L}_{01}\alpha_1+\alpha_1^{n-2}\mathcal{L}_{01}\alpha_1^2+\cdots+\mathcal{L}_{01}\alpha_1^n]\\&+c^{n-2}c_2^2[\alpha_1^{n-2}\mathcal{L}_{01}\alpha_1\mathcal{L}_{12}\alpha_1+\alpha_1^{n-3}(\mathcal{L}_{01}\alpha_1\mathcal{L}_{12}\alpha_1^2+\mathcal{L}_{01}\alpha_1^2\mathcal{L}_{12}\alpha_1)\\&+\cdots+(\mathcal{L}_{01}\alpha_1\mathcal{L}_{12}\alpha_1^{n-1}+\cdots+\mathcal{L}_{01}\alpha_1^{n-1}\mathcal{L}_{12}\alpha_1)]\\&+\cdots+c_2^n[\mathcal{L}_{01}\alpha_1\mathcal{L}_{12}\alpha_1\cdots\mathcal{L}_{(n-1)n}\alpha_1]\}\vec{E}_{\rm un}'.\end{aligned} \qquad (2.12)$$

If we now sum over all n and group the new series in terms of powers of c_2 (i.e., orders of $\mathcal{L}$), we find that,

$$\vec{E}'(\vec{r}_0)=\sum_{n=0}^\infty \vec{E}_n'=S\{1+c_2\mathcal{L}_{01}S\alpha_1 + c_2^2\mathcal{L}_{01}S\alpha_1\mathcal{L}_{12}S\alpha_1+\cdots\}\vec{E}_{\rm un}', \qquad (2.13a)$$

where S is the power series

$$S=\sum_{n=0}^\infty (c\alpha_1)^n=\frac{1}{1-c\alpha_1}, \qquad (2.13b)$$

which is evaluated at the position vector prescribed by the preceding $\mathcal{L}_{ij}$ operator.

We will now define a new function to describe the inhomogeneous medium,

$$\beta(\vec{r})\equiv c_1\alpha_1(\vec{r})/[1-c\alpha_1(\vec{r})]. \qquad (2.14)$$

Then the expansion for the effective field is

$$\vec{E}'=(1-c\alpha_1)^{-1}(1+\mathcal{L}\beta+\mathcal{L}\beta\mathcal{L}\beta+\cdots)\vec{E}_{\rm un}', \qquad (2.15)$$

where we have dropped the subscripts on the operator $\mathcal{L}$ given by Eq. (2.11). Note that each term contains all powers of α_1, the original fluctuation. Finally, Eq. (2.15) obviously reduces to the following closed form:

$$\vec{E}'=(1-c\alpha_1)^{-1}(1-\mathcal{L}\beta)^{-1}\vec{E}_{\rm un}', \qquad (2.16)$$

where one must determine the operator $(1-\mathcal{L}\beta)^{-1}$.

Equations (2.15) and (2.16) are the principal results of the present section. In Eq. (2.15) each order of $\mathcal{L}\beta$ corresponds to the summation of an order of scattering contributing to the effective field. The consequence of reducing the operator to a single integral is that all the fields are augmented by self-fields to all orders of α_1. That is, the leading term of the expansion goes from $\vec{E}_{\rm un}'$ to $(1-c\alpha_1)^{-1}\vec{E}_{\rm un}'$. The former is the uniform-medium field, the latter is the uniform-medium field plus the self-fields involving all powers of α_1. Similarly, within a multiple scattering term (e.g., $\mathcal{L}\beta\mathcal{L}\beta\ldots\mathcal{L}\beta\vec{E}_{\rm un}'$), any integral now has the form of $\mathcal{L}\beta\vec{F}$, where F stands for the vector field resulting from the operators to the right. $\vec{F}(\vec{r}')$ is the field at $\vec{r}'$ due to the prescribed number of scatterings elsewhere in the medium. Within each integral, the field has been augmented by a factor of $(1-c\alpha_1)^{-1}$ which accounts for all orders of self-fields.

III. VECTOR FIELD $L\beta\vec{F}$

In order to use Eq. (2.15) to determine the effective field in an inhomogeneous medium, one must determine the vector field $\mathcal{L}\beta\vec{F}$, where the operator $\mathcal{L}$ is defined by Eq. (2.11) and $\vec{F}$ is understood to be the field resulting from all the scattering operators acting on $\vec{E}'_{\text{un}}$ to the right of present one. The derivatives of the curl curl in the integrand of Eq. (2.11) act only on the argument of G_{un}. Thus, with the change of variables, $\vec{\rho}=\vec{r}-\vec{r}'$, we may write

$$\nabla_\rho\times\nabla_\rho\times G_{\text{un}}(\rho)\vec{F}=\nabla_\rho[\nabla_\rho\cdot G_{\text{un}}(\rho)\vec{F}]-\nabla^2_\rho[G_{\text{un}}(\rho)\vec{F}]. \quad (3.1a)$$

But,

$$-\nabla^2_\rho[G_{\text{un}}(\rho)\vec{F}]=-\vec{F}\,\nabla^2_\rho G_{\text{un}}(\rho)=+\vec{F}[k^2_{\text{un}}G_{\text{un}}(\rho)+4\pi\delta(\rho)]\to k^2_{\text{un}}\vec{F}G_{\text{un}}(\rho), \quad (3.1b)$$

since $\rho=0$ is excluded from the integral.

Then considering the vector components of $\vec{F}$, and hence of $\nabla_\rho[\nabla_\rho\cdot G_{\text{un}}(\rho)\vec{F}]$, we can give the field $\mathcal{L}\beta\vec{F}$ in the tensor form

$$\mathcal{L}\beta\vec{F}=\vec{\vec{\mathcal{L}}}\cdot\beta\vec{F}, \quad (3.2a)$$

where

$$(\mathcal{L}\beta\vec{F})_i=(\vec{\vec{\mathcal{L}}}\cdot\beta\vec{F})_i=\int_\sigma d^3\rho\, G_{\text{un}}(\rho)g_{ij}(\vec{\rho})\times\beta(\vec{r}+\vec{\rho})F_j(\vec{r}+\vec{\rho}), \quad (3.2b)$$

and

$$g_{ij}(\vec{\rho})=\left(k^2_{\text{un}}+\frac{1}{\rho}(ik_{\text{un}}-1/\rho)\right)\delta_{ij}+\frac{\rho_i\rho_j}{\rho^2}\left(-k^2_{\text{un}}-\frac{3ik_{\text{un}}}{\rho}+\frac{3}{\rho^2}\right). \quad (3.2c)$$

In Eq. (3.2b) we have used the summation convection, all repeated subscripts are summed over the three coordinate directions. Equation (3.2b) gives the ith coordinate contribution to $\mathcal{L}\beta\vec{F}$ evaluated at the position $\vec{r}$. If we take the Fourier transform of βF_j, then the integral over coordinate space in Eq. (3.2b) may be evaluated. That is, we will write

$$\beta(\vec{r}+\vec{\rho})F_j(\vec{r}+\vec{\rho})=\int\frac{d^3\kappa}{(2\pi)^3}e^{i\vec{\kappa}\cdot\vec{\rho}}\,\mathfrak{F}_j(\vec{r},\vec{\kappa}), \quad (3.3)$$

where by the convolution theorem

$$\mathfrak{F}_j(\vec{r},\vec{\kappa})=\int\frac{d^3\kappa'}{(2\pi)^3}\beta(\vec{r},\vec{\kappa}-\vec{\kappa}')F_j(\vec{r},\vec{\kappa}'), \quad (3.4)$$

and $\beta(\vec{r},\vec{\kappa})$ and $F_j(\vec{r},\vec{\kappa})$ are the Fourier transforms

$$\beta(\vec{r},\vec{\kappa})=\int d^3\rho\, e^{-i\vec{\kappa}\cdot\vec{\rho}}\beta(\vec{r}+\vec{\rho}), \quad (3.5a)$$

$$F_j(\vec{r},\vec{\kappa})=\int d^3\rho\, e^{-i\vec{\kappa}\cdot\vec{\rho}}F_j(\vec{r}+\vec{\rho}). \quad (3.5b)$$

Note that all of these transforms are between ρ and κ space; all functions remain dependent upon the position $\vec{r}$. Within these equations that dependence may be viewed as fixing the origin; though, by writing the arguments in this fashion we are explicitly allowing for the inhomogeneous nature of the medium. Using Eq. (3.3) in Eq. (3.2b), the ith contribution to $\mathcal{L}\beta\vec{F}$ is

$$(\mathcal{L}\beta\vec{F})_i=\int\frac{d^3\kappa}{(2\pi)^3}\mathfrak{F}_j(\vec{r},\vec{\kappa})\int_\sigma d^3\rho\, e^{i\vec{\kappa}\cdot\vec{\rho}}G_{\text{un}}(\rho)g_{ij}(\vec{\rho}) \quad (3.6a)$$

$$=\int\frac{d^3\kappa}{(2\pi)^3}\mathfrak{F}_j(\vec{r},\vec{\kappa})I_{ij}(\vec{\kappa}). \quad (3.6b)$$

We have defined $I_{ij}(\vec{\kappa})$ to be the spatial integral in Eq. (3.6a), which is evaluated in the Appendix.

Using Eq. (A13), we rewrite Eq. (3.6b) as

$$(\mathcal{L}\beta\vec{F})_i=\int\frac{d^3\kappa}{(2\pi)^3}\left\{-\frac{8\pi}{3}\delta_{ij}+4\pi(\kappa^2\delta_{ij}-\kappa_i\kappa_j)\times\left[\mathcal{P}\left(\frac{1}{\kappa^2-k^2_{\text{un}}}\right)+\frac{i\pi}{2k_{\text{un}}}\delta(\kappa-k_{\text{un}})\right]\right\}\mathfrak{F}_j(\vec{r},\vec{\kappa}), \quad (3.7)$$

where $\mathcal{P}$ stands for the Cauchy principal value as in Eq. (A11). We may now choose the coordinate directions. If we use spherical coordinates with respect to the vector $\vec{\kappa}$, then the tensor $\vec{\vec{\mathcal{L}}}$ is diagonal, and the vector field $\mathcal{L}\beta\vec{F}$ may be written

$$\mathcal{L}\beta\vec{F}=4\pi\int\frac{d^3\kappa}{(2\pi)^3}\{[-\tfrac{2}{3}+\Delta(\kappa)][\hat{\theta}\,\mathfrak{F}_\theta(\vec{r},\vec{\kappa})+\hat{\phi}\,\mathfrak{F}_\phi(\vec{r},\vec{\kappa})]-\tfrac{2}{3}\hat{\kappa}\,\mathfrak{F}_\kappa(\vec{r},\vec{\kappa})\} \quad (3.8a)$$

$$=4\pi\left(-\frac{2}{3}\int\frac{d^3\kappa}{(2\pi)^3}\vec{\mathfrak{F}}(\vec{r},\vec{\kappa})+\int\frac{d^3\kappa}{(2\pi)^3}\Delta(\kappa)P_T(\vec{\kappa})\vec{\mathfrak{F}}(\vec{r},\vec{\kappa})\right), \quad (3.8b)$$

where

$$\Delta(\kappa)=\kappa^2\left[\mathcal{P}\left(\frac{1}{\kappa^2-k^2_{\text{un}}}\right)+\frac{i\pi}{2k_{\text{un}}}\delta(\kappa-k_{\text{un}})\right], \quad (3.8c)$$

$P_T(\vec{\kappa})$ is the transverse projection operator with respect to $\vec{\kappa}$, and $(\hat{\kappa}, \hat{\theta}, \hat{\phi})$ are the spherical unit vectors at the position $\vec{\kappa}$. Note in Eq. (3.8a) that there is no mixing among the transverse and longitudinal components of the field with respect to the $\hat{\kappa}$ direction.

Let us consider the second integral in Eq. (3.8b). It has the form

$$4\pi\int \frac{d^3\kappa}{(2\pi)^3}\Delta(\kappa)P_T\vec{\mathfrak{F}}(\vec{r},\vec{\kappa}) = \frac{1}{2\pi^2}\int_0^\infty \kappa^4 d\kappa \oint d\Omega\left[\mathcal{P}\left(\frac{1}{\kappa^2-k_{un}^2}\right)+\frac{i\pi}{2k_{un}}\delta(\kappa-k_{un})\right]P_T\vec{\mathfrak{F}}(\vec{r},\vec{\kappa})$$
$$= ik_{un}^3\langle P_T\vec{\mathfrak{F}}(\vec{r},k_{un})\rangle + \frac{2}{\pi}\lim_{\epsilon\to 0}\left(\int_0^{k_{un}-\epsilon} d\kappa \frac{\kappa^4\langle P_T\vec{\mathfrak{F}}(\vec{r},\kappa)\rangle}{\kappa^2-k_{un}^2} + \int_{k_{un}+\epsilon}^\infty d\kappa\frac{\kappa^4\langle P_T\vec{\mathfrak{F}}(\vec{r},\kappa)\rangle}{\kappa^2-k_{un}^2}\right), \quad (3.9a)$$

where we have used

$$4\pi\langle P_T\vec{\mathfrak{F}}(\vec{r},\kappa)\rangle \equiv \oint d\Omega\, P_T\vec{\mathfrak{F}}(\vec{r},\vec{\kappa}). \quad (3.9b)$$

From Eq. (3.9a) we see the role of the function $\Delta(\kappa)$, within an integral in κ space. It produces a summation that is weighted in favor of wave vectors of magnitude similar to k_{un}. That is, the second integral of Eq. (3.8b) draws primarily from the transverse components of $\vec{\mathfrak{F}}(\vec{r},\vec{\kappa})$ that have $\kappa \simeq k_{un}$. In a perturbative sense, the field $\vec{F}$ will have Fourier wave vectors similar to $\vec{k}_{un}$. Combining these two conditions and using Eq. (3.4), we see that the weighting function $\Delta(\kappa)$ causes contributions to the field that draw upon a limited range of Fourier components of the fluctuation function β. That is, the most significant components will have wave vectors $(\vec{Q})$ that obey the following approximate selection rules:

$$\vec{Q} \simeq \vec{\kappa} - \vec{k}_{un} \text{ with } \kappa \simeq k_{un}. \quad (3.10)$$

This constitutes a loose form of the Bragg condition. Hence, we will refer to all field contributions that involve the weighting function $\Delta(\kappa)$ as being Bragg-like.

The first integral of Eq. (3.8b) is decidedly different. It involves no weighting function; and therefore, draws indiscriminately from all Fourier components of β. Contributions of this sort will determine the background effective field.

In writing the general form of the field $\mathcal{L}\beta F$, it will be useful to use the Cartesian coordinates $(\hat{x}, \hat{y}, \hat{z})$, with $\hat{z}$ and $\vec{k}_{un}$ aligned. Then the tensor $\mathcal{L}$ is not diagonal, though it is symmetrical. A generalized expression for the Cartesian components of the field $\mathcal{L}\beta\vec{F}$ may be written from Eq. (3.7) as

$$(\mathcal{L}\beta\vec{F})_i = -\frac{8\pi}{3}\int\frac{d^3\kappa}{(2\pi)^3}\mathfrak{F}_i + 4\pi\int\frac{d^3\kappa}{(2\pi)^3}\Delta(\kappa)\{[1-(\hat{\kappa})_i^2]\mathfrak{F}_i - (\hat{\kappa})_i(\hat{\kappa})_j\mathfrak{F}_j - (\hat{\kappa})_i(\hat{\kappa})_k\mathfrak{F}_k\}. \quad (3.11)$$

Here i denotes one Cartesian component and j and k, the other two and $(\hat{\kappa})_i = \hat{i}\cdot\hat{\kappa}$, etc. Again, we find two types of summations in κ space contributing to the field. In Eq. (3.11), the first integral is over all of κ space with no wave vector-dependent weighting function and results in a contribution to the background field. The second integral contains not only $\Delta(\kappa)$, but also geometric terms that resulted from our change to Cartesian coordinates. Referring to Eq. (3.11), we will write the field $\mathcal{L}\beta\vec{F}$ in tensor form as

$$\mathcal{L}\beta\vec{F} = 4\pi\int\frac{d^3\kappa}{(2\pi)^3}\vec{\gamma}(\vec{\kappa})\cdot\vec{\mathfrak{F}}(\vec{r},\vec{\kappa}), \quad (3.12a)$$

where the components of the tensor are given by

$$\gamma_{ij}(\vec{\kappa}) = [-\tfrac{2}{3}+\Delta(\kappa)]\delta_{ij} - \Delta(\kappa)(\hat{\kappa})_i(\hat{\kappa})_j. \quad (3.12b)$$

IV. GENERAL TERM IN THE EXPANSION FOR THE EFFECTIVE FIELD

The vector field $\mathcal{L}\beta[\mathcal{L}\beta\cdots\mathcal{L}\beta\vec{E}'_{un}] = \mathcal{L}\beta\vec{F}$ is a general term of the expansion for the effective field, Eq. (2.15). Equations (3.8) and (3.11) specify $\mathcal{L}\beta\vec{F}$ in terms of summations involving the Fourier transform $\vec{\mathfrak{F}}(\vec{r},\vec{\kappa})$, Eq. (3.4). In this section we will first write $\mathfrak{F}$ in terms of Fourier transforms of powers of the polarizability fluctuations, $\alpha_1(\vec{r})$, to then derive the general form of the field $\mathcal{L}\beta\vec{F}$. From Eq. (2.14), we write

$$\beta(\vec{r}) = \frac{c_2\alpha_1(\vec{r})}{1-c\alpha_1(\vec{r})} = c_2\sum_{n=0}^\infty c^n\alpha_1^{n+1}(\vec{r}), \quad (4.1)$$

where we want to evaluate

$$\beta(\vec{r},\vec{\kappa}) = \int d^3\rho\, e^{-i\vec{\kappa}\cdot\vec{\rho}}\beta(\vec{r}+\vec{\rho}), \quad (4.2)$$

in order to rewrite

$$\vec{\mathfrak{F}}(\vec{r},\vec{\kappa}) = \int\frac{d^3\kappa'}{(2\pi)^3}\beta(\vec{r},\vec{\kappa}')\vec{F}(\vec{r},\vec{\kappa}-\vec{\kappa}'). \quad (4.3)$$

In Eq. (4.3), $\vec{F}(\vec{r},\vec{\kappa})$ is the Fourier transform of the field $\vec{F} = \mathcal{L}\beta\cdots\mathcal{L}\beta\vec{E}'_{un}$. We will assume that $\alpha_1^m(\vec{r})$ has the Fourier transform $\alpha_1^{(m)}(\vec{r},\vec{\kappa})$ (note that in the transform the superscript in parentheses does not denote a power); that is,

$$\alpha_1^m(\vec{r}+\vec{\rho}) = \int\frac{d^3\kappa_m}{(2\pi)^3}e^{i\vec{\kappa}_m\cdot\vec{\rho}}\alpha_1^{(m)}(\vec{r},\vec{\kappa}_m). \quad (4.4)$$

Then, using Eqs. (4.1) and (4.4) in Eq. (4.2), we write

$$\beta(\vec{r},\vec{\kappa}) = c_2 \sum_n c^n \int \frac{d^3\kappa_{n+1}}{(2\pi)^3}\, \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}_{n+1}) \times \int d^3\rho\, e^{-i\vec{\rho}\cdot(\vec{\kappa}-\vec{\kappa}_{n+1})}$$

$$= c_2 \sum_n c^n \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}). \qquad (4.5)$$

Then, using Eq. (4.5) in Eq. (4.3), we write,

$$\mathfrak{F}(\vec{r},\vec{\kappa}) = c_2 \sum_n c^n \int \frac{d^3\kappa'}{(2\pi)^3}\, \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}')\vec{F}(\vec{r},\vec{\kappa}-\vec{\kappa}'). \qquad (4.6)$$

We will now give the expressions for contributions to the effective field due to first- and second-order scattering, at which point the form of the mth-order term will be apparent. From Eq. (2.13), the first-order term is $\mathcal{L}\beta\vec{E}'_{un}$; that is, $\vec{F}_1 = \vec{E}'_{un}$. The uniform-medium effective field is a plane wave if the incident wave is planar. Hence, we will write

$$\vec{F}_1 = \hat{e}\, E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}, \qquad (4.7a)$$

and

$$\vec{F}_1(\vec{r},\vec{\kappa}) = \vec{E}'_{un}(\vec{r})(2\pi)^3\delta(\vec{\kappa}-\vec{k}_{un}), \qquad (4.7b)$$

where $\vec{k}_{un}$ is the uniform-medium wave vector, extending in the $\hat{z}$ direction. Using Eq. (4.7b) in Eq. (4.6), we find that

$$\mathfrak{F}_1(\vec{r},\vec{\kappa}) = \vec{E}'_{un}(\vec{r})\, c_2 \sum_n c^n \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}-\vec{k}_{un}). \qquad (4.8)$$

Then using Eqs. (4.8) and (3.12a), the field $\mathcal{L}\beta\vec{E}'_{un}$ may be written as

$$\mathcal{L}\beta\vec{E}'_{un} = E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}\, 4\pi c_2 \times \sum_{n_1} c^{n_1} \int \frac{d^3\kappa_1}{(2\pi)^3}\, \alpha_1^{(n_1+1)}(\vec{r},\vec{\kappa}_1-\vec{k}_{un})\vec{\gamma}(\vec{\kappa}_1)\cdot\hat{e}. \qquad (4.9)$$

The second-order term is $\mathcal{L}\beta\mathcal{L}\beta\vec{E}'_{un} = \mathcal{L}\beta\vec{F}_2$, where $\vec{F}_2 = \mathcal{L}\beta\vec{E}'_{un}$ and is given by Eq. (4.9). First write out the transform $\vec{F}_2(\vec{r},\vec{\kappa})$ and then substitute it into Eq. (4.6) to find $\mathfrak{F}_2(\vec{r},\vec{\kappa})$. After propagating the δ functions through the resulting expression, we obtain

$$\mathfrak{F}_2(\vec{r},\vec{\kappa}) = E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}\, 4\pi c_2^2 \sum_{n_1 n_2} c^{n_1+n_2} \int \frac{d^3\kappa_2}{(2\pi)^3}\, \alpha_1^{(n_1+1)}(\vec{r},\vec{\kappa}-\vec{\kappa}_2)\, \alpha_1^{(n_2+1)}(\vec{r},\vec{\kappa}_2-\vec{k}_{un})\vec{\gamma}(\vec{\kappa}_2)\cdot\hat{e}. \qquad (4.10)$$

Again using Eq. (3.12a), the contribution to the effective field involving second-order scattering is

$$\mathcal{L}\beta\mathcal{L}\beta\vec{E}'_{un} = E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}(4\pi c_2)^2 \sum_{n_1 n_2} c^{n_1+n_2} \int \frac{d^3\kappa_1}{(2\pi)^3} \int \frac{d^3\kappa_2}{(2\pi)^3}\, \alpha_1^{(n_1+1)}(\vec{r},\vec{\kappa}_1-\vec{\kappa}_2)\, \alpha_1^{(n_2+1)}(\vec{r},\vec{\kappa}_2-\vec{k}_{un})\vec{\gamma}(\vec{\kappa}_1)\cdot\vec{\gamma}(\vec{\kappa}_2)\cdot\hat{e}. \qquad (4.11)$$

We may now generalize in the obvious way to write the mth order contribution as

$$\vec{E}'_m \equiv (\mathcal{L}\beta)^m\vec{E}'_{un} = E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}(4\pi c_2)^m \times \sum_{n_1\cdots n_m} c^{n_1+\cdots+n_m} \int \frac{d^3\kappa_1}{(2\pi)^3} \cdots \int \frac{d^3\kappa_m}{(2\pi)^3}\, \alpha_1^{(n_1+1)}(\vec{r},\vec{\kappa}_1-\vec{\kappa}_2)\cdots \alpha_1^{(n_{m-1}+1)}(\vec{r},\vec{\kappa}_{m-1}-\vec{\kappa}_m) \times \alpha_1^{(n_m+1)}(\vec{r},\vec{\kappa}_m-\vec{k}_{un})\,\vec{\gamma}(\vec{\kappa}_1)\cdots\vec{\gamma}(\vec{\kappa}_m)\cdot\hat{e}, \qquad (4.12a)$$

where we can now write Eq. (2.15) as

$$\vec{E}' = (1-c\alpha_1)^{-1}(\vec{E}'_{un} + \vec{E}'_1 + \vec{E}'_2 + \cdots). \qquad (4.12b)$$

Equations (4.12) are the major result of the present section. If the Fourier transform of all powers of the polarizability fluctuation function $\alpha_1(\vec{r})$ about all points in the medium are known, then the effective field is completely specified by these equations.

V. EFFECTIVE FIELD DUE TO A SINGLE-COSINE FOURIER COMPONENT OF THE POLARIZABILITY FLUCTUATION

Equations (4.12) specify the effective field in an inhomogeneous medium in terms of the Fourier transform

$$\alpha_1^{(n+1)}(\vec{r},\vec{\kappa}) = \int \rho d^3\, e^{-i\vec{\rho}\cdot\vec{\kappa}}\, \alpha_1^{n+1}(\vec{r}+\vec{\rho}), \qquad (5.1)$$

where $\alpha_1(\vec{r}+\vec{\rho})$ is the polarizability fluctuation at $\vec{r}+\vec{\rho}$, Eq. (2.4). In simple periodic media, the transforms of $\alpha_1(\vec{r})$ will have few Fourier components. If the primary cause of the polarizability fluctuations is a planar acoustical wave, then α_1 will have the same single-cosine Fourier component at all points. Even in random media, some components may reappear everywhere. For every coherence length of the medium (e.g., that associated with the nearest-neighbor distance) there will be a characteristic Fourier component of $\alpha_1(\vec{r})$. These considerations lead us to find the general contribution to the effective field for a single Fourier component of $\alpha_1(\vec{r})$ of wave vector $\vec{q}$.

We will assume that the fluctuating function has the form

$$\alpha_1(\vec{r}) = \alpha_1 \cos(\vec{q}\cdot\vec{r}) = \tfrac{1}{2}\alpha_1(e^{i\vec{q}\cdot\vec{r}} + e^{-i\vec{q}\cdot\vec{r}}). \quad (5.2)$$

Note that $\alpha_1(\vec{r})$ remains real at every point in space and that α_1 is now merely the fluctuation amplitude. We will have to restrict ourselves to placing the origin at a maximum of $\alpha_1(\vec{r})$. For an infinite beam in an infinite medium this should present no difficulties. Taking the $n+1$ power of $\alpha_1(\vec{r})$ in Eq. (5.2) and using it in Eq. (5.1), we may write the convoluted Fourier transform as

$$\alpha_1^{(n+1)}(\vec{r},\vec{\kappa}) = \left(\frac{\alpha_1}{2}\right)^{n+1} \int d^3\rho\, e^{-i\vec{\rho}\cdot\vec{\kappa}}[e^{i(n+1)\vec{q}\cdot(\vec{r}+\vec{\rho})} + (n+1)e^{i(n-1)\vec{q}\cdot(\vec{r}+\vec{\rho})} + \cdots + e^{-i(n+1)\vec{q}\cdot(\vec{r}+\vec{\rho})}]$$

$$= (2\pi)^3\left(\frac{\alpha_1}{2}\right)^{n+1} e^{i\vec{\kappa}\cdot\vec{r}}\{\delta[\vec{\kappa}-(n+1)\vec{q}] + (n+1)\delta[\vec{\kappa}-(n-1)\vec{q}] + \cdots + \delta[\vec{\kappa}+(n+1)\vec{q}]\}. \quad (5.3)$$

In our expression for the mth-order contribution to the effective field, Eq. (4.12a), each $\alpha_1^{(n+1)}(\vec{r},\vec{\kappa})$ is within a summation over n from zero to infinity. Hence, we should consider the sum

$$\sum_{n=0}^{\infty} c^n \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}) = (2\pi)^3 e^{i\vec{\kappa}\cdot\vec{r}}\left[\left(\frac{\alpha_1}{2}\right)[\delta(\vec{\kappa}-\vec{q}) + \delta(\vec{\kappa}+\vec{q})] + c\left(\frac{\alpha_1}{2}\right)^2[\delta(\vec{\kappa}-2\vec{q}) + 2\delta(\vec{\kappa}) + \delta(\vec{\kappa}+2\vec{q})]\right.$$

$$\left. + c^2\left(\frac{\alpha_1}{2}\right)^3[\delta(\vec{\kappa}-3\vec{q}) + 3\delta(\vec{\kappa}-\vec{q}) + 3\delta(\vec{\kappa}+\vec{q}) + \delta(\vec{\kappa}+3\vec{q})] + \cdots\right]; \quad (5.4a)$$

or, after reordering the summation in terms of the δ functions, we have

$$\sum_{n=0}^{\infty} c^n \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}) = \frac{(2\pi)^3}{c} e^{i\vec{\kappa}\cdot\vec{r}}\left\{[\delta(\vec{\kappa})]\left[2\left(\frac{c\alpha_1}{2}\right)^2 + 6\left(\frac{c\alpha_1}{2}\right)^4 + 20\left(\frac{c\alpha_1}{2}\right)^6 + \cdots\right]\right.$$

$$+ [\delta(\vec{\kappa}-\vec{q}) + \delta(\vec{\kappa}+\vec{q})]\left(\frac{c\alpha_1}{2}\right)\left[1 + 3\left(\frac{c\alpha_1}{2}\right)^2 + 10\left(\frac{c\alpha_1}{2}\right)^4 + 35\left(\frac{c\alpha_1}{2}\right)^6 + \cdots\right]$$

$$\left. + [\delta(\vec{\kappa}-2\vec{q}) + \delta(\vec{\kappa}+2\vec{q})]\left(\frac{c\alpha_1}{2}\right)^2\left[1 + 4\left(\frac{c\alpha_1}{2}\right)^2 + 15\left(\frac{c\alpha_1}{2}\right)^4 + 56\left(\frac{c\alpha_1}{2}\right)^6 + \cdots\right] + \cdots\right\}. \quad (5.4b)$$

Note that the coefficients of the power series in Eq. (5.4b) may be obtained by using the vertical columns of Pascal's triangle moving out from the center. Writing the power series in a general form, Eq. (5.4b) reduces to

$$\sum_{n=0}^{\infty} c^n \alpha_1^{(n+1)}(\vec{r},\vec{\kappa}) = \frac{(2\pi)^3}{c} e^{i\vec{\kappa}\cdot\vec{r}} \sum_{n=-\infty}^{\infty} \delta(\vec{\kappa}-n\vec{q})\left(\frac{c\alpha_1}{2}\right)^{|n|} s_{|n|}\left(\frac{c\alpha_1}{2}\right), \quad (5.5a)$$

where

$$s_n(x) \equiv \sum_{l=0}^{\infty} \frac{(2l+n)!}{l!(l+n)!} x^{2l} \quad \text{for } n>0, \quad (5.5b)$$

and

$$s_0(x) \equiv \left(\sum_{l=0}^{\infty} \frac{(2l)!}{2(l!)^2} x^{2l}\right) - \frac{1}{2}$$

$$= \tfrac{1}{2}[(1-4x^2)^{-1/2} - 1]. \quad (5.5c)$$

Note that $s_0(x)$ is small for small x; that is,

$$\lim_{x\to 0} s_0(x) = x^2. \quad (5.5d)$$

Special attention must be paid to s_0 because the Fourier transform of $\alpha_1^0(\vec{r})$ is not required; that is, in Eq. (5.4a), $\delta(\vec{\kappa})$ does not appear by itself, but rather first appears in the $n+1=2$ binomial expansion.

We may now use Eq. (5.5a) in Eq. (4.12a) to write an expression for the mth-order contribution to the effective field when the fluctuation has a single-cosine Fourier component. After making use of all the δ functions, we have

$$\vec{E}'_m(\vec{r};\vec{q}) = E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}\left(\frac{4\pi c_2}{c}\right)^m \sum_{n_1\cdots n_m=-\infty}^{+\infty}\left(\frac{c\alpha_1}{2}\right)^{|n_1|+\cdots+|n_m|}\left[s_{|n_1|}\left(\frac{c\alpha_1}{2}\right)\cdots s_{|n_m|}\left(\frac{c\alpha_1}{2}\right)\right] e^{i(n_1+\cdots+n_m)\vec{q}\cdot\vec{r}}$$
$$\times[\tilde{\gamma}(\vec{k}_{un}+(n_1+\cdots+n_m)\vec{q})\cdots\tilde{\gamma}(\vec{k}_{un}+n_m\vec{q})\cdot\hat{e}], \tag{5.6}$$

where the summations over $n_1\cdots n_m$ are all from $-\infty$ to $+\infty$. If we define the vector $\vec{e}_m$ as

$$\vec{e}_m = \sum_{n_1\cdots n_m=-\infty}^{+\infty}\left(\frac{c\alpha_1}{2}\right)^{|n_1|+\cdots+|n_m|}\left[s_{|n_1|}\left(\frac{c\alpha_1}{2}\right)\cdots s_{|n_m|}\left(\frac{c\alpha_1}{2}\right)\right] e^{i(n_1+\cdots+n_m)\vec{q}\cdot\vec{r}}$$
$$\times\{\tilde{\gamma}[\vec{k}_{un}+(n_1+\cdots+n_m)\vec{q}]\cdots\tilde{\gamma}(\vec{k}_{un}+n_m\vec{q})\cdot\hat{e}\}, \tag{5.7}$$

then, using Eqs. (5.7) and (5.6) in Eq. (4.12b), the effective field may be written

$$\vec{E}'(\vec{r};\vec{q}) = [1 - c\alpha_1(\vec{r})]^{-1}\left[\hat{e} + \sum_{m=1}^{\infty}\left(\frac{4\pi c_2}{c}\right)^m \vec{e}_m\right] E'_{un}\, e^{i\vec{k}_{un}\cdot\vec{r}}. \tag{5.8}$$

Because the effective field depends upon $\alpha_1(\vec{r})$ in a nonlinear fashion, the expressions of Eqs. (5.6)–(5.8) involve the wave vector $\vec{q}$ only as a parameter. If $\alpha_1(\vec{r})$ had more than one Fourier component, then transforms of powers of $\alpha_1(\vec{r})$ would involve cross products of these components. Such terms would describe scattering involving more than one Fourier wave vector. Hence, the effective field of Eq. (5.8) may not be summed in q space to give the complete field at $\vec{r}$; that is,

$$\vec{E}'(\vec{r}) \neq \int d^3q\, \vec{E}'(\vec{r};\vec{q}). \tag{5.9}$$

This summation is incomplete. This point bears upon the nature of the components of the tensor $\tilde{\gamma}$ that appear in Eqs. (5.6) and (5.7) and from Eqs. (3.12b) and (3.8c), have the form

$$\gamma_{ij}(\vec{k}_{un}+n\vec{q}) = \gamma_{ij}(\vec{\kappa}_n) = [-\tfrac{2}{3} + \Delta(\kappa_n)]\delta_{ij} - \Delta(\kappa_n)(\hat{\kappa}_n)_i(\hat{\kappa}_n)_j, \tag{5.10a}$$

where

$$\Delta(\kappa_n) = \mathcal{P}\left(\frac{\kappa_n^2}{\kappa_n^2 - k_{un}^2}\right) + \tfrac{1}{2} i\pi k_{un}\delta(\kappa_n - k_{un}), \tag{5.10b}$$

and where we have let $\vec{\kappa}_n = \vec{k}_{un} + n\vec{q}$. In Eq. (5.10a), the term that is independent of $\vec{\kappa}_n$ produces background contributions to the field in that it contains no $\vec{\kappa}_n$-dependent weighting function. The other terms contain the function $\Delta(\kappa_n)$ which discriminates in favor of wave vectors of magnitude similar to k_{un}. Hence we have referred to these terms as Bragg-like. However, note that in Eqs. (5.6)–(5.8) the tensors $\tilde{\gamma}$ are not contained within an integral over κ_n space; i.e., over q space. This means that the weighting function $\Delta(\kappa_n)$ appropriate for a single-fluctuation wave vector, has the form

$$\Delta_n \equiv \Delta(\kappa_n) = \frac{|\vec{k}_{un}+n\vec{q}|^2}{|\vec{k}_{un}+n\vec{q}|^2 - k_{un}^2} + \tfrac{1}{2} i\pi k_{un}\delta(|\vec{k}_{un}+n\vec{q}| - k_{un}) \tag{5.11a}$$

$$= \frac{(k_{un}/nq) + 2\hat{k}_{un}\cdot\hat{q} + (nq/k_{un})}{(2\hat{k}_{un}\cdot\hat{q} + nq/k_{un})} + \tfrac{1}{2} i\pi k_{un}\delta(|\vec{k}_{un}+n\vec{q}| - k_{un})$$
$$= 1 + \zeta_n(\vec{q},\vec{k}_{un}) + \tfrac{1}{2} i\pi k_{un}\delta(|\vec{k}_{un}+n\vec{q}| - k_{un}), \tag{5.11b}$$

where we have defined

$$\zeta_n(\vec{q},\vec{k}_{un}) \equiv \frac{(k_{un}/q)^2}{n[n + 2(k_{un}/q)\hat{k}_{un}\cdot\hat{q}]}. \tag{5.11c}$$

Note that q may never be zero. If it were, then by Eq. (5.2), $\alpha_1(\vec{r})$ would be a constant in coordinate space with a nonzero mean, which is not possible. In rewriting Δ_n as in Eq. (5.11b), another wave-vector-independent term has emerged. It will thus be useful to rewrite Eq. (5.10a) as

$$\gamma_{ij}(\vec{\kappa}_n) = (\tfrac{1}{3} + \Delta'_n)\delta_{ij} - (1 + \Delta'_n)(\hat{\kappa}_n)_i(\hat{\kappa}_n)_j, \tag{5.12a}$$

where

$$\Delta'_n \equiv \zeta_n(\vec{q},\vec{k}_{un}) + \tfrac{1}{2} i\pi k_{un}\delta(|\vec{k}_{un}+n\vec{q}| - k_{un}). \tag{5.12b}$$

We recognize the consequence of fulfilling the exact Bragg condition, $\kappa_n = |\vec{k}_{un}+n\vec{q}| = k_{un}$. The weighting function Δ'_n, Eq. (5.12b), becomes infinite.

When Eqs. (5.12), (5.5), and (5.7) are used in Eq. (5.8), one may calculate the effective field in the medium which fluctuates with the single-cosine Fourier wave vector $\vec{q}$. This is the principal result of the present section. In the majority of cases, Eq. (5.8) would be treated termwise with the effective field approximated by the truncated series. However, there are a few special cases which we will now investigate.

A. Bragg condition fulfilled

The Bragg condition is expressed as

$$|\vec{k}_{un}+n\vec{q}| = k_{un}, \tag{5.13}$$

where n is any integer. In the context of the present discussion, whenever the Bragg condition is satisfied, the weighting function Δ_n', Eq. (5.12b), becomes infinite. This is because we have assumed an infinite beam in an infinite medium, a medium which fluctuates with a single-cosine Fourier wave vector. The infinity is due to a naked one-dimensional δ function times $k_{\rm un}$. In a real medium, one would have a function that scales like the size of the medium and/or the beam times $k_{\rm un}$; hence, at the exact fulfillment of the Bragg condition, the weighting function becomes large, not infinite.

The easiest way that Eq. (5.13) may be obeyed is for n to be zero. This state must be considered because even powers of $\alpha_1(\vec{r}) \sim \cos(\vec{q}\cdot\vec{r})$ result in spatially constant terms over the infinite medium. Primary among such contributions will be the case when all n's in Eq. (5.7) are zero. We then may write

$$\vec{e}_m|_{{\rm all}\, n=0} = \hat{e}\left[s_0\left(\frac{c\alpha_1}{2}\right)\Delta_0'\right]^m , \tag{5.14a}$$

where we have assumed that the uniform field is transverse with the E field in the $\hat{e}$ direction and have recognized that here the tensor $\vec{\gamma}$ takes the form

$$\gamma_{ij}|_{{\rm all}\, n=0} = \Delta_0'(\delta_{ij} - \delta_{iz}\delta_{jz}) . \tag{5.14b}$$

Furthermore, using Eqs. (2.3), (2.7b), and (2.10b), we find that

$$\tfrac{1}{2}c\alpha_1 = \tfrac{1}{9}[(\epsilon_{\rm un} - 1)^2/\epsilon_{\rm un}](\alpha_1/\alpha_{\rm un}) . \tag{5.15}$$

Remembering that α_1 equals the polarizability fluctuation amplitude, by Eq. (5.15), it will often be the case that $\frac{1}{2}c\alpha_1 \ll 1$. Hence, we will use Eq. (5.5d) to approximate $s_0(\frac{1}{2}c\alpha_1)$ and write

$$\vec{e}_m|_{{\rm all}\, n=0} \simeq \hat{e}[(\tfrac{1}{2}c\alpha_1)^2\Delta_0']^m . \tag{5.16}$$

In the present discussion, we are regarding Δ_0' as a large dimensionless number that scales like the ratio of the length of the medium with the wavelength. Using Eq. (5.16) in Eq. (5.8), we find that the primary contribution to the effective field due to the constant components of the even powers of $\alpha_1(r)$ is

$$\vec{E}'(\vec{r};\vec{q})|_{{\rm all}\, n=0} \simeq \vec{E}_{\rm un}'(\vec{r})[1 - c\alpha_1(\vec{r})]^{-1} \times \sum_{m=0}^{\infty} (\pi c_2 c\alpha_1^2\Delta_0')^m \tag{5.17a}$$

$$= \frac{\vec{E}_{\rm un}'(\vec{r})}{[1 - c\alpha_1(\vec{r})](1 - \pi c_2 c\alpha_1^2\Delta_0')} \tag{5.17b}$$

if

$$\pi c_2 c\alpha_1^2\Delta_0' = \frac{1}{54}\frac{(\epsilon_{\rm un}-1)^3(\epsilon_{\rm un}+2)}{\epsilon_{\rm un}^2}\left(\frac{\alpha_1}{\alpha_{\rm un}}\right)^2 \Delta_0' < 1 . \tag{5.17c}$$

One may regard the contribution to the effective field of Eq. (5.17b) as involving a large-scale depolarization factor, in that the magnitude of this contribution depends upon the dimensions of the sample and or the beam, and does not involve the establishment of a standing wave due to the periodicity of the medium itself. This is true of all $n = 0$ contributions. The detailed consideration of such contributions would require a theoretical development that does not assume an infinite medium from the beginning. Realizing this we will henceforth ignore the singularities associated with $\vec{\gamma}(\vec{k}_{\rm un} + n\vec{q})$ when $n = 0$.

For some $\vec{q}$, there may be a nonzero integer n_B such that Eq. (5.13) is satisfied. For the sake of determining how the effective field scales in this situation, we will make the following approximations (all quite reasonable in a variety of real situations):

$$(\tfrac{1}{2}c\alpha_1) \ll 1 , \tag{5.18a}$$

hence

$$s_0(\tfrac{1}{2}c\alpha_1) \simeq (\tfrac{1}{2}c\alpha_1)^2 , \tag{5.18b}$$

$$s_n(\tfrac{1}{2}c\alpha_1) \simeq 1 \quad \text{for } n > 0 . \tag{5.18c}$$

Thus, from Eq. (5.7), the dominant contribution to $\vec{e}_m$ will occur when one n is equal to n_B and all the others are ± 1; that is, $\vec{e}_m$ scales like

$$e_m|_{\rm Bragg} \sim (\tfrac{1}{2}c\alpha_1)^{n_B + (m-1)} k_{\rm un}\,\delta[\,|\vec{k}_{\rm un} + n_B\vec{q}\,| - k_{\rm un}\,] . \tag{5.19}$$

All of the terms raised to the mth power in Eq. (5.19) will combine with the mth-power term in the summation in Eq. (5.8), and the summation will produce a constant factor not much different than unity. However, the other factors of Eq. (5.19) will factor out of the summation of Eq. (5.8) and will determine the scaling of the effective field when the Bragg condition is exactly obeyed; that is,

$$E'|_{\rm Bragg} \sim (\tfrac{1}{2}c\alpha_1)^{n_B - 1} k_{\rm un}\,\delta(\,|\vec{k}_{\rm un} + n_B\vec{q}\,| - k_{\rm un}) , \tag{5.20}$$

where as before, we read $k_{\rm un}$ times the δ function as merely a large number. Equation (5.20) reveals the value of fulfilling the Bragg condition with as few factors of $\vec{q}$ as possible in weakly varying media.

B. Background effective field

If the only coherence length is small enough relative to the wavelength, then $q > 2k_{\rm un}$ and the Bragg condition may never be satisfied. If $q \gg k_{\rm un}$, then by Eqs. (5.11c) and (5.12b), the weighting function Δ'_n may be ignored relative to the constant terms remembering that we are ignoring the singularities associated with $n=0$. That is, we may rewrite Eq. (5.12a) as

$$\gamma_{ij}(\vec{\kappa}_n)\,|_{q\gg k_{\rm un}} \simeq \tfrac{1}{3}[\delta_{ij} - 3(\hat{\kappa}_n)_i(\hat{\kappa}_n)_j], \tag{5.21a}$$

where

$$(\hat{\kappa}_n)_i = \hat{i}\cdot\left(\frac{\vec{k}_{\rm un}+n\vec{q}}{|\vec{k}_{\rm un}+n\vec{q}\,|}\right) \simeq (\hat{q})_i\,. \tag{5.21b}$$

In order to sum the series in Eq. (5.7) we will make the reasonable assumption that $\frac{1}{2}c\alpha_1 \ll 1$, so that $s_n(\frac{1}{2}c\alpha_1)\simeq 1$ for $n>0$ and $s_0(\frac{1}{2}c\alpha_1)\simeq(\frac{1}{2}c\alpha_1)^2$. Using these approximations and Eqs. (5.21) in (5.7), the vector $\vec{e}_m$ takes the form

$$\vec{e}_m\,|_{q\gg k_{\rm un}} \simeq \left[\sum_{n_1=1}^{\infty}\left(\frac{c\alpha_1}{2}e^{i\vec{q}\cdot\vec{r}}\right)^{n_1} + \sum_{n_1=1}^{\infty}\left(\frac{c\alpha_1}{2}e^{-i\vec{q}\cdot\vec{r}}\right)^{n_1} + \left(\frac{c\alpha_1}{2}\right)^2\right]\cdots\left[\sum_{n_m=1}^{\infty}\left(\frac{c\alpha_1}{2}e^{i\vec{q}\cdot\vec{r}}\right)^{n_m} + \sum_{n_m=1}^{\infty}\left(\frac{c\alpha_1}{2}e^{-i\vec{q}\cdot\vec{r}}\right)^{n_m} + \left(\frac{c\alpha_1}{2}\right)^2\right]\left(\frac{1}{3}\right)^m[\hat{e}+\hat{q}(\hat{q})_e b_m]$$

$$= [\hat{e}+\hat{q}(\hat{q})_e b_m]\left[\frac{1}{3}\left(\frac{c\alpha_1(\vec{r})}{1-c\alpha_1(\vec{r})+(\frac{1}{2}c\alpha_1)^2}+\left(\frac{c\alpha_1}{2}\right)^2\right)\right]^m$$

$$\simeq [\hat{e}+\hat{q}(\hat{q})_e b_m]\left(\frac{c\alpha_1(\vec{r})}{3[1-c\alpha_1(\vec{r})]}\right)^m, \tag{5.22a}$$

where b_m is given by the recursion relation

$$b_m = -2b_{m-1} - 3\,, \tag{5.22b}$$

with $b_0=0$. In order to obtain the relation in Eq. (5.22a) we have again assumed that the uniform-medium field is transverse with the E field in the $\hat{e}$ direction and we have used Eq. (5.2) to obtain the spatially varying fluctuation function $\alpha_1(\vec{r})$. Note that the only depolarized component of $\vec{e}_m$ is the $\hat{q}$ direction and includes the factor $(\hat{q})_e$. Equation (5.22a) may be used in Eq. (5.8) to obtain the effective field in the limit of $q\gg k_{\rm un}$ and small fluctuations; that is,

$$\vec{E}'(\vec{r};\vec{q})\,|_{q\gg k_{\rm un}} \simeq \left[\hat{e}[1-c\alpha_1(\vec{r})]^{-1}\sum_{m=0}^{\infty}\left(\frac{\frac{4}{3}\pi c_2\alpha_1(\vec{r})}{1-c\alpha_1(\vec{r})}\right)^m + \hat{q}(\hat{q})_e[1-c\alpha_1(\vec{r})]^{-1}\sum_{m=1}^{\infty}b_m\left(\frac{\frac{4}{3}\pi c_2\alpha_1(\vec{r})}{1-c\alpha_1(\vec{r})}\right)^m\right]E'_{\rm un}e^{i\vec{k}_{\rm un}\cdot\vec{r}}$$

$$= [A(\vec{r})\hat{e} + B(\vec{r})\hat{q}]E'_{\rm un}e^{i\vec{k}_{\rm un}\cdot\vec{r}}, \tag{5.23a}$$

where

$$A(\vec{r}) \equiv [1-c\alpha_1(\vec{r})]^{-1}\sum_{m=0}^{\infty}\left(\frac{\frac{4}{3}\pi c_2\alpha_1(\vec{r})}{1-c\alpha_1(\vec{r})}\right)^m = \left[1-\frac{(\epsilon_{\rm un}-1)}{3}\left(\frac{\alpha_1(\vec{r})}{\alpha_{\rm un}}\right)\right]^{-1} \tag{5.23b}$$

$$B(\vec{r}) = (\hat{q})_e[1-c\alpha_1(\vec{r})]^{-1}\sum_{m=1}^{\infty}b_m\left(\frac{\frac{4}{3}\pi c_2\alpha_1(\vec{r})}{1-c\alpha_1(\vec{r})}\right)^m. \tag{5.23c}$$

Rewriting the constants of $A(\vec{r})$ required Eqs. (2.3), (2.7b), and (2.10b). Also note that $A = 1+O(\alpha_1/\alpha_{\rm un})$; whereas, $B=0+O(\alpha_1/\alpha_{\rm un})$. The field given by Eqs. (5.23) is the background effective field. That is, we have ignored all Bragg-like contributions; this field is completely insensitive to how the fluctuation wave numbers compare with $k_{\rm un}$. Finally, if we ignore $B(\vec{r})$, or if $(\hat{q})_e=0$, then the background field becomes

$$\vec{E}'_{\rm background} \to A(\vec{r})\vec{E}'_{\rm un}(\vec{r})\,.$$

One might regard this field as the most basic nondepolarized background field to which all depolarized, q dependent, and Bragg-like contributions are added. It provides a far better approximation for the local field than the incident or uniform medium fields. The ensemble average of this field is presumably a good approximation of the macroscopic field. Note that the evaluation of the modulating function $A(\vec{r})$ only requires knowledge of the uniform-medium dielectric constant and the ratio of the local polarizability with the uniform-medium polarizability.

VI. DISCUSSION

This paper has considered the effective electric field in an infinite, polarizable, and inhomogeneous medium. We should now review the possible approximations for the effective field. It is assumed that the coherent, forward propagating field is the ensemble average (spatial and/or temporal average, depending on the nature of the fluctuations) of the effective field. This is the macroscopic field that propagates through the medium. Macroscopic parameters, such as the dielectric constant and attenuation length due to elastic scattering, describe the behavior of this field and fundamentally must depend upon the nature of the inhomogeneities. A related consideration is that knowledge of the effective field allows the straightforward determination of the macroscopic scattered field.

The first approximation for the effective field would be the Born approximation,

$$\vec{E}' \simeq \vec{E}_{\text{incident}} . \tag{6.1}$$

Here, the medium is made of scatterers situated in free space. This approximation is useful in situations of small scattering cross section (e.g., x-ray diffraction from core electrons). The scattered field from any one site must only occasionally interact with any other site. The next approximation would be to let the effective field be the uniform medium field,

$$\vec{E}' \simeq \vec{E}'_{\text{un}} . \tag{6.2a}$$

This is the effective field if the polarizability is assumed to be homogeneous. To this approximation, the incident field is extinguished and replaced with the uniform medium field that propagates with the wave vector

$$\vec{k}_{\text{un}} = (\epsilon_{\text{un}})^{1/2}\vec{k}_0 , \tag{6.2b}$$

where the uniform medium dielectric constant is related to the homogeneous polarizability by the Lorentz-Lorenz relation

$$\tfrac{4}{3}\pi\alpha_{\text{un}} = (\epsilon_{\text{un}} - 1)/(\epsilon_{\text{un}} + 2) . \tag{6.2c}$$

This approximation is commonly used when the wavelength is much greater than the range over which the polarizability varies (e.g., optical fields in molecular substances).

We know that the effective field must be modulated by local variations in the polarizability. As outlined above, the nature of this modulation determines how macroscopic parameters depend upon the inhomogeneities. In general there are two approaches to dealing with effective fields in inhomogeneous media, multiple scattering, and perturbative treatments. There exists a large literature in both areas; the reader is referred to Refs. 9–11 for a review of the multiple-scattering approach and Refs. 12–14 for a review of the perturbative approach. In Ref. 4 our perturbative approach is compared to that of others. The present paper represents an extension and generalization of that approach. In particular, we wrote a general expression for the effective field, Eqs. (4.12), considering all orders of scattering, that only assumed that the fluctuations could be Fourier analyzed. Truncating this series results in our next approximation for the effective field

$$\vec{E}' \simeq (1 - c\alpha_1)^{-1} (\vec{E}'_{\text{un}} + \vec{E}'_1 + \cdots + \vec{E}'_m) , \tag{6.3}$$

where $\vec{E}'_m$ is given by Eq. (4.12a). This approximation has the definite virtue relative to other results,[9–14] of honestly including all multiple-scattering contributions to the vector field to the mth order. A proposed way of utilizing this result would be to simulate the inhomogeneous medium, starting with only statistical information about the inhomogeneities. One could then compute all of the required Fourier components and approximate the effective field using Eq. (6.3). This could be repeated at various points in the medium or in a time-dependent situation, at various times. The ensemble average of these fields would be the macroscopic field of the medium. And, the variation in the field itself would show how the effective field is modulated by the inhomogeneities. A less general but more tractable approximation results when the inhomogeneities have a single-cosine Fourier component. In this case $\vec{E}'_m \to \vec{E}'_m(\vec{r};\vec{q})$, which is given by Eq. (5.6). One would then consider multiple scatterings to the mth order involving a single component of the inhomogeneity distribution.

Finally, we should consider what we have called the background field as an approximation to the effective field,

$$\vec{E}'_b(r) = \left[1 - \frac{\epsilon_{\text{un}} - 1}{3}\left(\frac{\alpha_1(\vec{r})}{\alpha_{\text{un}}}\right)\right]^{-1} \vec{E}'_{\text{un}}(\vec{r}) . \tag{6.4}$$

This field includes all wavelength-independent corrections to the uniform medium field due to the inhomogeneities. It thus provides the most basic description of how the effective field is modulated by the inhomogeneities. Other authors[4,12–15] consider corrections of this type. Relative to these treatments, Eq. (6.4) is distinguished by the facts that: (i) it is an explicit expression resulting from consideration of all orders of corrections and not a correction that results from consideration of only the lowest-order multiple scattering and (ii) it results from a careful treatment of the self-field singularities of contributions to the inhomogeneous vector field.

We may use the background field to write new expressions for the effective macroscopic parameters of an inhomogeneous medium in the limit of long wavelengths. The dipole moment of a macroscopically small volume at $\vec{r}$ is

$$\vec{p}(\vec{r}) \simeq \alpha(\vec{r})\vec{E}_b'(\vec{r}), \tag{6.5a}$$

and thus the polarization is

$$\begin{aligned}\vec{P} &= \frac{1}{V}\int_V \vec{p}\, d^3r \\ &\simeq \vec{E}_{un}'\left\{\frac{1}{V}\int_V \left[\frac{\alpha_{un}+\alpha_1(\vec{r})}{1-\frac{1}{3}(\epsilon_{un}-1)(\alpha_1(\vec{r})/\alpha_{un})}\right] d^3r\right\} \\ &= \vec{E}_{un}'\left\langle\frac{\alpha_{un}+\alpha_1}{1-\frac{1}{3}(\epsilon_{un}-1)(\alpha_1/\alpha_{un})}\right\rangle_V . \end{aligned}\tag{6.5b}$$

In Eq. (6.5b) we have assumed that the wavelength is large enough so that the uniform-medium field is constant over the volume V which is only large enough to include many inhomogeneities (i.e., $\langle\alpha_1\rangle_V \simeq 0$). Equation (6.5b) may be rewritten to find the effective polarizability, that is defined to be the ratio of the polarization to the uniform-medium field

$$\alpha' \equiv P/E_{un}' \simeq \alpha_{un} + (3/4\pi)\eta, \tag{6.6a}$$

where

$$\eta = \left\langle\frac{\frac{1}{3}(\epsilon_{un}-1)(\alpha_1/\alpha_{un})}{1-\frac{1}{3}(\epsilon_{un}-1)(\alpha_1/\alpha_{un})}\right\rangle . \tag{6.6b}$$

This polarizability is perturbed from the uniform polarizability by accounting for averages of all powers of the fluctuation. We can now derive a macroscopic dielectric constant for the inhomogeneous medium. The average effective field should be related to the macroscopic $\vec{E}$ and $\vec{P}$ fields in the standard way; that is,

$$\langle\vec{E}'\rangle = \vec{E} + \tfrac{4}{3}\pi\vec{P} . \tag{6.7}$$

If we use the background field for the effective field, Eq. (6.4), then Eq. (6.7) may be rewritten

$$\begin{aligned}\vec{E} + \frac{4\pi}{3}\vec{P} &= \frac{\vec{P}}{\alpha'}\left\langle\frac{1}{1-\frac{1}{3}(\epsilon_{un}-1)(\alpha_1/\alpha_{un})}\right\rangle \\ &= \vec{P}\left(\frac{1+\eta}{\alpha_{un}+(3/4\pi)\eta}\right). \end{aligned}\tag{6.8}$$

Equation (6.8) may be solved for the polarization which is used to write an expression for the effective dielectric constant

$$\epsilon' = 1 + 4\pi P/E ; \tag{6.9}$$

after some algebra,

$$\epsilon' = \epsilon_{un} + (\epsilon_{un}+2)\eta , \tag{6.10}$$

where η is given by Eq. (6.6b).

Equation (6.10) gives the exact macroscopic dielectric constant for a medium in which the inhomogeneities are completely uncorrelated over distances comparable to a wavelength. Note that this expression has the following characteristics: (i) of course, $\epsilon' \to \epsilon_{un}$ as $\alpha_1 \to 0$, (ii) for all $\eta > 0$, $\epsilon' > \epsilon_{un}$, (iii) the inhomogeneities are characterized by the relative fluctuation (α_1/α_{un}) multiplied by the scaling factor $\frac{1}{3}(\epsilon_{un}-1)$, and (iv) if the magnitude of the fluctuations is small enough so that only the mean-squared term is significant and $\epsilon_{un} \simeq 1$, then Eq. (6.10) may be approximated as

$$\epsilon' \simeq 1 + 3\langle(\tfrac{4}{3}\pi\alpha_1)^2\rangle . \tag{6.11}$$

Point (ii) is of some significance. It says that the coherent wave propagates faster in a homogeneous medium than in an inhomogeneous medium that has an average polarizability equal to that of the homogeneous medium. Some authors[14,15] who treat only second-order scattering do not conform to this physical requirement.

As the above discussion demonstrates, knowing the effective field allows one to determine how the coherent wave propagates through the medium. Correspondingly, it also allows the straightforward determination of the scattered field; that is,

$$\vec{E}_{sc}(\vec{r}) = \int_\sigma \nabla_r \times \nabla_r \times \frac{e^{ik_0|\vec{r}-\vec{r}_1|}}{|\vec{r}-\vec{r}_1|}\, \alpha(\vec{r}_1)\vec{E}'(\vec{r}_1)\, d^3r_1 . \tag{6.12}$$

If the approximation for the effective field takes the form

$$\vec{E}'(\vec{r}_1) = f(\vec{r}_1)\vec{E}_{un}'(\vec{r}_1), \tag{6.13}$$

[such as in the case of the background field, Eq. (6.4)] then when $\vec{r}$ is in the far field of the scattering volume

$$E_{sc} \sim \int_{\text{sc vol}} d^3r_1\, e^{i\vec{r}_1\cdot\vec{s}} f(\vec{r}_1), \tag{6.14a}$$

and the scattered irradiance is proportional to the Fourier transform of the two-point correlation function of f; that is,

$$I_{sc} \sim \int_{\text{sc vol}} d^3\rho\, e^{i\vec{\rho}\cdot\vec{s}} S(\vec{\rho}), \tag{6.14b}$$

where

$$S(\vec{\rho}) = \langle f(\vec{r})f(\vec{r}+\vec{\rho})\rangle . \tag{6.14c}$$

In Eqs. (6.14a) and (6.14b) the scattering vector $\vec{s}$ is

$$\vec{s} = \vec{k}_{un} - k_0\hat{m} , \tag{6.14d}$$

where $\hat{m}$ gives the propagation direction of the scattered field. Finally, it must be noted that throughout this paper we have assumed an infinite medium, irradiated with an infinite incident beam. Whereas, the integrals of Eqs. (6.14a) and (6.14b) are over a real, finite volume. The differences between infinite and finite integrals will cause the scattered field to be modulated. The finite size of the medium and/or the incident beam introduces an additional length to the fluctuation distribution, from which coherent scattering will occur. This effect becomes significant as the size of the medium and/or the beam becomes comparable to a wavelength.

APPENDIX

In this appendix we will evaluate the following integral[8]:

$$I_{ij}(\vec{\kappa}) = \lim_{\epsilon\to 0}\int_{\rho>\epsilon} d^3\rho\, g_{ij}(\vec{\rho})(e^{ik_{\rm un}\rho}/\rho)e^{i\vec{\kappa}\cdot\vec{\rho}}, \quad \text{(A1a)}$$

where

$$g_{ij}(\vec{\rho}) = \left(k_{\rm un}^2 + \frac{1}{\rho}(ik_{\rm un} - 1/\rho)\right)\delta_{ij} + \frac{\rho_i\rho_j}{\rho^2}\left(-k_{\rm un}^2 - \frac{3ik_{\rm un}}{\rho} + \frac{3}{\rho^2}\right). \quad \text{(A1b)}$$

First, we will replace $k_{\rm un}$ with $k' = k_{\rm un} + i\mu$, in both $G_{\rm un}(\rho) = e^{ik_{\rm un}\rho}/\rho$ and in $g_{ij}(\vec{\rho})$, where μ is real and positive. We will let $\mu \to 0+$ at the end of the calculation. Now, as $\rho\to\infty$, $G_{\rm un}$ goes to zero fast enough so that there will be no contribution to I_{ij} from a surface at infinity. Next, we note the following about the terms in the integrand of Eq. (A1a):

$$g_{ij}(\vec{\rho})e^{ik'\rho}/\rho = (\nabla_i\nabla_j - \delta_{ij}\nabla^2)e^{ik'\rho}/\rho = \nabla_l\nabla_m(\delta_{li}\delta_{mj} - \delta_{lm}\delta_{ij})e^{ik'\rho}/\rho, \quad \text{for all } \rho > 0, \quad \text{(A2a)}$$

and

$$\int d^3\rho\, e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho}/\rho = \frac{4\pi}{\kappa^2 - k'^2} \quad \text{if } \mathrm{Im}(k') > 0. \quad \text{(A2b)}$$

With Eq. (A2a), Eq. (A1a) may be rewritten

$$I_{ij}(\vec{\kappa}) = \lim_{\epsilon\to 0}(\delta_{li}\delta_{mj} - \delta_{lm}\delta_{ij}) \times \int_{\rho>\epsilon} d^3\rho\, e^{i\vec{\kappa}\cdot\vec{\rho}}\nabla_l\nabla_m(e^{ik'\rho}/\rho). \quad \text{(A3)}$$

We now integrate by parts; that is,

$$\begin{aligned} I_{ij} &= \lim_{\epsilon\to 0}(\delta_{li}\delta_{mj} - \delta_{lm}\delta_{ij})\int_{\rho>\epsilon} d^3\rho\left[\nabla_l\left(e^{i\vec{\kappa}\cdot\vec{\rho}}\nabla_m(e^{ik'\rho}/\rho)\right) - i\kappa_l e^{i\vec{\kappa}\cdot\vec{\rho}}\nabla_m(e^{ik'\rho}/\rho)\right] \\ &= \lim_{\epsilon\to 0}(\delta_{li}\delta_{mj} - \delta_{lm}\delta_{ij})\left[\int_{\rho>\epsilon} d^3\rho\left\{\nabla_l\left[e^{i\vec{\kappa}\cdot\vec{\rho}}(\hat{\rho})_m\left(\frac{ik'e^{ik'\rho}}{\rho} - \frac{e^{ik'\rho}}{\rho^2}\right)\right] - i\kappa_l\nabla_m\left(\frac{e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho}}{\rho}\right)\right\}\right. \\ &\qquad \left. - \kappa_l\kappa_m\int_{\rho>\epsilon} d^3\rho\,\frac{e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho}}{\rho}\right], \end{aligned} \quad \text{(A4a)}$$

where

$$\kappa_l = \hat{l}\cdot\vec{\kappa} \text{ and } (\hat{\rho})_m = \hat{m}\cdot\hat{\rho}. \quad \text{(A4b)}$$

We will use the generalized Stoke's theorem,

$$\int_{\rho>\epsilon} d^3\rho\,\nabla_l f(\vec{\rho}) = -\oint_{\rho>\epsilon}\rho^2\,d\Omega(\hat{\rho})_l f(\vec{\rho}) + \oint_{\rho\to\infty}\rho^2\,d\Omega(\hat{\rho})_l f(\vec{\rho}), \quad \text{(A5)}$$

to rewrite the first term of Eq. (A4a), where there are surface integrals over the small surface at ϵ as well as a large surface at infinity. However, because of the decaying nature of the Green's function, $e^{ik'\rho}/\rho$, always present in $f(\rho)$ in Eq. (A5), the latter integral will contribute nothing. Thus, we obtain

$$I_{ij} = \lim_{\epsilon\to 0}(\delta_{li}\delta_{mj} - \delta_{lm}\delta_{ij})\left(-\oint_{\rho=\epsilon} d\Omega(\hat{\rho})_l(\hat{\rho})_m e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho}(ik'\rho - 1) + i\kappa_l\oint_{\rho=\epsilon}\rho^2\,d\Omega\,\frac{(\hat{\rho})_m e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho}}{\rho} - \kappa_l\kappa_m\int_{\rho>\epsilon} d^3\rho\,\frac{e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho}}{\rho}\right). \quad \text{(A6)}$$

The only term from the surface integrals that does not vanish as $\epsilon\to 0$ is

$$\oint d\Omega(\hat{\rho})_l(\hat{\rho})_m e^{i\vec{\kappa}\cdot\vec{\rho}}e^{ik'\rho} \to \oint d\Omega(\hat{\rho})_l(\hat{\rho})_m = \frac{4\pi}{3}\delta_{lm}. \quad \text{(A7)}$$

Then using Eqs. (A2b) and (A7) in Eq. (A6), we may write

$$I_{ij} = (\delta_{li}\delta_{mj} - \delta_{lm}\delta_{ij})\left(\frac{4\pi}{3}\delta_{lm} - \kappa_l\kappa_m\frac{4\pi}{\kappa^2 - k'^2}\right). \quad \text{(A8)}$$

Remembering to sum over repeated indices, we obtain

$$I_{ij}=\frac{4\pi}{3}[\delta_{ij}-\delta_{ij}(1+1+1)]$$

$$-\frac{4\pi}{\kappa^2-k'^2}[\kappa_i\kappa_j-\delta_{ij}(\kappa_l^2+\kappa_m^2+\kappa_n^2)]$$

$$=-\frac{8\pi}{3}\delta_{ij}+4\pi\frac{\kappa^2\delta_{ij}-\kappa_i\kappa_j}{\kappa^2-k'^2}\,. \qquad \text{(A9)}$$

Now we must make the substitution, $k'=k_{\text{un}}+i\mu$, and take the limit $\mu\to 0+$; that is,

$$\frac{1}{\kappa^2-k'^2}=\frac{1}{\kappa^2-k_{\text{un}}^2+\mu^2-2ik_{\text{un}}\mu}\,,$$

and

$$\lim_{\mu\to 0+}\left(\frac{1}{(\kappa^2-k_{\text{un}}^2+\mu^2)-2ik_{\text{un}}\mu}\right)=\mathcal{P}\left(\frac{1}{\kappa^2-k_{\text{un}}^2}\right)+i\pi\delta(\kappa^2-k_{\text{un}}^2)\,, \qquad \text{(A10)}$$

where we assumed that k_{un} can never be negative (i.e., only forward propagation occurs). Also $\mathcal{P}$ stands for the Cauchy principal value; that is, within an integral over κ, we have

$$\int_0^\infty d\kappa\,\mathcal{P}\left(\frac{1}{\kappa^2-k_{\text{un}}^2}\right)f(\kappa)=\lim_{\epsilon\to 0+}\left[\int_0^{k_{\text{un}}-\epsilon}\frac{f(\kappa)\,d\kappa}{\kappa^2-k_{\text{un}}^2}+\int_{k_{\text{un}}+\epsilon}^\infty\frac{f(\kappa)\,d\kappa}{\kappa^2-k_{\text{un}}^2}\right]. \qquad \text{(A11)}$$

Finally, since κ is never negative, in Eq. (A10) we can make the substitution

$$\delta(\kappa^2-k_{\text{un}}^2)=(1/2k_{\text{un}})\delta(\kappa-k_{\text{un}})\,. \qquad \text{(A12)}$$

Then, using Eqs. (A10) and (A12), we rewrite Eq. (A9) to obtain our result,

$$I_{ij}(\vec{\kappa})=-\tfrac{8}{3}\pi\delta_{ij}+4\pi(\kappa^2\delta_{ij}-\kappa_i\kappa_j)\times\left[\mathcal{P}\left(\frac{1}{\kappa^2-k_{\text{un}}^2}\right)+\frac{i\pi}{2k_{\text{un}}}\delta(\kappa-k_{\text{un}})\right]. \qquad \text{(A13)}$$

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