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Validity and significance of time-dependent Hartree approximation for a one-dimensional system of bosons with attractive δ-function interactions

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The Hartree (mean-field) approximation to the description of the scattering of two heavy bound states is studied in a model consisting of bosons with attractive δ-function interactions. The approximation is derived from matrix elements of the Heisenberg equations of motion of the system and requires careful enumeration of all necessary approximations. The arguments made are verified by comparison of exact and approximate scattering amplitudes. The derivation also yields the physical significance of the amplitude which satisfies the Hartree approximation; it is a Fourier sum over amplitudes for different channels, which can in prinicple be recovered individually. The approach is not restricted to the model studied.

I. INTRODUCTION

The present note is a byproduct of the authors' investigation of the quantization of solitons.[1-5] It was stimulated by a recent note by Yoon and Negele[6] who studied the validity of the time-dependent Hartree-Fock approximation (TDHF)[7] for an exactly soluble one-dimensional model of interacting bosons.[8-11] In the language of second quantization, this model is summarized by its Hamiltonian

$$H=\frac{1}{2}\int dx\,\frac{d}{dx}\psi^{\dagger}(x)\frac{d}{dx}\psi(x) - \tfrac{1}{2}K\int dx\,\psi^{\dagger}(x)\psi^{\dagger}(x)\psi(x)\psi(x), \tag{1.1}$$

where we have chosen $\hbar = m$ (mass of particle) = 1 and $\psi, \psi^{\dagger}$ satisfy the commutation relations

$$[\psi(x), \psi^{\dagger}(y)] = \delta(x-y). \tag{1.2}$$

Any interest which the questions to be discussed here may have relates to the application of TDHF to the nuclear many-body problem, in particular to the study of heavy-ion collisions.[12-17] Since the growth of technology in this area outstrips the growth of fundamental understanding, even a modest additional effort to redress the balance may have some value.

For the model under investigation, for which only elastic scattering occurs as an energy-conserving process, it was shown by Yoon and Negele[6] that the phase shift emerging from a Hartree approximation for the scattering of two composite particles each with n bosons agrees with the exact phase shift as $n \to \infty$. In addition to this criterion for validity of the approximation, an auxiliary standard discussion of the effect of quantum-mechanical spreading of the incident wave packets suggests that a second criterion for validity is that the relative velocity must be high. This condition failed to emerge clearly from the detailed considerations, however, and was thus left as an open question.

A second open question in this model (and in general) concerns initial conditions. In the product wave function describing the collision, when the two composites are far removed from each other, should one choose single-particle functions localized in one place or the other, or functions which are with equal probability at either location? Phenomenologically, it is found that only the latter choice yields a simple equation with known solution. Comparison with the exact solution referred to above was made for this choice.

To these questions, raised by the previous authors[6] we add two more: It is well understood[18] that TDHF, in general, describes a superposition of amplitudes for elastic and inelastic processes. What is this superposition and how can individual amplitudes be disentangled, if desired? Finally, how can the leading corrections to TDHF be chosen and possibly added to the calculation?

We shall address in detail all questions raised except the last, to which we allude briefly at the end of the introduction, giving references to the (scant) literature. The technical details of this aspect are not germaine to the aims of the present work; we are, in any event planning a systematic account of this subject as part of the problem of quantization of soliton solutions.

To address the remaining questions, we shall approach the theory in a manner different from the current mode in nuclear physics. Conventionally, one derives a mean-field approximation by the application of the variational principle for the time-dependent Schrödinger equation to a dynamically uncorrelated wave function. One has guessed the wave function and therefore the only basis for error estimation in general is to *add* complications to the trial function and see what it gives. (A con-

ceivable alternative, which has been studied for the present model,[19] is to develop diagram expansions for the collision problem.)

In the line we shall follow, one studies suitably chosen matrix elements of the "field equations"

$$i\partial_t\psi(x,t) = -\tfrac{1}{2}\partial_x^2\psi(x,t) - K|\psi^\dagger(x,t)|^2\psi(x,t). \quad (1.3)$$

It is known that the *classical* partial differential equation which has the *same form* as (1.3), the so-called nonlinear Schrödinger equation, has an infinite number of soliton[20,21] solutions representing, in order of analytic complexity, the motion of a single lump of field, and the collision of two or more such lumps. Since these should be the classical limits of the quantum theory defined by (1.1) and (1.2), the common equation they satisfy should emerge from the application of suitable forms of theoretical surgery to the matrix elements of (1.3). Our main thrust is that even though a common differential equation will ultimately emerge, it makes sense to consider each physical problem (bound state or collision) separately since conditions for the validity of the equation may not be the same for each problem. Furthermore these conditions emerge quite independently of the specific form of the solution (but are, of course, verified by it) and some of the arguments carry over to more general theories.

It is convenient to begin our discussion in Sec. II by presenting the one- and two-soliton solution and some of their properties in a form which is suitable for later comparison.

In Sec. III we apply the basic technique of studying the matrix element of (1.3) to the bound state of n particles. This problem has been well studied previously[2,22,23] and a technique for expanding the binding energy in powers of n^{-1} and for interpreting the single-particle amplitude has been established. We do not repeat these developments. We do point out that in contrast to the classical wave which propagates without dispersion, the associated quantum amplitude spreads in time, and thus we find that the classical description is valid only for times t sensibly smaller than nt_{orbit}, where t_{orbit} is the period of a bound orbit. Because $t_{\text{orbit}} \sim n^{-2}$ in this unusual model, t becomes shorter as n increases. This is in contrast with a saturating system, where $t_{\text{orbit}} \sim$ const and the classical description is valid over increasing time intervals as n increases.[24]

In Sec. IV we study Eq. (1.3) for the collision of two bound states. For the validity of the classical limit, we find, in addition to the previous restrictions, a further condition that the momentum transfer in the collision be small compared to the relative momentum of the collision. This is seen, however, to be equivalent to the condition of no sensible spreading of the wave packets over the time interval between preparation of the initial wave packet and detection of the final wave packet.

In Sec. V we compare the known exact phase shift or time delay with the classical approximation to it for the collision of a bound state of n_1 particles with a bound state of n_2 particles and verify that they agree under the conditions derived.

Two of the other problems raised are settled in Sec. IV. By identifying a quantum scattering function with the two-soliton solution, we may take the asymptotic structure of the latter—containing a one-soliton amplitude in each collision partner—to be decisive for yielding the correct initial conditions for TDHF. Furthermore, we demonstrate that the two-soliton solution generates a double Fourier series in time, each coefficient describing a different scattering channel (in the semiclassical approximation). It is pointed out that by solving the TDHF for a suitably dense set of initial *phases* for the individual solitons, one can, in effect, perform the inverse transformation to obtain individual reaction amplitudes. Both resolutions of this paragraph are not confined to the special model considered.

Finally within the model considered, one can generate quantum corrections in a systematic way. This has in effect been described in previous work.[25] For the first correction, one is instructed to study the random-phase approximation. Though quite similar considerations should obtain in more realistic problems, the strong excitation in such cases of collective modes has no counterpart in the present model, and therefore the consideration of more realistic models becomes necessary. This is planned for future work within a purely nuclear context.

It has been customary to extract the time delay, which is the energy derivative of the phase shift, from the classical solution. In the Appendix, we show that the classical solution also contains the derivative of the phase shift with respect to particle number.

II. NONLINEAR SCHRODINGER EQUATION: CLASSICAL SOLUTIONS

The classical nonlinear Schrödinger equation (NLSE), which we study in the form

$$i\partial_t\Psi(x,t) + \tfrac{1}{2}\partial_{xx}\Psi(x,t) + K|\Psi(x,t)|^2\Psi(x,t) = 0, \quad (2.1)$$

has an infinite number of "soliton" solutions which have been given in convenient form by Hirota.[21] The most elementary solution, the solitary wave, describing the propagation of a single lump or amplitude with velocity u, has the form

$$\Psi_1(x,t) = N(x,t)/D(x,t), \tag{2.2}$$

where

$$N(x,t) = \exp[i\zeta(x,t) + \kappa z], \tag{2.3}$$

$$D(x,t) = 1 + a(11^*)\exp[2\kappa z], \tag{2.4}$$

with

$$\zeta = ux - \tfrac{1}{2}u^2 t + \tfrac{1}{2}\kappa^2 t + \theta, \tag{2.5}$$

$$z = x - ut + \delta, \tag{2.6}$$

$$a(11^*) = K/(2\kappa)^2. \tag{2.7}$$

In addition to the velocity u, the solution depends on the real constants κ, θ, and δ. This can be rewritten as

$$\Psi_1(x,t) = \frac{\exp[i\zeta(x,t)]}{2[a(11^*)]^{1/2}\cosh\kappa(x-ut)}, \tag{2.8}$$

if we eliminate δ by the condition $[a(11^*)]^{1/2} = \exp(-\delta)$, as is convenient.

We shall also wish to study the two-soliton solution

$$\Psi_{12}(x,t) = N_{12}/D_{12}, \tag{2.9}$$

where

$$N_{12} = N_1(x,t) + N_2(x,t) + a(121^*)N_2(x,t)|N_1(x,t)|^2 + a(122^*)N_1(x,t)|N_2(x,t)|^2, \tag{2.10}$$

$$D_{12} = 1 + a(11^*)|N_1|^2 + a(22^*)|N_2|^2 + |N_1|^2|N_2|^2 a(121^*2^*) + a(12^*)N_2^* N_1 + a(21^*)N_1^* N_2. \tag{2.11}$$

Here N_1 and N_2 are each of the form (2.3) but the quantities κ, u, θ carry identifying subscripts. Also

$$a(ij^*) = K[(\kappa_i + \kappa_j) + i(u_i - u_j)]^{-2}, \tag{2.12}$$

$$a(ij) = K^{-1}[(\kappa_i - \kappa_j) + i(u_i - u_j)]^2 = a(i^*j^*)^*, \tag{2.13}$$

$$a(ijk^*) = a(ij)\,a(ik^*)\,a(jk^*), \tag{2.14}$$

$$a(121^*2^*) = a(12)\,a(11^*)\,a(12^*)\,a(21^*)\,a(22^*)\,a(1^*2^*) = |a(12)|^2|a(12^*)|^2 a(11^*)a(22^*). \tag{2.15}$$

The physical significance of Ψ_{12} is clarified by considering the two limits $t \to \mp\infty$. Let $u_1 > 0$, $u_2 < 0$ and

$$u \equiv u_1 - u_2. \tag{2.16}$$

From (2.9)–(2.15), we then find

$$\Psi_{12}(x, t \to -\infty) = \frac{\exp[i\zeta_1(x,t) + \kappa_1(x-u_1 t)]}{1 + a(11^*)\exp[2\kappa_1(x-u_1 t)]} + \frac{\exp[i\zeta_2(x,t) + \kappa_2(x-u_2 t)]}{1 + a(22^*)\exp[2\kappa_2(x-u_2 t)]}, \tag{2.17}$$

which represents two nonoverlapping solitary waves approaching each other from afar. For $t \to +\infty$, we find

$$\Psi(x, t \to \infty) = \frac{\exp\{i\zeta_1 + \kappa_1[x - u_1(t-\Delta t_1)]\}\exp[2i(\alpha_1 - \alpha_2)]}{1 + a(11^*)\exp\{2\kappa_1[x-u_1(t-\Delta t_1)]\}} + \frac{\exp\{i\zeta_2 + \kappa_2[x - u_2(t-\Delta t_2)]\}\exp[2i(\alpha_1 + \alpha_2)]}{1 + a(22^*)\exp\{2\kappa_2[x-u_2(t-\Delta t_2)]\}}, \tag{2.18}$$

where

$$\kappa_1 u_1 \Delta t_1 = \kappa_2 u_2 \Delta t_2 = -\ln\left[\frac{(\kappa_1+\kappa_2)^2 + u^2}{(\kappa_1-\kappa_2)^2 + u^2}\right], \tag{2.19}$$

$$\alpha_1 = \tan^{-1}\left(\frac{u}{\kappa_1 - \kappa_2}\right), \quad \alpha_2 = \tan^{-1}\left(\frac{u}{\kappa_1 + \kappa_2}\right). \tag{2.20}$$

The expression (2.18) represents two nonoverlapping solitary waves, receding from one another. Each has been shifted in phase compared to its incident form and, as well, has suffered a time delay. In Sec. V we shall compare the time delay Δt, with the exact value for the associated quantum many-body problem. The reason for the shifts in phase and their significance is discussed in the Appendix, where it will emerge that these quantities contain physical information equivalent to that in the time-delay function.

We take due note of one further property of $\Psi_{12}(x,t)$ which will interest us in the sequel. If we introduce the forms of N_1 and N_2 into (2.10) and (2.11), we notice that the latter can be rewritten as

$$N_{12} = \mathfrak{N}_1 e^{i\zeta_1} + \mathfrak{N}_2 e^{i\zeta_2}, \tag{2.21}$$

$$D_{12} = \mathfrak{D}_0 + \mathfrak{D}_1 e^{i(\zeta_1 - \zeta_2)} + \mathfrak{D}_1^* e^{i(\zeta_2 - \zeta_1)}, \tag{2.22}$$

where the coefficients are functions of $\kappa_1, \kappa_2, u_1, u_2, x - u_1 t$, and $x - u_2 t$. Thus if formally we expand D_{12}^{-1} about $\mathfrak{D}_0^{-1}$, we obtain the structure

$$\Psi_{12}(x,t) = \sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty} \delta_{\nu_1+\nu_2-1} e^{i(\nu_1\zeta_1 + \nu_2\zeta_2)} \times \Phi_{\nu_1\nu_2}^{\kappa_1\kappa_2}(u_1, u_2, x-u_1 t, x-u_2 t), \tag{2.23}$$

which has the form of a double Fourier series in the variables ζ_1, ζ_2. We shall see in Sec. IV that the coefficients Φ contain information about quantum scattering in individual channels.

III. N-PARTICLE BOUND STATE AS SOLITARY WAVE

We next show how the classical results of the previous section can be derived as limiting cases of a quantum theory and constitute Hartree approximations to the latter. This is simplest to do for the solitary wave, which we therefore take as initial example.[26]

We start from the quantum field equation of the NLSE, Eq. (1.3). This field theory describes bosons interacting on the line with an attractive δ-function potential. It possesses for each value n of the number of bosons exactly one bound state with energy

$$E_n = -\tfrac{1}{24}K(n^3 - n). \tag{3.1}$$

Furthermore, all S-matrix elements are known and will be quoted as needed in Sec. V.

The bound state of n particles moving with momentum p, velocity $u \equiv p/n$, designated $|n(p)\rangle$, has energy $E_n + p^2/2n$. We study the matrix element

$$\Psi^n_{1p}(x,t) = \int \frac{dk}{2\pi} \langle (n-1)(p-k) \,|\, \psi(x,t) \,|\, n(p)\rangle, \tag{3.2}$$

and seek conditions under which (1.3) becomes approximately an equation for the amplitude (3.2). We first consider the limit $t = 0$.

This procedure entails a series of approximations. *In the present instance every one of these approximations can be justified.* Of these the most difficult to justify for more elaborate models is the dynamical assumption[27]

$$C(x,t) \equiv \int \frac{dk}{2\pi} \langle n-1(p-k) \,\|\, \psi(x,t)|^2\, \psi(x,t) \,|\, n(p)\rangle$$
$$\cong \int \frac{dk}{2\pi}\frac{dp'}{2\pi}\frac{dp'''}{2\pi} \langle n-1(p-k) \,|\, \psi^\dagger(x,t) \,|\, n-2(p-p'-p'')\rangle \, \langle n-2(p-p'-p'') \,|\, \psi(x,t)$$
$$\times |\, n-1(p-p')\rangle \langle n-1(p-p') \,|\, \psi(x,t) \,|\, n(p)\rangle . \tag{3.3}$$

Here we have omitted all intermediate states involving virtual breakup into two or more particles. As shown previously,[2] the most important omission, for large n are the intermediate states $|n'(p'), k\rangle$ describing the scattering of a single particle by the bound state $n'(=n-1$ or $n-2$ depending on the term involved); for large n these yield contributions which are $O(n^{-1})$ compared to the terms retained. For the present model these corrections are known in detail.

Further evaluation of (3.3) is relatively straightforward and will be seen again to involve errors which are relatively $O(n^{-1})$, but here because of kinematical rather than dynamical approximations. We first replace $n \to n+1$ in the first two factors of C. We subsequently utilize the translational and Galilean invariance of the present model to write,

$$\langle n'(p') \,|\, \psi(x,0) \,|\, n(p)\rangle$$
$$= \exp\left[i\left(\frac{n-n'}{n}\right)qx\right] \left\langle n'\left(p' - \frac{n'}{n}q\right)\right|$$
$$\times \psi(x,0) \,|\, n(p-q)\rangle . \tag{3.4}$$

Several applications of (3.4) plus displacements of the variables of integration yield the form

$$C(x,0) = \int \frac{dk}{2\pi}\frac{dp'}{2\pi} \left\{ \exp[i(k+p')x/n] \langle n(p) \,|\, \psi^\dagger(x,0) \left| (n-1)\left[p+k-\left(\frac{k+p'}{n}\right)\right]\right\rangle \right\}$$
$$\times \langle n-1(p-p') \,|\, \psi(x,0) \,|\, n(p)\rangle \, \Psi_{1p}(x,0) \cong |\Psi_{1p}(x,0)|^2 \Psi_{1p}(x,0). \tag{3.5}$$

The final approximation consisted in observing that because the curly bracket can be written as

$$\{\cdots\} = e^{ikx} \langle n(p) \,|\, \psi^\dagger(0,0) \left| (n-1)\left[p+k-\left(\frac{k+p'}{n}\right)\right]\right\rangle \cong e^{ikx} \langle n(p) \,|\, \psi^\dagger(0,0) \,|\, (n-1)(p+k)\rangle$$
$$= \langle n(p) \,|\, \psi^\dagger(x,0) \,|\, (n-1)(p+k)\rangle , \tag{3.6}$$

only in the combination shown is it permissible to carry out the expansion in powers of n^{-1}.

To proceed to the required case $t \neq 0$, we could start all over and carry through the discussion using the appropriate generalization of (3.4). It is more instructive to proceed as follows: For $t=0$ let us also set $p=0$. Replacing $i\partial_t$ by the commutation with the Hamiltonian H we have

$$\int \frac{dk}{2\pi} \langle (n-1)(-k) \,|\, i\partial_t \psi(x,0) \,|\, n(0) \rangle$$

$$= (E_n - E_{n-1}) \Psi^n_{10}(x,0) \cong \omega(n) \Psi^n_{10}(x,0), \quad (3.7)$$

where we set

$$E_n - E_{n-1} \cong \frac{dE_n}{dn} \equiv \omega(n). \quad (3.8)$$

Combined with (3.5), it follows that $\Psi^n_{10}(x,0)$ satisfies the time-independent equation,

$$[\omega(n) + \tfrac{1}{2}\partial_{xx}] \Psi^n_{10}(x,0) + K \,|\, \Psi^n_{10}(x,0) \,|^2 \Psi^n_{10}(x,0) = 0. \quad (3.9)$$

This has as solution the function (2.8) for $t=u=0$ if we identify the real parameter κ as

$$\kappa^2 = 2 \,|\, \omega(n) \,|. \quad (3.10)$$

Finally we are ready to consider the full amplitude (3.2) and ask for the conditions under which it has the form (2.8). From the definition, utilizing time- and space-translation invariance followed by Galilean invariance, we find straightforwardly

$$\Psi^n_{1,p}(x,t) = e^{-i\zeta(x,t)} \int \frac{dk}{2\pi} \exp[-ik(x-ut)] \times \exp[ik^2 t/(n-1)] \times \langle n-1(k) \,|\, \psi(0,0) \,|\, n(0) \rangle, \quad (3.11)$$

where

$$\zeta = ux - \omega(n)t - \tfrac{1}{2}u^2 t. \quad (3.12)$$

Comparison with (2.5) and (3.10) shows that this is the phase factor sought. Under the condition that we may ignore the second exponential factor under the integral, i.e., for times such that $k^2 t/n \ll 1$, we have therefore

$$\Psi^n_{1,p}(x,t) = e^{i\zeta(x,t)} \Psi^n_{10}(x-ut,0), \quad (3.13)$$

which is indeed the solution of (2.1) given in (2.8).

The factor we have ignored leads to the quantum-mechanical spreading of the wave packet. Since from (2.8) and (3.10) $\bar{k}^2 \sim |\, \omega(n) \,|$, our condition becomes

$$t < n t_{\text{orbit}}, \quad (3.14)$$

where $t_{\text{orbit}} \sim |\, \omega(n) \,|^{-1}$ is the characteristic "orbiting time" for a particle in the bound state.

In summary, the propagating bound state may be described by a solitary wave [whose quantum significance is given by (3.2)], which is the solution of the Hartree approximation for n sufficiently large and for times sufficiently small. The latter condition turns into a restriction on permissible relative momenta when we consider the collision of two such bound states, since for the corresponding Hartree approximation to obtain, we must restrict permissible *collision times* to satisfy the condition (3.14).

IV. COLLISION OF COMPOSITES AS TWO-SOLITON SOLUTION

We next seek the quantum analog of the two-soliton solution $\Psi_{12}(x,t)$. The obvious first candidate for a quantum amplitude with this classical limit is, by analogy with (3.2),

$$\Psi^{n_1 n_2}_{12,\nu_1\nu_2}(x,t) = \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \langle (n_1 - \nu_1)(p_1 - k_1), (n_2 - \nu_2)(p_2 - k_2) \,|\, \psi(x,t) \,|\, n_1(p_1), n_2(p_2) \rangle. \quad (4.1)$$

This cannot be correct, however, since besides the dependence of Ψ on n_1, n_2, $u_1 = p_1/n_1$, and $u_2 = p_2/n_2$, the latter suppressed in the notation, there is a dependence on the two integers ν_1 and ν_2 which satisfy the constraint $\nu_1 + \nu_2 = 1$. The comparison with (2.23) is suggestive, and let us therefore begin with the analysis corresponding to (3.11)–(3.13), though here we must proceed a bit more cautiously.

By the application of time- and space-translation invariance and the approximation (3.8), we obtain

$$\Psi^{n_1 n_2}_{12,\nu_1\nu_2}(x,t) = \exp\{-it\nu_1[\omega(n_1) - \tfrac{1}{2}u_1^2] - it\nu_2[\omega(n_2) - \tfrac{1}{2}u_2^2]\} \times \int \frac{dk_1}{2\pi} \frac{dk_2}{2\pi} \exp[i(k_1+k_2)x - it(u_1 k_1 + u_2 k_2)] \exp\{it[(1/2n_1)k_1^2 + (1/2n_2)k_2^2]\} \times \langle (n_1 - \nu_1)(p_1 - k_1)(n_2 - \nu_2)(p_2 - k_2) \,|\, \psi(0,0) \,|\, n_1(p_1), n_2(p_2) \rangle \quad (4.2)$$

We enforce the condition (3.14) by immediately dropping the second exponential under the integral. We

then define

$$\langle (n_1-\nu_1)(p_1-k_1),\,(n_2-\nu_2)(p_2-k_2)\,|\,\psi(0,0)\,|\,n_1(p_1),n_2(p_2)\rangle$$
$$\equiv \int dy_1\,dy_2 \exp[-ik_1y_1-ik_2y_2+i\nu_1u_1y_1+i\nu_2u_2y_2]\Phi^{n_1n_2}_{\nu_1\nu_2}(u_1,u_2,y_1,y_2). \quad (4.3)$$

The stimulus for this particular definition is that in the absence of interaction between the composites and consequently also for large separations, Φ becomes independent of u_1 and u_2, in consequence of Galilean invariance.

Because we have ignored the quantum-mechanical spreading of the wave packet (4.2) becomes with the help of (4.3),

$$\Psi^{n_1n_2}_{12,\nu_1\nu_2}(x,t)=\exp(i\nu_1\zeta_1+i\nu_2\zeta_2)\times\Phi^{n_1n_2}_{\nu_1\nu_2}(u_1,u_2,x-u_1t,x-u_2t), \quad (4.4)$$

where ζ_1 and ζ_2 are expressions of type (3.12). To obtain (2.23) it suggests itself that we multiply (4.4) by $\exp(i\nu_1\theta_1+i\nu_2\theta_2)$ and sum, subject to the restriction imposed by particle conservation, over ν_1, ν_2. The quantity

$$\Psi^{n_1n_2}_{12}(x,t,\theta_1,\theta_2)=\sum_{\nu_1,\nu_2}\exp[i\nu_1\theta_1+i\nu_2\theta_2]\times\Psi^{n_1n_2}_{12,\nu_1\nu_2}(x,t) \quad (4.5)$$

is a candidate for solution of the Hartree or mean-field approximation.

Let us for the moment accept this result, which will be established below. The structure found does not in fact depend on the special, soluble model under study. It illustrates, moreover, a feature of mean-field approximations for scattering which has received considerable attention in the corresponding nuclear case, namely, that the solution does not describe a single reaction: This has been termed the problem of "spurious cross-channel correlations."[18] The solution to the problem of disentangling the various amplitudes is implicit in (4.5). In a numerical situation, where one integrates from some starting t ($t=-\infty$), the phases θ_1 and θ_2 must be specified as part of the initial conditions. (Actually only one nontrivial phase is needed in the present case.) If the solutions are computed for a sufficiently dense set of initial phases, it is thus possible to reconstruct the individual amplitudes as Fourier expansion coefficients using θ_1 and θ_2 as "times."

We turn then to the problem of proving that (4.5) satisfies the classical field equation. We study the quantity[27]

$$C_{p_1p_2}(x,t)=\int\frac{dk_1}{2\pi}\frac{dk_2}{2\pi}\,\langle(n_1-\nu_1)(p_1-k_1),(n_2-\nu_2)(p_2-k_2)\,|\,|\psi(x,t)|^2\psi(x,t)\,|\,n_1(p_1),n_2(p_2)\rangle$$
$$\cong\int\frac{dk_1}{2\pi}\cdots\frac{dp_2''}{2\pi}\,\langle n_1(p_1-k_1),(n_2(p_2-k_2)\,|\,\psi^\dagger(x,t)\,|\,n_1+\nu_1-\nu_1'-\nu_1''(p_1-p_1'-p_1''),(1\rightarrow 2)\rangle$$
$$\times\langle n_1-\nu_1''(p_1-p_1'-p_1''),(1\rightarrow 2)\,|\,\psi(x,t)\,|\,n_1(p_1-p_1'),(1\rightarrow 2)\rangle$$
$$\times\langle n_1-\nu_1'(p_1-p_1'),(1\rightarrow 2)\,|\,\psi(x,t)\,|\,n_1(p_1)n_2(p_2)\rangle. \quad (4.6)$$

Here we have already carried out a first essential step of shifting the "n values," which entails errors of relative order n_1^{-1} and n_2^{-1}. For the further simplification, imagine that we are in the center of mass system $p_2=-p_1$. Taking a clue from the structure of the classical solutions displayed in Sec. II, we can be certain that the form factors will not support momentum transfers any larger than, e.g.,

$$|p_1'|\lesssim|\omega(n_1)|. \quad (4.7)$$

Therefore if we require

$$|\omega(n_1)|\sim|\omega(n_2)|\ll p_1^2, \quad (4.8)$$

we can *decouple* the various momentum integrals in (4.6) by replacing $p_1\rightarrow p_1+p_1'$, $p_2\rightarrow p_2+p_2'$ in the second factor and by making analogous but slightly more complex shifts in the first factor. We thereby obtain the result

$$C_{12}(x,t) \cong \sum_{\substack{\nu_1'\nu_2'\nu_1''\nu_2''}} \Psi^{*n_1n_2}_{\nu_1-\nu_1'-\nu_1'',\nu_2-\nu_2'-\nu_2''}(x,t) \times \Psi^{n_1n_2}_{\nu_1''\nu_2''}(x,t)\Psi^{n_1n_2}_{\nu_1'\nu_2'}(x,t) \equiv [\,|\Psi^{*n_1n_2}_{12}(x,t)|^2 \Psi^{n_1n_2}_{12}(x,t)]_{\nu_1\nu_2}. \quad (4.9)$$

Forming the sum (4.5), we see in consequence of (4.9) that the latter does indeed satisfy Eq. (2.1) and is indeed the two-soliton solution.

We summarize the conditions for the validity of this result: (i) $n_1, n_2 \gg 1$; (ii) $t_{\text{collision}} \ll n t_{\text{orbit}}$; (iii) $[|\omega(n)|]^{1/2} \ll p$. In fact (ii) and (iii) are equivalent conditions, as we see from the relations

$$t_{\text{collision}} \sim 1/(|\omega|)^{1/2} u = n/(|\omega|)^{1/2} p, \quad (4.10)$$

and

$$t_{\text{orbit}} \sim 1/|\omega|. \quad (4.11)$$

V. COMPARISON WITH EXACT TIME DELAY

Having established that the solitary wave described the bound state in the large n limit, it was a foregone conclusion that the leading term in the binding energy could be obtained from it, and this was indeed the case. Having further established that the two-soliton solution describes the collision of two bound states in the limit of large n and large relative momentum, it should again be a foregone conclusion that the phase shift or time delay is properly given by the classical function in this limit. In this section we shall carry out this verification.[28]

The exact S-matrix element[10,11] for the scattering of a composite n_1 by a composite n_2 ($n_1 > n_2$, e.g.) with relative velocity u is given by the formula, applicable to the center-of-mass system

$$S_{n_1n_2}(u) = \exp[2i\delta_{n_1n_2}(u)] = \frac{[u+in_+Ki]}{[u-in_+Ki]} \frac{[u+in_-Ki]}{[u-in_-Ki]} \prod_{r=n_-+1}^{n_++1} \frac{[u+rKi]^2}{[u-rKi]^2}$$

$$= \exp[2i(\delta_{n_+} + \delta_{n_-} + \sum_{r=n_-+1}^{n_++1} 2\delta_r)], \quad (5.1)$$

$$\delta_r = \tan^{-1}(rK/u), \quad (5.2)$$

$$n_\pm = \tfrac{1}{2}(n_1 \pm n_2). \quad (5.3)$$

The time delay is *defined* by the relation

$$\Delta t = \frac{d}{dE} 2\delta(E), \quad (5.4)$$

where

$$E = \tfrac{1}{2}\bar{m}u^2, \quad \bar{m} = n_1n_2/(n_1+n_2), \quad (5.5)$$

($\bar{m}$ is the reduced mass). We thereby compute from (5.1)–(5.5)

$$(\Delta t)_{\text{exact}} = -\frac{2}{\bar{m}uK}\left(\sum_{r=n_-}^{n_++1} \frac{r}{(u/K)^2+r^2} + \sum_{r=n_-+1}^{n_+} \frac{r}{(u/K)^2+r^2}\right). \quad (5.6)$$

If we can approximate to the sums by integrals, we obtain

$$(\Delta t)_{\text{class}} \cong -\frac{2}{\bar{m}uK}\int_{n_-}^{n_+} \frac{2x\,dx}{x^2+(u/K)^2} = -\frac{2}{\bar{m}uK}\ln\left(\frac{K^2n_+^2+u^2}{K^2n_-^2+u^2}\right). \quad (5.7)$$

It is easily shown that this agrees with (2.19) for Δt_1 if we utilize the formulas

$$\kappa_i = \tfrac{1}{2}n_iK, \quad i=1,2, \quad (5.8)$$

$$\kappa_1u_1 = \tfrac{1}{2}Kn_1u_1 = \tfrac{1}{2}\bar{m}uK/2. \quad (5.9)$$

The examination of a standard approximation formula such as

$$\sum_{r=n_-}^{n_+} f(r) \cong \int_{n_-}^{n_+} f(x)\,dx + \tfrac{1}{2}[f(n_-)+f(n_+)] - \tfrac{1}{2}[f'(n_-)-f'(n_+)] + \cdots, \quad (5.10)$$

where prime means derivative, shows that if we assume that n_- remains of order unity as $n_+ \to \infty$, that a finite error remains in the replacement of the sums in (5.6) by the integral in (5.7) unless $u \to \infty$. This is in accord with expectations, since what occurs in the expressions is the ratio

$$u/K \sim p/[\omega(n)]^{1/2}, \quad (5.11)$$

required to be large compared to unity. Thus we require both conditions derived in Sec. IV.

In the present model, p can be as large as we wish, but it is precisely here where one fails most clearly to mirror reality. For in nature, we must expect considerations such as those given in this paper to break down when a condition such

$$p/n \gtrsim [\omega(n)]^{1/2} \quad (5.12)$$

obtains, i.e., when we pass thresholds for particle knockout by individual particle collisions. For then the collective picture inherent in the TDHF approach must certainly prove inadequate.

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APPENDIX

In this Appendix we shall derive the asymptotic structure found in Eqs. (2.18)–(2.20) for the two-

soliton solution from purely quantum-mechanical considerations. We shall thereby establish that the phases α_1 and α_2 may also be used to reconstruct the phase shift. We shall in fact show that

$$\alpha_1(u, n_-) - \alpha_2(u, n_+) \cong \delta_{n_1,n_2}(u) - \delta_{n_1-1,n_2}(u) \cong \frac{\partial}{\partial n_1}\delta_{n_1,n_2}(u). \tag{A1}$$

A corresponding derivation would show that

$$(\alpha_1+\alpha_2) \propto \frac{\partial}{\partial n_2}\delta_{n_1n_2}(u). \tag{A2}$$

From (A1) and (A2), we can reconstruct $\delta_{n_1n_2}(u)$ at the semiclassical level.

We shall outline the computation of the quantity

$$I(x,t) = \lim_{x,t\to+\infty} \Psi^{n_1n_2}_{12,\nu_1\nu_2}(x,t) = \lim \int dk_1\,dk_2\, \langle n_1 - \nu_1(p_1-k_1), n_2 - \nu_2(p_2-k_2) | \times \psi(x,t) | n_1(p_1)n_2(p_2)\rangle. \tag{A3}$$

In fact, for the model under study, the only nonvanishing elements occur for $\nu_1=1, \nu_2=0$ and $\nu_1=0, \nu_2=1$. We consider the former. We utilize the form

$$|n_1(p_1), n_2(p_2)\rangle = e^{i(p_1+p_2)x}\int \psi^\dagger(x_1+x)\cdots\psi^\dagger(x_{n_1}+x)\,\psi^\dagger(y_1+x)\cdots\psi^\dagger(y_{n_2}+x)\,|\mathrm{vac}\rangle \exp[ip_1X(n_1)+ip_2X(n_2)] \times \Phi_{n_1}(\xi_1\cdots\xi_{n_1-1})\Phi_{n_2}(\eta_1\cdots\eta_{n_2-1})\,F^{n_1n_2}_u[X(n_1)-X(n_2);\xi_i,\eta_j], \tag{A4}$$

where x is an arbitrary point; the coordinates are defined as follows:

$$\begin{aligned} X(n_1) &= n_1^{-1}[x_1+\cdots+x_{n_1}],\\ X(n_2) &= n_2^{-1}[y_1+\cdots+y_{n_2}],\\ \xi_i &= x_1 - x_i, \quad i=2,\ldots,n_1,\\ \eta_i &= y_1 - y_i, \quad i=2,\ldots,n_2. \end{aligned} \tag{A5}$$

Φ_{n_1} and Φ_{n_2} are the unique bound-state wave functions and F is the wave function of relative motion, normalized so that

$$\begin{aligned} &\lim[X(n_1)-X(n_2)\to-\infty]\,F = 1,\\ &\lim[X(n_1)-X(n_2)\to+\infty]\,F = \exp[(2i\delta_{n_1n_2}(u)]. \end{aligned} \tag{A6}$$

We then proceed to the calculation of (A3). We first replace $\psi(x,t)$ by

$$\psi(x,t) = e^{iHt}\psi(x)e^{-iHt} \to \exp\{it[E_{n_1}(p_1)+E_{n_2}(p_2) - E_{n_1-1}(p_1-k_1) - E_{n_2}(p_2-k_2)]\}\,\psi(x). \tag{A7}$$

We next commute $\psi(x)$ through to the right to annihilate $|\mathrm{vac}\rangle$. With the result, we then compute the overlap of the basis vectors of initial and final state. Of the numerous contributions which occur, the only set that interests us is the one in which $\psi(x)$ annihilates a particle in the complex n_1 and in the overlap the remaining n_1-1 particles are required to coincide with the n_1-1 complex in the final state, the complexes n_2 in initial and final state also coinciding. These are clearly the terms of interest. At this stage the calculation takes the form (up to overall factors, which we incorporate into the bound-state wave functions)

$$\begin{aligned} I(x,t) = \lim_{x,t\to+\infty} &\int dk_1\,dk_2 \exp\{-it[E_{n_1}(p_1)+E_{n_2}(p_2) - E_{n_1-1}(p_1-k_1) - E_{n_2}(p_2-k_2)]\}\\ &\times \int dx_2\cdots dx_{n_1-1}dy_1\cdots dy_{n_2} \exp\{ik_1x + ik_2X(n_2) + i[(k_1-p_1/n_1)(x_2+\cdots+x_{n_1})/(n_1-1)]\}\\ &\times \Phi^*_{n_1-1}(x_2-x_3,\ldots,x_2-x_{n_1})\Phi_{n_1}(-x_2,\ldots,-x_{n_1})\,|\,\Phi_{n_2}(\eta_i)\,|^2\\ &\times F^{*n_1-1,n_2}_{(p_1-k_1)/(n_1-1)-(p_2-k_2)/n_2}[x+X(n_1-1)-X(n_2);\xi;\eta)F^{n_1n_2}_u[x-x_1/n+X(n_1)-X(n_2);-x_2\cdots-x_{n_1};\eta]. \end{aligned} \tag{A8}$$

The integrations over $y_1 \cdots y_{n_2}, k_2$ become trivial as soon as we put in the limit (A6). With the definition

$$\int dx_2 \cdots dx_{n_1} \exp[i(k_1 - u_1)(x_2 + \cdots + x_{n_1})/(n_1 - 1)] \times \Phi^*_{n-1}(x_2 - x_3, \ldots, x_2 - x_{n_1}) \Phi_{n_1}(-x_2 \cdots - x_{n_1}) = \tilde{\phi}(k_1 - p_1/n_1), \quad \text{(A9)}$$

and a semiclassical evaluation of the energy difference factor in (A8), ignoring, as in the text, the spreading of the wave packet, we obtain at the next stage

$$I(x,t) = \int dk \exp[ik(x - u_1 t) - it(\omega - \tfrac{1}{2}u_1^2)]\, \tilde{\phi}(k - p_1/n_1) \times \exp\left\{2i\delta_{n_1 n_2}(u) - \delta_{n_1 - 1, n_2}\left[u - \frac{1}{(n_1 - 1)}\left(k - \frac{p_1}{n_1}\right)\right]\right\}. \quad \text{(A10)}$$

To do the integral over k we expand δ_{n_1-1,n_2} about $\delta_{n_1-1,n_2}(u)$, keeping only first-order terms. With the final definition

$$\phi(x) = \int dk\, e^{ikx} \tilde{\phi}(k), \quad \text{(A11)}$$

we find

$$I(x,t) = \phi(x - u_1(t - \Delta t)) \exp[i\zeta(x,t,u_1)] \times \exp\{2i[\delta_{n_1 n_2}(u) - \delta_{n_1 - 1, n_2}(u)]\}, \quad \text{(A12)}$$

where

$$\Delta t = \frac{2}{n_1 u_1} \frac{d\delta_{n_1 n_2}(u)}{du}. \quad \text{(A13)}$$

This is of the form of the first term of (2.18) (up to a phase), as it should be. This allows us to make the identification (A1), which we proceed to check by utilizing the results quoted in Sec. V.

For this purpose Eq. (5.1) may be approximated in the semiclassical limit as

$$\delta_{n_1 n_2}(n) \cong \int_{n_-}^{n_+} dx\, 2\delta(x), \qquad \delta(x) = \tan^{-1}(xK/u). \quad \text{(A14)}$$

Thus

$$\frac{\partial}{\partial n_1}\delta_{n_1 n_2}(u) = \delta(n_+) - \delta(n_-) = \tan^{-1}(n_+ K/u) - \tan^{-1}(n_- K/u) = \tan^{-1}(u/n_- K) - \tan^{-1}(u/n_+ K) = \alpha_1 - \alpha_2, \quad \text{(A15)}$$

as required.

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context.

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