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Covariant Wigner function approach for relativistic quantum plasmas

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In this paper a general formalism for the treatment of relativistic quantum plasmas is given. It is manifestly covariant and rests on the use of a (covariant) relativistic Wigner function. Here it is applied to the particular case where spin effects are neglected (in most astrophysical applications this is a good approximation): a relativistic quantum Bogoliubov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy is given. The Vlasov approximation (Hartree approximation) is then considered and dispersion relations are obtained. Limiting cases (relativistic nonquantum high-temperature plasma and relativistic degenerate zero-temperature plasma) obtained previously by other authors are found anew. Finally, the formalism given appears to be much simpler and physically more transparent than many-body techniques used elsewhere.

I. INTRODUCTION

Since the discovery of pulsars (1967) the study of relativistic quantum electrodynamical plasmas has been in "highly desirable" demand, particularly when dealing with descriptions of pulsar's magnetosphere. Indeed, the extreme conditions (large magnetic fields and, more important, near the crust of a *rotating* neutron star, large electric fields) prevailing in such objects give rise to many quantum effects (particle production, plasma instabilities due to the possible existence of a positron beam, etc.) as witnessed by the well-known model of Ruderman and Sutherland.[1] This physical framework constitutes the main reason why we have undertaken a study of relativistic quantum electrodynamical plasmas. Also, it is clear that such a subject has an interest of its own.

At this point it should be specified that the application to a real problem—such as the one alluded to above—demands much effort from a theoretical point of view and needs, of course, many further calculations. With this paper we have a more modest aim; We desire to give some *methods* for dealing with relativistic quantum electrodynamical plasmas and also to investigate some of their most elementary consequences.

Our paper is of course not the first one on the subject. However, those papers[2] that have been beyond the simpler considerations rest on rather heavy, many-body techniques, which are generally not extremely transparent, at least for the plasma physicist. Here we use a manifestly covariant formalism which bears a strong resemblance with the one used in the description of classical relativistic plasmas.[3] This formalism is based on the use of a covariant Wigner function, although non-manifestly-covariant formalisms have already been considered.[2] It allows the use of methods that are quite familiar to the plasma physicist.

Among the various articles on relativistic plasmas we must first mention the paper by Tsytovich[4] who studied the dispersion relations in detail. He did not derive his equations from a Vlasov-like equation itself derived from an appropriate hierarchy as we do here. Instead he started from the usual quantum electrodynamical expression of the polarization tensor at order e^2 in which he inserted the zeroth-order (in e^2) Green's function of the electrons.

Later, Melrose[5] used the same method, in an explicitly covariant manner, to obtain the *nonlinear* response of the relativistic quantum plasma. To this end he gave a general diagrammatic technique which he illustrated in various cases (Cherenkov effect, scattering of photons, photon splitting). In the present paper we shall not be concerned with the nonlinear response of the plasma: it can be dealt with as in the nonrelativistic case. This is an advantage of our methods.

Unlike Tsytovich or Melrosé, Bezzerides and Dubois[6] developed a general formalism based on the use of the Green's function formulation of nonequilibrium statistical mechanics. So far, this work constitutes the most complete and far reaching study in the field. However, their approach is not always very simple. In particular the link with the classical kinetic theory is not manifest from the beginning. Moreover the mathematical apparatus seems, in our opinion, to be more complicated than necessary. We briefly come back to the first point at the end of Sec. II.

After giving the basic formalism (Sec. II) we discuss the possible covariant quantum Vlasov equations (Sec. III) which we use in the derivation of dispersion relations (Sec IV). In Sec. V we show how these dispersion relations reduce to previous results in the limiting cases of (i) relativistic classical plasmas and (ii) relativistic extreme degenerate plasmas. We also give the first quantum corrections.

Notations and conventions

The metric tensor is endowed with signature $+---$. Four-vectors are designate either with indices or without. For instance, $x\cdot p\equiv x^\mu p_\mu$ where the Einstein summation convention on repeated indices has been used. We have also used the following notation:

$$\varphi\overleftrightarrow{\partial}\psi\equiv\varphi\cdot\partial\psi-(\partial\varphi)\cdot\psi\,,$$

where ∂ is a partial derivative and (φ,ψ) are arbitrary functions. $\epsilon^{\mu\nu\alpha\beta}$ is, as usual, the completely antisymmetrical pseudotensor. Finally, except where otherwise stated, we use a system of units such that the speed of light and Planck's constant divided by 2π are set equal to 1.

II. BASIC NOTATIONS OF RELATIVISTIC QUANTUM STATISTICAL MECHANICS

Relativistic quantum statistical mechanics has been discussed and used elsewhere[2,7] so that in this section we only give the minimum number of definitions and properties necessary for a good understanding of that which follows.

(i) The first notion we must define is that of a covariant *one-particle Wigner function*[8]: two- or three-, etc., particle Wigner functions are easy—although some care is needed—generalizations[2] of this one-particle function. We have to recall that the Wigner function is a kind of quantum distribution function (thus not necessarily possitive or even real) useful for calculating average values of observables. We must add that there is no unique definition[9] and that it may be regarded as a mere intermediate in the calculations. However, its strong resemblance with a classical distribution function allows the use of standard methods of classical statistical mechanics and furthermore this suggests, for instance, approximations.

In the relativistic case, the Wigner distribution is no more unique that it is in the classical (i.e., Newtonian) case and useful definitions other than the one given below can also be considered.[10] Here we shall use a modification of Carruther's and Zachariasen's definition[11] valid for spin-$\frac{1}{2}$ particles (electrons...). A more general definition is given in Ref. 2 and is used elsewhere.

We define the covariant Wigner distribution[11] by

$$f^\lambda(x,p)=\frac{1}{(2\pi)^4}\int d_4R\,\exp(-ip\cdot R)\times\langle\bar{\psi}(x+\tfrac{1}{2}R)\gamma^\lambda\psi(x-\tfrac{1}{2}R)\rangle\,,\qquad(2.1)$$

where ψ is the electron/positron field ($\bar{\psi}\equiv\psi^{*T}\gamma^0$; $*$ being the operation of complex conjugation; T being the transposition) and where γ^λ are usual Dirac matrices. In Eq. (2.1) the brackets $\langle\;\rangle$ indicate a quantum statistical average defined through

$$\langle\hat{A}\rangle\equiv\mathrm{Tr}\{\rho\hat{A}\}\,,\qquad(2.2)$$

A being an arbitrary operator, where ρ is the density operator representing the statistical state of the system; Tr is the trace.

This $f^\lambda(x,p)$ can be used to calculate some average values. As an illustration, we have

$$\int d_4p\,f^\lambda(x,p)=\langle\bar{\psi}(x+\tfrac{1}{2}R)\gamma^\lambda\psi(x-\tfrac{1}{2}R)\rangle\big|_{R=0}=\langle\bar{\psi}(x)\gamma^\lambda\psi(x)\rangle\,,\qquad(2.3)$$

which is just the average four-current[12] $J^\lambda(x)$ of the electron plasma. Similarly, one finds

$$\int d_4p\,p^\mu f^\lambda(x,p)=\tfrac{1}{2}i\langle\bar{\psi}(x)\overleftrightarrow{\partial}^\mu\gamma^\lambda\psi(x)\rangle\,,\qquad(2.4)$$

which represents the average momentum-energy tensor[12] of the plasma.

At this point it should be noted that in all the above equations p^μ *is not* the four-momentum of a particle but rather its *representation*[2,9] in Minkowski four-dimensional space: to each Wigner distribution is associated a particular representation of operators by functions of x and p.

With $f^\lambda(x,p)$ we can calculate the average value of any observable $\hat{A}$ which *does not depend* on spin variables (otherwise we should use a more general quantum distribution[2] than the one given above) through

$$\langle\hat{A}\rangle_\Sigma=\int_\Sigma d\Sigma_\lambda\int d_4p\,A(x,p)f^\lambda(x,p)\,,\qquad(2.5)$$

where $A(x,p)$ is the *function* associated[2] with the observable $\hat{A}$. In Eq. (2.5) Σ is an arbitrary spacelike hypersurface (i.e., the "time" at which the average value of $\hat{A}$ is computed) which may or may not be reduced to $t=$ const; $d\Sigma_\lambda$ is the usual differential form "element of the three-surface embedded in a four-dimensional space":

$$d\Sigma_\lambda=(1/3!)\epsilon_{\lambda\mu\nu\rho}\,dx^\mu\wedge dx^\nu\wedge dx^\rho\,.\qquad(2.6)$$

In fact, when dealing with specific physical problems, we are generally not interested in average values of $\hat{A}$ at "time" Σ but rather in the "current of property $\hat{A}$" such as those given in Eqs. (2.3) and (2.4).

(ii) As to the physical interpretation of $f^\lambda(x,p)$ we just mention that it represents more or less a *one-charge* distribution function rather than a one-particle one. It should also be emphasized that it contains vacuum contributions which will have to be substracted via, e.g., a normal ordering of the Dirac fields involved in definition (2.1). Such a new definition leads to the vacuum polarization.

(iii) The usual Dirac current operator

$$J^\lambda(x)=\overline{\psi}(x)\gamma^\lambda\psi(x)\,, \tag{2.7}$$

(thus including vacuum contributions, to be substracted elsewhere) can be broken into two parts with the help of *Gordon's decomposition*:

$$\hat{J}^\lambda(x)=(i/2m)\{\overline{\psi}(x)\overleftrightarrow{\partial}^\lambda\psi(x)-\partial_\mu[\overline{\psi}(x)\sigma^{\mu\lambda}\psi(x)]\}\,, \tag{2.8}$$

with

$$\sigma^{\mu\lambda}=\tfrac{1}{2}(\gamma^\mu\gamma^\lambda-\gamma^\lambda\gamma^\mu)\,. \tag{2.9}$$

In Eq. (2.8) the first term is the *convective* part of the current, while the second term represents the *spin contribution.*

In a completely similar manner $f^\lambda(x,p)$ can also be decomposed. Using Dirac's equations

$$\{\gamma^\mu\cdot[i\partial_\mu-eA_\mu(x)]-m\}\psi(x)=0\,, \tag{2.10a}$$

$$\overline{\psi}(x)\cdot\{\gamma^\mu[i\overleftarrow{\partial}_\mu+eA_\mu(x)]+m\}=0\,, \tag{2.10b}$$

it turns out that

$$\overline{\psi}(x+\tfrac{1}{2}R)\gamma^\lambda\psi(x-\tfrac{1}{2}R)=\frac{i}{2m}\overline{\psi}(x+\tfrac{1}{2}R)\overleftrightarrow{\partial}^\lambda\psi(x-\tfrac{1}{2}R)-\frac{e}{m}S^\lambda(x,R)\overline{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R)$$
$$-\frac{e}{m}D_\mu(x,R)\overline{\psi}(x+\tfrac{1}{2}R)\sigma^{\mu\lambda}\psi(x-\tfrac{1}{2}R)-\frac{i}{2m}\partial_\mu[\overline{\psi}(x+\tfrac{1}{2}R)\sigma^{\mu\lambda}\psi(x-\tfrac{1}{2}R)]\,, \tag{2.11}$$

where we have used the notations

$$S^\lambda(x,R)\equiv\tfrac{1}{2}[A^\lambda(x+\tfrac{1}{2}R)+A^\lambda(x-\tfrac{1}{2}R)]\,, \tag{2.12}$$

$$D^\lambda(x,R)\equiv\tfrac{1}{2}[A^\lambda(x+\tfrac{1}{2}R)-A^\lambda(x-\tfrac{1}{2}R)]\,, \tag{2.13}$$

$A^\lambda(x)$ being the electromagnetic (quantized) field.

If we neglect the spin contributions, or rather the magnetic-moment contributions to energy, current, etc., Eq. (2.11) reduces to

$$\overline{\psi}(x+\tfrac{1}{2}R)\gamma^\lambda\psi(x-\tfrac{1}{2}R)\simeq(i/2m)\overline{\psi}(x+\tfrac{1}{2}R)\overleftrightarrow{\partial}^\lambda\psi(x-\tfrac{1}{2}R)-(e/m)S^\lambda(x,R)\times\overline{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R)\,. \tag{2.14}$$

In fact, this approximation is rather good in most astrophysical situations. We come back to this point in Sec. V.

With the approximation (2.14), $f^\lambda(x,p)$ can be written as

$$f^\lambda(x,p)\simeq\frac{p^\lambda}{m}f(x,p)-\frac{e}{m}\int d_4p'\,\hat{S}^\lambda(x,p')f(x,p-p')\,, \tag{2.15}$$

where

$$f(x,p)=\frac{1}{(2\pi)^4}\int d_4R\,\exp(-ip\cdot R)\times\langle\overline{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R)\rangle\,. \tag{2.16}$$

In Eq. (2.15), $\hat{S}(x,p)$ is the Fourier transform of $S(x,R)$; x being *fixed.*

A simple consequence of Eq. (2.15), which we use in the following, is that the current is given by

$$J^\lambda(x)=e\int d_4p\,f^\lambda(x,p)\simeq e\int d_4p\,\frac{p^\lambda}{m}f(x,p)-\frac{e^2}{m}\int d_4p\,d_4p'\,\hat{S}^\lambda(x,p')f(x,p-p')\,. \tag{2.17}$$

When the electromagnetic field $A^\lambda(x)$ is an *external* constant electromagnetic field [i.e., when

$$A^\lambda(x)=\tfrac{1}{2}F^\lambda_\mu x^\mu\,, \tag{2.18}$$

in the Lorentz gauge], then Eq. (2.17) reduces to

$$J^\lambda(x)\simeq e\int d_4p\,\frac{p^\lambda-eA^\lambda(x)}{m}f(x,p)\,, \tag{2.19}$$

i.e., to the usual nonquantum expression.[3]

(iv) From Dirac's equations (2.10a) and (2.10b) one can easily find an equation satisfied by $f^\lambda(x,p)$; it reads

$$\partial_\lambda f^\lambda(x,p)=2ie\int d_4p'\,f^\lambda(x,p')\hat{D}_\lambda(x,p-p')\,. \tag{2.20}$$

Of course this equation alone is not sufficient to determine the four quantities $f^\lambda(x,p)$. A true one-particle equation[2] not only involves $f^\lambda(x,p)$ and $f(x,p)$ but also more general functions. However, when magnetic-moment effects (for brevity we shall say, when spin effects) are neglected, Eq. (2.20) reduces to

$$\frac{p^\lambda}{m}\partial_\lambda f(x,p) - \frac{e}{m}\int d_4p'\{[\partial_\lambda \hat{S}^\lambda(x,p')]f(x,p-p')+\hat{S}^\lambda(x,p')\partial_\lambda f(x,p-p')\}$$
$$= \frac{2ie}{m}\int d_4p' p'^\lambda \hat{D}_\lambda(x,p-p')f(x,p') - \frac{2ie^2}{m}\int d_4p'\, d_4p''\, \hat{S}^\lambda(x,p'')f(x,p'-p'')\hat{D}_\lambda(x,p-p'), \quad (2.21)$$

i.e., Eq. (2.20) reduces to only *one* equation for the only unknown function $f(x,p)$. This equation is very similar to the one obtained in the nonrelativistic (but quantum) case.[13]

When $A^\lambda(x)$ leads to an external constant field [Eq. (2.18)], then Eq. (2.21) becomes

$$\frac{p^\lambda - eA^\lambda(x)}{m}\partial_\lambda f(x,p) = \frac{e}{m}F^{\mu\nu}[p_\nu - eA_\nu(x)]\frac{\partial}{\partial p^\mu}f(x,p), \quad (2.22)$$

i.e., the usual one-particle relativistic Liouville equation.[3] However, it should be noted that this equation (2.22) is still a *quantum* equation: $f(x,p)$ is *not* positive definite.

(v) Let us now briefly compare the above treatment and parts of the results of Bezzerides and Dubois.[6] First our Wigner function $f(x,p)$ is essentially a Fourier transform of the trace (on spinor indices) of their $g^<$ [Ref. 6, Eq. (2.25a)]. This object is introduced here from the very begining and is linked with the neglect of spin effects. In Ref. 6 a similar approximation is performed when an averaging operation is performed [Ref. 6, Eq. (3.16c)]. Our Valsov equation (2.21) is very similar to analogous nonrelativistic and quantum equations obtained by other authors[13]; moreover no ansatz other than the usual Valsov one has been used in its derivation. Furthermore, a slowly-varying-field approximation of our Eq. (2.21) leads to the more conventional Eq. (2.22) [Ref. 6, Eq. (3.24)].

III. RELATIVISTIC QUANTUM VLASOV EQUATION

Before deriving the relativistic and quantum form of the usual Vlasov equation let us first specify our basic dynamics more precisely. It should be clear that—in order to deal with a relativistic quantum electron plasma—we must work within the frame of quantum electrodynamics of which two equations [Eqs. (2.10a) and (2.10b)] have already been written. To these equations we must add the equation satisfied by the electromagnetic field $A^\mu(x)$,

$$\Box A^\mu(x) = 4\pi e\bar{\psi}(x)\gamma^\mu\psi(x), \quad (3.1)$$

which have to be supplemented by a gauge condition. At this point we *neglect the spin*, i.e., we use the approximation (2.15) for $f^\lambda(x,p)$.

As usual in plasma physics we are mainly interested in the collective behavior of the plasma and hence we discuss briefly the ways a Vlasov equation can be derived.

(i) Firstly, we can couple the covariant quantum one-particle Liouville equation (2.21) with the average value of Eq. (3.1), i.e., with

$$\Box\langle A^\lambda(x)\rangle = \frac{4\pi e}{m}\int d_4p\, p^\lambda f(x,p) - \frac{4\pi e^2}{m}\int d_4p\, d_4p' \langle \hat{S}^\lambda(x,p')\rangle f(x,p-p'), \quad (3.2)$$

obtaining thereby what might be called a *phenomenological Vlasov equation.*

(ii) Secondly, we can use a covariant BBGKY hierarchy[2,10] and then perform a truncation by neglecting correlations. The generating equations of such a hierarchy are simply constituted of Eq. (3.1) and also Eq. (2.21) where the average operation $\langle\ \rangle$ *has not been taken;* i.e., they read

$$\Box A^\lambda(x) = \frac{4\pi e}{m}\int d_4p\, p^\lambda \tilde{f}(x,p) - \frac{4\pi e^2}{m}\int d_4p\, d_4p'\, \hat{S}^\lambda(x,p')\tilde{f}(x,p-p'), \quad (3.3a)$$

$$p^\lambda\partial_\lambda \tilde{f}(x,p) - e\int d_4p'[\partial_\lambda\hat{S}^\lambda(x,p')\tilde{f}(x,p-p') + \hat{S}^\lambda(x,p')\partial_\lambda\tilde{f}(x,p-p')]$$
$$= 2ie\int d_4p' p'^\lambda \hat{D}_\lambda(x,p-p')\tilde{f}(x,p') - 2e^2\int d_4p'\, d_4p''\, \hat{S}^\lambda(x,p'')\hat{D}_\lambda(x,p-p')\tilde{f}(x,p'-p''), \quad (3.3b)$$

where we have used the notation

$$\tilde{f}(x,p) = \frac{1}{(2\pi)^4}\int d_4R\, \exp(-ip\cdot R)\bar{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R); \quad (3.4)$$

$\tilde{f}(x,p)$ is still an operator and, by construction,

$$f(x,p) = \langle\tilde{f}(x,p)\rangle. \quad (3.5)$$

In Eqs. (3.3a) and (3.3b) the symbols $S^\lambda\tilde{f}$ or $D^\lambda\tilde{f}$ must be understood with the following sense:

$$S^\lambda(\text{or } D^\lambda)\tilde{f}=A^\lambda(x+\tfrac{1}{2}R)\tilde{f}\pm\tilde{f}A^\lambda(x-\tfrac{1}{2}R)\,,$$

since the operators A^λ and $\tilde{f}$ *do not* commute.

Equations (3.3a) and (3.3b) are the relativistic quantum analog of the Klimontovich[14] set of equations generating the BBGKY hierarchy. As usual, taking the average value of both sides of these equations yields an equation connecting $f(x,p)$ and higher-order terms like $\langle \tilde{f}(x,p)A^\lambda(x')\rangle$, etc. These higher-order terms are themselves linked to terms like $\langle \tilde{f}(x,p)\tilde{f}(x',p')A^\lambda(x'')\rangle$, etc.

The Vlasov equations (2.21) and (3.2) are then recovered under the assumption that correlations are negligibly small and therefore that terms like $\langle \tilde{f}(x,p)A^\lambda(x')\rangle$ factorize:

$$\langle \tilde{f}(x,p)A^\lambda(x')\rangle \simeq f(x,p)\langle A^\lambda(x')\rangle\,. \tag{3.6}$$

However, although the two sets of equations are formally identical, they differ on one point; the mass involved in the second set of Vlasov equations (i.e., the one arising from the hierarchy) is the *bare mass* of the electron. A similar situation also arises in the classical case[3] and after a renormalization of the nonquantum relativistic BBGKY hierarchy the two Vlasov equation become identical. Here the renormalization process is much more complex.

At this point it should be remarked that in either case *exchange terms* are not considered; we dealt only with the Hartree approximation. This is due to the *ansatz* (3.6) used to truncate the hierarchy. However, if we want to consider exchange forces at this approximation (Hartree-Fock approximation) we must (i) neglect correlations due to *interactions* and (ii) keep correlations due to Fermi-Dirac statistics. This has apparently not been considered by previous authors.[15] To do this we can use a method employed elsewhere[2] that consists of formally solving Eq. (3.1) for $A^\lambda(x)$. Doing so, we obtain

$$A^\mu(x)=A^\mu_{in}(x)+4\pi e\bar{\psi}(x)\gamma^\mu\psi(x)*D(x)\,, \tag{3.7}$$

where $A^\mu_{in}(x)$ is a solution of the *homogeneous* wave equation which accounts either for a free radiation field and/or an external electromagnetic field; $D(x)$ is an appropriate Green's function which, in our case, has to be a *retarded* photon propagator; the symbol $*$ here means the usual convolution product. Inserting Eq. (3.7) into Eq. (3.3b) we obtain (see Ref. 2) a rather involved generating equation for the relativistic quantum BBGKY hierarchy that *now* explicitly contains products like

$$\tilde{f}(x,p)\tilde{f}(x',p') \text{ and } \tilde{f}(x,p)\tilde{f}(x',p')\tilde{f}(x'',p'')\,. \tag{3.8}$$

This generating equation is of course equivalent to Eq. (3.3b) and also involves products like

$$A^\lambda_{in}(x)\tilde{f}(x',p'),\quad \text{etc.} \tag{3.9}$$

These terms account for radiation emission or absorption.[13] The average value of those terms like (3.8) gives rise to many-particle distribution functions like $f_2(x,p;\,x',p'), f_3(x,p;\,x',p';\,x'',p''),\ldots$ and more complex functions[2] similar to those introduced in the nonquantum but relativistic case.[3] The splitting of f_2 and f_3 into correlated and uncorrelated parts (in a dynamical sense only, therefore taking account of exchange correlations) is now possible,[13] and a new relativistic quantum Vlasov equation is obtained (see Ref. 2 for a similar case).

Another point deserves a brief discussion. Radiative corrections are not taken into account in the previous approaches. In order to deal with such corrections a semiphenomenological treatment consists in coupling Dirac's equations with radiative corrections (due to Schwinger[16]) with Eq. (3.1). In fact we have to realize that besides the usual plasma parameter—giving rise at the lowest order to the Vlasov equation—there also exists the fine-structure constant; this latter constant—an independent dimensionless parameter—is responsible for radiative terms.

IV. BASIC DISPERSION RELATIONS

In order to obtain a deeper understanding of the various effects arising in the relativistic quantum mechanical case, and in order to obtain the nonquantum limit, we discuss the simplest case considered in Sec. III. By adding more complicated effects such as spin, exchange effects, radiative corrections, pair creation or annihilation, external electromagnetic fields, etc., more sophisticated dispersion relations can be obtained.

Consequently, let us start with the semiphenomenological relativistic quantum Vlasov system (3.2) and (2.21) which we linearize around an equilibrium state (to be specified later)

$$\begin{aligned} &f(x,p)\simeq f_{eq}(p)+f^{(1)}(x,p)\,,\\ &\langle A^\mu(x)\rangle \simeq 0+A^{\mu(1)}(x)\,. \end{aligned} \tag{4.1}$$

We obtain

$$\begin{aligned}\Box A^{\lambda(1)}(x)=\frac{4\pi e}{m}\int d_4p\, p^\lambda f^{(1)}(x,p)\\ -\frac{4\pi e^2}{m}\int d_4p\, d_4p'\, \hat{S}^{\lambda(1)}(x,p')f_{eq}(p-p')\,,\end{aligned} \tag{4.2}$$

and

$$p^\lambda\partial_\lambda f^{(1)}(x,p)-2ie\int d_4p'\, p'^\lambda \hat{D}^{(1)}_\lambda(x,p-p')f_{eq}(p')=0\,, \tag{4.3}$$

which is much simpler than the original nonlinear kinetic equation. In Eq. (4.3), $\hat{D}_\lambda^{(1)}(x, p-p')$ is defined as $\hat{D}_\lambda(x,p-p')$ [Eq. (2.13)] except that $A_\lambda^{(1)}(x)$ is involved instead of $A_\lambda(x)$. Still denoting Fourier transforms in x space by a caret, Eqs. (4.2) and (4.3) provide

$$ik_\lambda p^\lambda \hat{f}^{(1)}(k,p) = 2ie\int d_4p'\, d_4k'\, p'^\lambda \hat{D}_\lambda(k-k', p-p') \times \delta^{(4)}(k') f_{\text{eq}}(p'), \quad (4.4)$$

$$-k_\mu k^\mu \hat{A}^{\lambda(1)}(k) = \frac{4\pi e}{m}\int d_4p\, p^\lambda f^{(1)}(k,p) - \frac{4\pi e^2}{m}\int d_4p\, d_4p'\, d_4k' \hat{S}^\lambda(k-k',p') \times \delta^{(4)}(k') f_{\text{eq}}(p-p'), \quad (4.5)$$

where we have taken account of the invariance under space-time translations of the equilibrium distribution

$$\hat{f}_{\text{eq}}(k,p) = \delta^{(4)}(k) f_{\text{eq}}(p). \quad (4.6)$$

Notice that, more explicitly, we also have

$$\hat{D}_\lambda(k,p) = \tfrac{1}{2}[\hat{A}_\lambda^{(1)}(k)\delta^{(4)}(p-\tfrac{1}{2}k) - \hat{A}_\lambda^{(1)}(k)\delta^{(4)}(p+\tfrac{1}{2}k)], \quad (4.7)$$

$$\hat{S}_\lambda(k,p) = \tfrac{1}{2}[\hat{A}_\lambda^{(1)}(k)\delta^{(4)}(p-\tfrac{1}{2}k) + \hat{A}_\lambda^{(1)}(k)\delta^{(4)}(p+\tfrac{1}{2}k)].$$

From Eq. (4.4) and (4.5) it follows that

$$\hat{f}^{(1)}(k,p) = \frac{e}{k_\lambda p^\lambda}\hat{A}_\mu^{(1)}(k)\int d_4p'\, p'^\mu f_{\text{eq}}(p') \times [\delta^{(4)}(p-p'-\tfrac{1}{2}k) - \delta^{(4)}(p-p'+\tfrac{1}{2}k)], \quad (4.8)$$

$$-k_\mu k^\mu \hat{A}^{\lambda(1)}(k) = \frac{4\pi e}{m}\int d_4p\, p^\lambda \hat{f}^{(1)}(k,p) - \frac{2\pi e^2}{m}\int d_4p\, d_4p' \hat{A}^{\lambda(1)}(k) f_{\text{eq}}(p-p') \times [\delta^{(4)}(p'-\tfrac{1}{2}k) + \delta^{(4)}(p'+\tfrac{1}{2}k)]. \quad (4.9)$$

In Eq. (4.8) we implicitly used the Landau's prescription for the calculation of $(p_\mu k^\mu)^{-1}$; i.e.,

$$\frac{1}{p_\mu k^\mu} \equiv \delta^-(p_\mu k^\mu) = \lim_{\epsilon\to 0}\frac{1}{p_\mu k^\mu + i\epsilon}. \quad (4.10)$$

Similarly, the calculation of $(k_\mu k^\mu)^{-1}$ implicitly contains a small imaginary part so that it represents nothing but the Fourier transform of the *retarded photon propagator*,[12]

$$D_R(x) = -\delta(x^0 - |\vec{x}|)/|\vec{x}|. \quad (4.11)$$

Now inserting $f^{(1)}(k,p)$ from Eq. (4.8) into Eq. (4.9) we get

$$\left(-k_\mu k^\mu + \frac{4\pi e^2}{m}\int d_4p f_{\text{eq}}(p)\right)\hat{A}^{\lambda(1)}(k) = -\frac{4\pi e^2}{m}\int d_4p \frac{p^\lambda p^\mu}{k_\mu p^\mu}[f_{\text{eq}}(p+\tfrac{1}{2}k) - f_{\text{eq}}(p-\tfrac{1}{2}k)]\hat{A}_\mu^{(1)}(k), \quad (4.12)$$

where we have used the Lorentz gauge condition

$$k_\mu \hat{A}^{\mu(1)}(k) = 0, \quad (4.13)$$

valid for the *average* field (i.e., for a nonquantum field) considered in Eq. (4.1).

As usual the homogeneous equation (4.12) has nonvanishing solutions only when its determinant is zero *and* when taking the gauge condition (4.13) into account. This provides the dispersion relations

$$\text{Det}\{[\Omega_p^2 - k_\mu k^\mu]g^{\lambda\sigma} + \omega_p^2 K^{\lambda\sigma}\} = 0, \quad k_\mu \hat{A}^{\mu(1)}(k) = 0 \quad (4.14)$$

(ω_p^2 being the usual plasma frequency, i.e., $\omega_p^2 = 4\pi e^2 n_{\text{eq}}/m$) and where

$$\Omega_p^2 = \frac{4\pi e^2}{m}\int d_4p\, f_{\text{eq}}(p) \quad (4.15)$$

is the *relativistic quantum plasma frequency* and where we have set

$$K^{\lambda\sigma} = \frac{1}{n_{\text{eq}}}\int d_4p \frac{p^\lambda p^\sigma}{k_\mu p^\mu}[f_{\text{eq}}(p+\tfrac{1}{2}k) - f_{\text{eq}}(p-\tfrac{1}{2}k)], \quad (4.16)$$

n_{eq} being the equilibrium invariant numerical density of particles.

Note that Ω_p^2 is related to the usual (squared) plasma frequency through

$$\frac{\Omega_p^2}{\omega_p^2} = \frac{1}{n_{\text{eq}}}\int d_4p f_{\text{eq}}(p), \quad (4.17)$$

and that the integral appearing in the above relation *is not* the plasma density. For instance when $f_{\text{eq}}(p)$ is chosen to be a relativistic Fermi-Dirac distribution function, then a glance at Eq. (A13) shows that

$$\Omega_p^2/\omega_p^2 = T^\mu{}_\mu/mn_{\text{eq}}, \quad (4.18)$$

$T^\mu{}_\mu$ being the trace of the equilibrium momentum energy tensor.

In a frame of reference where the equilibrium four-velocity of the plasma reduces to (1, 0, 0, 0) and for waves propagating along the third axis, i.e., for $k \equiv (\omega, 0, 0, k_3)$, Eqs. (4.14) become

$$1 - \frac{\Omega_p^2}{\omega^2 - \vec{k}^2} + \frac{\omega_p^2}{\omega^2 - \vec{k}^2}K^{11} = 0 \quad (4.19)$$

(transverse modes);

$$1-\frac{\Omega_p^2}{\omega^2-\vec{k}^2}-\frac{\omega_p^2}{\omega^2-\vec{k}^2}K^{00}+\frac{\omega}{|\vec{k}|}\frac{\omega_p^2}{\omega^2-\vec{k}^2}K^{30}=0 \quad (4.20)$$

(longitudinal modes). Moreover, from Eq. (4.18) one can easily see that the polarization tensor $\Pi^{\mu\nu}(k)$, defined through

$$\hat{J}^{\mu(1)}(k)=\Pi^{\mu\nu}(k)\hat{A}_\nu^{(1)}(k), \quad (4.21)$$

is given by

$$4\pi\Pi^{\mu\nu}(k)=-(\omega_p^2K^{\mu\nu}+\Omega_p^2 g^{\mu\nu}). \quad (4.22)$$

As usual the Lorentz gauge condition and the charge conservation equation lead to

$$k_\mu\Pi^{\mu\nu}(k)=0, \quad (4.23)$$

which is satisfied by Eq. (4.22).

Although $f_{eq}(p)$ has not been specified in this section (in fact, any homogeneous and uniform invariant distribution function satisfying usual mathematical requirements such as integrability, etc., can be chosen), most of the applications require the use of the relativistic Fermi-Dirac distribution function. In our case the corresponding covariant Wigner function is given by

$$f_{eq}(p)=\frac{2m}{(2\pi)^3}\int\frac{d_3p'}{|p_0'|}\left(\frac{\delta^{(4)}(p-p')}{\exp[\beta(u^\mu p'_\mu-\epsilon_f)]+1}+\frac{\delta^{(4)}(p+p')}{\exp[\beta(\epsilon_f-u_\mu p'^\mu)]+1}\right), \quad (4.24)$$

(see the Appendix) where $\beta\equiv(kT)^{-1}$, ϵ_f being the Fermi energy. In Eq. (4.24) the second term corresponds to the *positrons* present in the plasma. In $f_{eq}(p)$ we have eliminated an irrelevant vacuum contribution.

V. RESULTS AND DISCUSSION

Let us now apply the results of the preceding sections to several important cases: (i) nonquantum approximation, (ii) first quantum correction, (iii) degenerate plasma.

(i) Let us begin with the nonquantum limit and first consider the dispersion relation (4.19) for transverse modes. For a nondegenerate plasma $f_{eq}(p)$ reduces to the usual Jüttner-Synge distribution[17] and therefore the relativistic quantum plasma frequency Ω_p^2 reduces to the one obtained elsewhere,[3] i.e., to

$$\Omega_p^2=\omega_p^2K_1(m\beta)/K_2(m\beta), \quad (5.1)$$

the K_i's being the Kelvin's functions[17] which are connected to the usual modified Bessel's functions. As to the other term involved in Eq. (4.19), it contains K^{11} which should be considered—in the classical limit—in the low-frequency and long-wavelength approximation. In other words, in Eq. (4.16) we must take

$$f_{eq}(p\pm\tfrac{1}{2}k)\simeq f_{eq}(p)\pm\tfrac{1}{2}k^\lambda\frac{\partial}{\partial p^\lambda}f_{eq}(p). \quad (5.2)$$

Doing so and integrating by parts the resulting integral, we are left with the dispersion equation found elsewhere[3] (neglecting of course the radiation-reaction terms).

The longitudinal modes provided by Eq. (4.20) can similarly be treated and the nonquantum limit[3] is also obtained. However it is simpler to check that Eqs. (4.14) joined to Eqs. (5.1) and (5.2) lead to the classical results.[3]

(ii) Let us now give the first quantum correction. Here we have to specify more precisely what is meant by "quantum correction." It is of course clear that a necessary quantum correction arises from the effect of the Pauli principle. Such a correction is provided by the expansion of the relativistic Fermi-Dirac distribution function, i.e., by

$$\{1+\exp[\beta(p_\mu u^\mu-\epsilon_f)]\}^{-1}\simeq\exp[-\beta(p_\mu u^\mu-\epsilon_f)]\{1-\exp[-\beta(p_\mu u^\mu-\epsilon_f)]\}. \quad (5.3)$$

As usual this expansion comes from a comparison of the thermal de Broglie wavelength and of the interparticle distance.[18] Besides this, there also exists another quantum correction connected with the consideration of moderately high wavelengths and frequencies as compared to the thermal de Broglie wavelength. For instance, we could take

$$f_{eq}(p\pm\tfrac{1}{2}k)\simeq f_{eq}(p)\pm\frac{k^\lambda}{2}\frac{\partial}{\partial p^\lambda}f_{eq}(p)+\frac{k^\lambda k^\mu}{4}\frac{1}{2!}\frac{\partial^2 f_{eq}(p)}{\partial p^\lambda\partial p^\mu}, \quad (5.4)$$

however, it is simpler to keep the full expression. We could perfectly well consider a diluted quantum plasma; in other words, we could neglect the effect of the Pauli exclusion principle, while keeping the finite momentum transfer effects. The last quantum correction to be included deals with positrons possibly present in a high temperature ($kT\sim mc^2$) plasma. Strictly speaking, positron creation is a quantum effect by itself, arising *even* in a classical relativistic plasma. They can be taken into account by simply adding one more component in the classical relativistic plasma, and thus this is not a purely quantum correction, *at least at low frequencies*. In fact the positrons play—at very high frequencies ($\hbar\omega_p\simeq mc^2$)—a very important role in the damping of plasma waves. We come back to this point elsewhere in connection with the problem of vacuum polarization.

These *three* kinds of quantum corrections reflect the fact that we have basically *three* independent parameters in the problem constructed from ϵ_f, kT, mc^2, and $\hbar\omega_p$.

Since the nonquantum dispersion relations [Refs. 3(b) and 3(c)] have the same *form* as the quantum ones [Eqs. (4.19) and (4.20)] except that the $K^{\mu\nu}$ does not have the same value, it is sufficient to calculate the first quantum corrections to this last tensor. In order to compare the quantum case with the classical results, we also have to neglect the contributions of the positrons. These contributions are in fact trivial and would have to be compared with the classical results in which positrons would have been included.

First we have to normalize the approximate distribution function (5.3) via the use of Eq. (A12). We find

$$2\sinh\beta\epsilon_f \frac{m^2K_2(m\beta)}{\pi^2\beta} - 2\sinh 2\beta\epsilon_f \frac{m^2K_2(2m\beta)}{2\pi^2\beta} = n_{\text{eq}}, \tag{5.5}$$

where n_{eq} is the invariant charge density of the plasma. Had we neglected the contribution of positrons, we would have found $\exp(\beta\epsilon_f)$ instead of $\sinh\beta\epsilon_f$. Note that the small parameter (in order to obtain the first correction due to the Pauli's exclusion principle) is $\exp(\beta\epsilon_f)$. Thus, neglecting the positrons in Eq. (5.5) we obtain

$$\exp(\beta\epsilon_f) \simeq \frac{n_{\text{eq}}}{A(\beta)} + n_{\text{eq}}^2 \frac{A(2\beta)}{A^3(\beta)}, \tag{5.6}$$

where we have

$$A(\beta) = m^2K_2(m\beta)/\pi^2\beta. \tag{5.7}$$

In Eq. (5.6) the first term of the right-hand side is the classical term (arising in the Jüttner-Synge distribution[17]).

Let us now come back to the tensor $K^{\mu\nu}$ of Eqs. (4.19) and (4.20). Inserting Eqs. (5.3), (5.4), and (5.6) into Eq. (4.16), we get

$$K^{\lambda\sigma} = \frac{\exp(\beta\epsilon_f)m^2K_2(m\beta)}{\pi^2 n_{\text{eq}}\beta}\left[k^\lambda I^\sigma(\beta) + k^\sigma I^\lambda(\beta) + k^\rho k_\rho I^{\sigma\lambda}(\beta)\right] - (\text{similar terms with } \beta \to 2\beta). \tag{5.8}$$

In this equation the tensors $I^\sigma(\beta)$ and $I^{\lambda\sigma}(\beta)$ are given by the integrals

$$I^\sigma(\beta) = \frac{1}{n_{\text{eq}}}\int d_4\xi \frac{\xi^\sigma}{k^\alpha\xi_\alpha}\mathfrak{N}_{\text{eq}}(\xi^\rho), \tag{5.9}$$

$$I^{\lambda\sigma}(\beta) = -\frac{1}{n_{\text{eq}}}\int d_4\xi \frac{\xi^\lambda\xi^\sigma}{(k_\alpha\xi^\alpha)^2}\mathfrak{N}_{\text{eq}}(\xi^\rho), \tag{5.10}$$

where $\mathfrak{N}_{\text{eq}}(\xi^\rho)$ is the Jüttner-Synge equilibrium distribution

$$\mathfrak{N}_{\text{eq}}(\xi^\rho) = 2\theta(\xi^0)\delta(\xi^\alpha\xi_\alpha - 1)\frac{n_{\text{eq}}m\beta}{4\pi K_2(m\beta)}\exp(-\beta u_\alpha\xi^\alpha). \tag{5.11}$$

These integrals have been studied in details in Refs. 3(b) and 3(c). Now, using Eq. (5.6), the first quantum correction $\delta K^{\lambda\sigma}$ to $K^{\lambda\sigma}_{\text{class}}$ is found to be

$$\delta K^{\lambda\sigma} = \frac{n_{\text{eq}}A(2\beta)}{A^2(\beta)}\left[k^\lambda I^\sigma(\beta) + k^\sigma I^\lambda(\beta) + k^\rho k_\rho I^{\sigma\lambda}(\beta)\right] - \left[k^\lambda I^\sigma(2\beta) + k^\sigma I^\lambda(2\beta) + k^\rho k_\rho I^{\sigma\lambda}(2\beta)\right]. \tag{5.12}$$

(iii) Let us now consider the completely degenerate case, i.e., $T = 0\,^\circ\text{K}$. We should like to find anew results already obtained by Jancovici,[19] who used many-body techniques. Here we content ourselves to rederive his results in the approximation where the frequencies are much smaller than the Fermi energy and where the wavelengths are much smaller than the Fermi momentum of the plasma:

$$\omega \ll \epsilon_f \quad\text{and}\quad k \ll f \equiv (\epsilon_f^2 - m^2)^{1/2}. \tag{5.13}$$

Inserting the zero-temperature relativistic Fermi-Dirac distribution

$$f_{\text{eq}}(p) = \frac{2m}{(2\pi)^3}\int d_4p'\,\delta^{(4)}(p-p')2\theta(p'^0) \times \delta(p'^2 - m^2)\theta(\epsilon_f - p_0') \tag{5.14}$$

(where θ is the Heaviside step function and where the positron part is not present at $T = 0\,^\circ\text{K}$) in the expression of K^{11} [see Eq. (4.16)] occurring in the dispersion equation (4.19), we obtain

$$K^{11} = \frac{m}{4\pi^3 n_{\text{eq}}}\int d_3p\, p_1^2\left(\frac{\theta(\epsilon_f - E_{p+k/2})}{E_{p+k/2}}\,\frac{1}{\omega(E_{p+k/2} - \frac{1}{2}\omega) - \vec{k}\cdot\vec{p}} - \frac{\theta(\epsilon_f - E_{p-k/2})}{E_{p-k/2}}\,\frac{1}{\omega(E_{p-k/2} + \frac{1}{2}\omega) - \vec{k}\cdot\vec{p}}\right), \tag{5.15}$$

where $E_p \equiv (p^2 + m^2)^{1/2}$.

Taking account of the approximations (5.13), we can write

$$\frac{1}{\omega(E_{p\pm k/2} \mp \frac{1}{2}\omega) - \vec{k}\cdot\vec{p}} \sim \frac{1 \pm \omega/2E_p}{\omega E_p - \vec{k}\cdot\vec{p}} \tag{5.16}$$

which, once introduced into Eq. (5.15), provides

$$K^{11} = \frac{m}{4\pi^3 n_{\text{eq}}}\int d_3p\, \frac{p_1^2}{\omega E_p - \vec{k}\cdot\vec{p}} \left[\frac{\theta(\epsilon_f - E_p)}{E_p}\,\frac{\omega}{E_p} + \frac{\vec{k}\cdot\vec{p}}{E_p}\,\frac{\partial}{\partial E_p}\left(\frac{\theta(\epsilon_f - E_p)}{E_p}\right)\right] \tag{5.17}$$

after taking the first-order terms in the expansion of the remaining factors. Performing the angular integrations in this last equation gives

$$K^{11}=\frac{m}{4\pi^2 n_{\rm eq}}\int dp\,\frac{p^4}{E_p^2}\left[\frac{\partial\theta(\epsilon_f-E_p)}{\partial E_p}\left(-\frac{4}{3}+A(1-A^2)\ln\left|\frac{1+A}{1-A}\right|+2A^2\right)+\frac{4}{3}\frac{\theta(\epsilon_f-E_p)}{E_p}\right], \tag{5.18}$$

where we have set

$$A=\omega E_p/kp\,. \tag{5.19}$$

Notice that the derivatives in Eqs. (5.17) and (5.18) are to be taken in the sense of distribution theory, i.e.,

$$\frac{\partial}{\partial E_p}\,\theta(\epsilon_f-E_p)=-\delta(\epsilon_f-E_p)\,. \tag{5.20}$$

The remaining p integration is straightforward and the final result yields the dispersion relation

$$1-\frac{2e^2}{\pi(\omega^2-k^2)}\frac{f^3}{\epsilon_f}\left[\frac{\omega\epsilon_f}{2kf}\left(1-\frac{\omega^2\epsilon_f^2}{k^2f^2}\right)\times\ln\left|\frac{\omega\epsilon_f+kf}{\omega\epsilon_f-kf}\right|+\frac{\omega^2\epsilon_f^2}{k^2f^2}\right]=0. \tag{5.21}$$

The real part of the transverse dielectric constant is then given by

$$\mathrm{Re}\,\epsilon^T(\omega,k)=1-\frac{2e^2f^3}{\pi\omega^2\epsilon_f}\left[\frac{\omega\epsilon_f}{2kf}\left(1-\frac{\omega^2\epsilon_f^2}{k^2f^2}\right)\times\ln\left|\frac{\omega\epsilon_f+kf}{\omega\epsilon_f-kf}\right|+\frac{\omega^2\epsilon_f^2}{k^2f^2}\right]. \tag{5.22}$$

This expression does not look like Jancovici's Eq. (66)[19] because he used a different definition for the dielectric constant. His definition is related to our's (taken from standard plasma physics[20]) through

$$\epsilon_{\rm Janco.}=(k^2-\omega^2\epsilon)/(k^2-\omega^2), \tag{5.23}$$

(see the footnote on p. 438 of Ref. 19).

Once Eq. (5.22) is inserted into Eq. (5.23) Jancovici's result is recovered. Similarly, the imaginary part is found to be

$$\mathrm{Im}\,\epsilon^T(\omega,k)=\frac{e^2f^3}{k\omega}\left(\frac{\omega^2\epsilon_f^2}{k^2f^2}-1\right), \tag{5.24}$$

in the case where $\omega<fk/\epsilon_f$ and $\mathrm{Im}\,\epsilon^T=0$ otherwise. Using once more Eq. (5.23) Jancovici's result is recovered.

Similarly, the longitudinal modes are those obtained by Jancovici.

Let us now briefly discuss this dispersion relation. First let us make f tend to zero (equivalently, $\epsilon_f\to m$ or $n_{\rm eg}\to 0$). Then the dielectric constant tends to 1 as it should be. Moreover this shows that there is no extra contribution coming from the vacuum; our phenomenological Vlasov equation is, in a sense, renormalized. Furthermore, this sheds some light on Jancovici's (renormalized) results. They are valid (as are our's) only when there is no pair creation (which would damp those plasma waves for which $\hbar\omega\gtrsim mc^2$); there is an implicit assumption that only those frequencies such that

$$\hbar\omega\ll mc^2 \tag{5.25}$$

are considered. Another remark is that it might be surprising that our results are identical to those obtained by Jancovici since we neglected spin effects, whereas they are fully taken into account in Ref. 19. This is, in fact, due to our approximation (5.13); a dimensional analysis of the supplementary term introduced by spin effects shows that this term is small when Eq. (5.13) is satisfied.

APPENDIX: THE EQUILIBRIUM WIGNER FUNCTION

Let us briefly sketch the derivation of the equilibrium Wigner function with no interaction. Essentially we have to calculate

$$\langle :\overline{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R):\rangle=\mathrm{Tr}[\rho:\overline{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R):] \tag{A1}$$

where the density operator ρ is given by

$$\rho=Z^{-1}\exp(-\beta u_\mu P^\mu+\beta\epsilon_f Q), \tag{A2}$$

with

$$Z\equiv Z(\beta,\epsilon_f)=\mathrm{Tr}[\exp(-\beta u_\mu P^\mu+\beta\epsilon_f Q)]. \tag{A3}$$

In Eq. (A3) P^μ is the momentum-energy operator

$$P^\mu=\int_\Sigma T^{\mu\nu}d\Sigma_\nu=\int_\Sigma \tfrac{1}{2}i:\overline{\psi}\gamma^\nu\overleftrightarrow{\partial}^\mu\psi:d\Sigma_\nu, \tag{A4}$$

(Σ being an arbitrary spacelike three-surface) and Q the total charge operator

$$Q=\int_\Sigma J^\mu d\Sigma_\mu=\int_\Sigma:\overline{\psi}\gamma^\mu\psi:d\Sigma_\mu. \tag{A5}$$

In Eqs. (A1), (A4), and (A5) we have introduced a normal ordering of the various operators involved in order to eliminate irrelevant vacuum contributions.[12] With more conventional (transparent) notations the exponential in Eq. (A2) can be written as[12]

$$\exp\left(-\beta\sum_p\sum_{s=1,2}[(E_p-\epsilon_f)b_s^\dagger(p)b_s(p)+(E_p+\epsilon_f)d_s^\dagger(p)d_s(p)]\right), \tag{A6}$$

in the frame of reference where $u^\mu\equiv(1,0,0,0)$. In

Eq. (A6) and in all subsequent equations we use the notations of Ref. 12. For instance the $b, b^\dagger$ (the $d, d^\dagger$) are the electron (positron) annihilation and creation operators.

From

$$\psi(x)=\frac{1}{(2\pi)^{3/2}}\int d_3p\left(\frac{m}{E_p}\right)^{1/2}\sum_{s=1,2}\{\}, \tag{A7}$$

with

$$\{\}=b_s(\vec{p})w^s(\vec{p})\exp(-ip\cdot x)+d_s^\dagger(\vec{p})v^s(\vec{p})\exp(-ip\cdot x) \tag{A8}$$

(see Ref. 12, pp. 224–225) and a similar relation for $\bar{\psi}(x)$, we get

$$
\begin{aligned}
:\bar{\psi}(x+\tfrac{1}{2}R)\psi(x-\tfrac{1}{2}R):=\frac{1}{(2\pi)^3}\sum_{p,p'}\sum_{s,s'}\left(\frac{m^2}{E_p\cdot E_{p'}}\right)^{1/2}
&\times\{\bar{w}_s(p')v_s(p)\exp[ip'\cdot(x+\tfrac{1}{2}R)+ip\cdot(x-\tfrac{1}{2}R)]b_s^\dagger(p')d_s^\dagger(p)\\
&+\bar{v}_s(p')w_s(p)\exp[-ip'\cdot(x+\tfrac{1}{2}R)-ip\cdot(x-\tfrac{1}{2}R)]d_s(p')b_s(p)\\
&+\bar{w}_s(p')w_s(p)\exp[ip'\cdot(x+\tfrac{1}{2}R)-ip\cdot(x-\tfrac{1}{2}R)]b_s^\dagger(p')b_s(p)\\
&-\bar{v}_s(p')v_s(p)\exp[-ip'\cdot(x+\tfrac{1}{2}R)+ip\cdot(x-\tfrac{1}{2}R)]d_s^\dagger(p')d_s(p)\}.
\end{aligned} \tag{A9}
$$

Taking account of the orthogonality relations between the free spinors v and w (Ref. 12, p. 87) the cross terms (db and $b^\dagger d^\dagger$) in Eq. (A9) vanishes in the process of averaging while the other terms lead to one-particle wave functions to be weighted by a factor

$$[\exp(\beta E_p-\beta\epsilon_f)+1]^{-1} \tag{A10}$$

for the electrons and a factor

$$[\exp(-\beta E_p-\beta\epsilon_f)+1]^{-1}\quad (E_p<0) \tag{A11}$$

for the positrons. This is due, as usual when there is no interaction, to the fact that we consider only an average value of a *one-particle* operator. Taking now the Fourier transform of the average value of Eq. (A9) the final result (4.24) is obtained.

Although the form (4.24) for $f_{eq}(p)$ is not the usual one, let us show that it actually leads to conventional expressions on the example of the four-current and of the momentum-energy tensor. They are given by

$$
\begin{aligned}
J^\mu&=\int d_4p\,\frac{p^\mu}{m}f_{eq}(p)\\
&=\frac{1}{4\pi^3}\int\frac{d_3p}{E_p}p^\mu\left(\frac{1}{\exp[\beta(E_p-\epsilon_f)]+1}-\frac{1}{\exp[\beta(E_p+\epsilon_f)]+1}\right),
\end{aligned} \tag{A12}
$$

$$
\begin{aligned}
T^{\mu\nu}&=\int d_4p\,\frac{p^\mu p^\nu}{m}f_{eq}(p)\\
&=\frac{1}{4\pi^3}\int\frac{d_3p}{E_p}p^\mu p^\nu\left(\frac{1}{\exp[\beta(E_p-\epsilon_f)]+1}+\frac{1}{\exp[\beta(E_p+\epsilon_f)]+1}\right).
\end{aligned} \tag{A13}
$$

They are completely identical to the conventional expressions[21] except that they also contain the correct contributions of the positrons.

Finally it should be emphasized that this $f_{eq}(p)$ is normalized—as it should be—to the *total* charge of the plasma (which is a constant of the motion) and not to the total number of particles (which is essentially variable).

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