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Semiclassical and quantum theory of the bistability in lasers containing saturable absorbers. II

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We consider the time-evolution equation for the Glauber distribution function of the field emitted by a laser with saturable absorber and investigate the role of the terms with derivatives of order higher than second. It is found that the main effect of these terms is the following: When the relative saturability of the passive atoms with respect to the active atoms is very high, the (first-order-like) transition is much sharper than predicted by the Fokker-Planck equation.

I. INTRODUCTION

In a previous paper,[1] to be referred to hereafter as I, we have studied the dynamics of a laser containing two cells, one amplifying and one absorbing (laser with saturatable absorber). We have given both a semiclassical and a quantum-mechanical analysis. The latter treatment is based on a suitably approximate Fokker-Planck equation, which is obtained roughly as follows. The starting point is a generalized von Neumann equation which describes the interaction of a single damped mode with the active and passive atoms. Then assuming that the atoms constitute a dilute system and that the atomic relaxation times are much smaller than the field relaxation time, one gets a closed time-evolution equation for the Glauber quasiprobability distribution $P(\beta, \beta^*, t)$ of the field alone. This equation contains derivatives of all orders in β, β^*. By neglecting the terms with derivatives of order higher than second, the equation reduces to a Fokker-Planck equation, which gives a fairly complete physical description of the problem. The drift term of this equation is directly linked to the semiclassical description, while the diffusion term describes the quantum-mechanical fluctuations.

The aim of the present paper is to give a detailed mathematical analysis of the additional quantum mechanical effects arising from the terms with derivatives of order higher than second. In Sec. II, we recall the von Neumann equation and the derivation of the closed time-evolution equation for $P(\beta, \beta^*, t)$. This equation is solved exactly in the stationary situation in Sec. III. Unfortunately, this solution, which is expressed in terms of hypergeometric functions, is not suitable for the numerical calculation of the mean values. Hence, in Sec. IV we give an approximate solution of the steady-state equation. This solution holds when the saturation parameter $\bar{S}$ of the absorbing atoms is much larger than the saturation parameter S of the active atoms. As shown in I, this condition guarantees that the system can exhibit a bistable behavior. Finally, in Sec. V, we evaluate numerically the first two moments of this stationary distribution and compare them with the same quantities calculated from the stationary solution of the Fokker-Planck equation. It turns out that the terms with derivatives of order higher than second have the effect of strongly reducing the width of the transition region, thereby making the (first-order-like) transition threshold much sharper.

II. QUANTUM-MECHANICAL TIME-EVOLUTION EQUATION FOR THE FIELD

In I, we considered the interaction of a single running mode of the laser cavity with a system of two-level atoms of two different species. The N atoms of the first species are pumped to a positive inversion, so that they are active atoms. The $\bar{N}$ atoms of the second species are also pumped, but in such a way that their inversion remains negative, so that they are absorbing atoms. The active (passive) atoms are labeled by the letter p (q) where $p=1,\ldots,N$ ($q=1,2,\ldots,\bar{N}$). To the pth active atom are associated the raising and lowering operators $a^\dagger(p), a(p)$. The combined action of atomic decay and pumping on the pth atom is characterized by the three parameters $\gamma_\perp(p)$ (transverse relaxation time), $\gamma_\parallel(p)$ (longitudinal relaxation time), and $\sigma(p)$ (unsaturated inversion).

Of course corresponding operators $A^\dagger(q), A(q)$ and parameters $\overline{\gamma}_\perp(q), \overline{\gamma}_\parallel(q), \overline{\sigma}(q)$ are associated to each of the passive atoms; we distinguish the parameters which refer to the passive atoms by simply putting a bar on them. Note that $\overline{\sigma}(q)$ is negative for all $q=1,2,\ldots,\overline{N}$.

Let us consider the statistical operator $\mathcal{W}(t)$ of the system atoms + running mode. More precisely, the running mode is described in the Glauber diagonal representation, labeled by the continuous amplitude variable β. Hence, we write

$$\mathcal{W}(t) = \int d_2\beta \,|\beta\rangle\langle\beta|\, W(\beta,\beta^*,t)\,, \tag{2.1}$$

where $W(\beta,\beta^*,t)$ is an operator in the atomic Hilbert space alone. $W(\beta,\beta^*,t)$ obeys a generalized von Neumann equation

$$i\hbar\partial_t W(\beta,\beta^*,t) = (L_A + L_F + L_{AF} + i\Lambda_A + i\Lambda_F)W(\beta,\beta^*,t)\,. \tag{2.2}$$

The explicit expressions of L_A, L_F, etc. are given in I, to which we refer for all details. Here, we limit ourselves to explain the meaning of the various terms in Eq. (2.2). L_A and L_F describe the free motion of the field and of the atoms, respectively. L_{AF} arises from the interaction of the atoms (active and passive) with the field in the dipole and rotating wave approximation. The interaction of the pth atom with the running mode is characterized by a coupling constant $g(p)$. Λ_A describes the atomic decay and pumping. Finally Λ_F takes into account that the photons escape from the cavity; the rate of escape will be given by a parameter κ. We have analyzed[1] Eq. (2.2) with the following assumptions: (a) Low concentration of active and passive atoms, (b) $\kappa \ll \gamma_\perp, \gamma_\parallel$ for all active and passive atoms.

Under these conditions, the von Neumann equation (2.2) for the full operator $W(\beta,\beta^*,t)$ reduces to a closed set of equations for the c-number quantities

$$\begin{aligned} P(\beta,\beta^*,t) &= e^{iL_F t/\hbar}\,\mathrm{Tr}\,W(\beta,\beta^*,t)\,,\\ C(p,\beta,\beta^*,t) &= e^{iL_F t/\hbar}\,\mathrm{Tr}[a^\dagger(p)a(p)W(\beta,\beta^*,t)]\,,\\ \overline{C}(q,\beta,\beta^*,t) &= e^{iL_F t/\hbar}\,\mathrm{Tr}[A^\dagger(q)A(q)W(\beta,\beta^*,t)]\,, \end{aligned} \tag{2.3}$$

where Tr means the trace over the complete atomic Hilbert space. $P(\beta,\beta^*,t)$ is the Glauber quasiprobability distribution function of the field, while C and $\overline{C}$ are two auxiliary distributions. In the present paper we shall consider the simplest situation:

(i) The active (passive) atoms are identical and homogeneously distributed within the active (passive) region.

(ii) The transition frequencies of the active and passive atoms are equal and coincide with the frequency of the running mode.

Under conditions (i), (ii) the equations for P, C, and $\overline{C}$ are [see Eqs. (4.9) of I]:

$$\begin{aligned}\kappa^{-1}\partial_t P(\beta,\beta^*,t) = {}&\left(1+\frac{1}{\sigma_T}+\frac{1}{\overline{\sigma}_T}\right)\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)P(\beta,\beta^*,t)+\frac{2}{\sigma_T}\left(\frac{\partial^2}{\partial\beta^*\partial\beta}-\frac{\partial}{\partial\beta}\beta-\frac{\partial}{\partial\beta^*}\beta^*\right)C(\beta,\beta^*,t)\\ &+\frac{2}{\overline{\sigma}_T}\left(\frac{\partial^2}{\partial\beta^*\partial\beta}-\frac{\partial}{\partial\beta}\beta-\frac{\partial}{\partial\beta^*}\beta^*\right)\overline{C}(\beta,\beta^*,t)\,,\end{aligned} \tag{2.4a}$$

$$\tfrac{1}{2}(1+\sigma+S|\beta|^2)P(\beta,\beta^*,t) = \left[1+S\left(|\beta|^2-\frac{1}{4}\beta\frac{\partial}{\partial\beta}-\frac{1}{4}\beta^*\frac{\partial}{\partial\beta^*}\right)\right]C(\beta,\beta^*,t)\,,\quad \sigma_T=\frac{\kappa\gamma_\perp}{N|g|^2}\,,\quad S=\frac{4|g|^2}{\gamma_\perp\gamma_\parallel}\,. \tag{2.4b}$$

The equation for $\overline{C}$ is immediately obtained from Eq. (2.4b) by simply putting a bar on all atomic quantities (i.e., σ, S). On the basis of assumption (i), we have dropped the labels p, q everywhere. The parameter σ_T is the threshold inversion per atom when the laser contains active atoms only, S is the saturation parameter of the active atoms. The quantities $C, \overline{C}$ can be immediately eliminated thereby obtaining the time-evolution equation for the field alone,

$$\begin{aligned}\kappa^{-1}\partial_t P(\beta,\beta^*,t) &= \Lambda P(\beta,\beta^*,t)\,,\\ \Lambda &= \left(1+\frac{1}{\sigma_T}+\frac{1}{\overline{\sigma}_T}\right)\left(\frac{\partial}{\partial\beta}\beta+\frac{\partial}{\partial\beta^*}\beta^*\right)+\Lambda_1+\overline{\Lambda}_1\,,\end{aligned} \tag{2.5}$$

where

$$\Lambda_1 = \frac{1}{\sigma_T}\left(\frac{\partial^2}{\partial\beta^*\partial\beta}-\frac{\partial}{\partial\beta}\beta-\frac{\partial}{\partial\beta^*}\beta^*\right)\left[1+S\left(|\beta|^2-\frac{1}{4}\beta\frac{\partial}{\partial\beta}-\frac{1}{4}\beta^*\frac{\partial}{\partial\beta^*}\right)\right]^{-1}(1+\sigma+S|\beta|^2) \tag{2.6}$$

and $\overline{\Lambda}_1$ is obtained from Λ_1 by putting a bar on all the atomic quantities. Clearly, Eq. (2.5) is not a Fokker-Planck equation, since it contains derivatives of all orders in β, β^* due to the presence of the inverse operator

$$\left[1+S\left(|\beta|^2-\frac{1}{4}\beta\frac{\partial}{\partial\beta}-\frac{1}{4}\beta^*\frac{\partial}{\partial\beta^*}\right)\right]^{-1} \text{ in } \Lambda_1 .$$

In I we neglected the derivative terms in Eq. (2.4b); within such an approximation the field equation (2.5) reduces to a Fokker-Planck equation with first- and second-order derivatives only. In particular, we have calculated the stationary solution of such a Fokker-Planck equation, which is given by [see Eq. (4.13) of I]

$$P^{(\mathrm{FP})}(z')=\mathfrak{N}_{\mathrm{FP}}[e^{-z'}(z'+S^{-1})^{\sigma/S\sigma_T}(z'+\overline{S}^{-1})^{\overline{\sigma}/\overline{S}\overline{\sigma}_T}]^{S\epsilon}, \tag{2.7}$$

where

$$z'=\beta^*\beta, \quad \epsilon=\frac{2S^{-1}}{(1+\sigma)/\sigma_T+(1+\overline{\sigma})/\overline{\sigma}_T}$$

and $\mathfrak{N}_{\mathrm{FP}}$ is a suitable normalization constant. In the present paper, we study the additional effects arising from the terms with derivatives of order higher than second in Eq. (2.5). We shall limit ourselves to the stationary situation. Contrary to I, in the present paper we shall not use normalized coordinates because this is not suitable when one keeps the derivative terms in Eq. (2.4b). However, the connection with the symbols used in I is simply given by

$$z=Sz', \quad a=S/\overline{S}, \quad A=\sigma/\sigma_T, \quad C=1-\overline{\sigma}/\overline{\sigma}_T. \tag{2.8}$$

III. STATIONARY SOLUTION: EXACT TREATMENT

We shall follow the procedure devised in Ref. 2 for the laser with active atoms only. Hence, we start from Eqs. (2.4a), (2.4b) and set $\partial_t P(\beta,\beta^*,t)=0$. Since in the stationary situation $P, C, \overline{C}$ are functions of the modulus of β only, we use $z'=\beta^*\beta$ as the independent variable. Equations (2.4a), (2.4b) become

$$\left(1+\frac{1}{\sigma_T}+\frac{1}{\overline{\sigma}_T}\right)P(z')=\frac{2}{\sigma_T}\left(C(z')-\frac{1}{2}\frac{dC}{dz'}\right)+\frac{2}{\overline{\sigma}_T}\left(\overline{C}(z')-\frac{1}{2}\frac{d\overline{C}}{dz'}\right), \tag{3.1a}$$

$$\tfrac{1}{2}(1+\sigma+Sz')P(z')=\left[C(z')+Sz'\left(C(z')-\frac{1}{2}\frac{dC}{dz'}\right)\right], \tag{3.1b}$$

$$\tfrac{1}{2}(1+\overline{\sigma}+\overline{S}z')P(z')=\left[\overline{C}(z')+\overline{S}z'\left(\overline{C}(z')-\frac{1}{2}\frac{d\overline{C}}{dz'}\right)\right]. \tag{3.1c}$$

By eliminating the derivative terms between Eqs. (3.1a), (3.1b), and (3.1c) finds the relationship

$$2[ZC(z')+\overline{Z}\,\overline{C}(z')]=(H-z')P(z'), \tag{3.2}$$

where

$$Z=\frac{1}{S\sigma_T}=\frac{N\gamma_\parallel}{4\kappa}, \quad H=h+\overline{h}, \quad h=Z(1+\sigma). \tag{3.3}$$

(a) *The case* $S=\overline{S}$. As shown in I, for $S=\overline{S}$ [which corresponds to $a=1$, see Eq. (2.8)] no bistable situation is possible. In this case, our system of equations can be trivially solved. In fact, using (3.3) Eq. (3.10) for $\overline{S}=S$ can be rewritten as follows:

$$\left(1+\frac{1}{\sigma_T}+\frac{1}{\overline{\sigma}_T}\right)P(z')=2S\left(1-\frac{1}{2}\frac{d}{dz'}\right)[ZC(z')+\overline{Z}\,\overline{C}(z')]. \tag{3.4}$$

Hence, with Eq. (3.2) one gets the equation

$$(z'-H)\frac{dP}{dz'}=2(z'-z_+-\tfrac{1}{2})P, \tag{3.5}$$

where z_+ is the semiclassical intensity above threshold for $S=\overline{S}$ [see Eq. (3.24) of I and Eq. (2.8)],

$$z_+=S^{-1}(\sigma/\sigma_T+\overline{\sigma}/\overline{\sigma}_T-1). \tag{3.6}$$

The normalizable solution of Eq. (3.5) is

$$P(z')=\begin{cases}\mathfrak{N}e^{2z'}(H-z')^{2(H-z_+-1/2)} & \text{for } 0\le z'\le H,\\ 0 \quad \text{for } z'\ge H,\end{cases} \tag{3.7}$$

where $\mathfrak{N}$ is the normalization constant. In this case, $P(z')$ has the same structure as the stationary distribution for the laser with active atoms only. Hence, we refer to Ref. 2 for the analysis of distribution (3.7) as well as for the comparison with distribution (2.7) specialized to the case $S=\overline{S}$.

(b) *General case*. Introducing the new quantities

$$\left.\begin{matrix}\tilde{P}(z')\\ \tilde{C}(z')\\ \tilde{\overline{C}}(z')\end{matrix}\right\}=e^{-2z'}\times\left\{\begin{matrix}P(z'),\\ C(z'),\\ \overline{C}(z').\end{matrix}\right. \tag{3.8}$$

Equations (3.1b), (3.1c), and (3.2) become

$$\tfrac{1}{2}(1+\sigma+Sz')\tilde{P}(z')=\left(\tilde{C}(z')-\frac{S}{2}z'\frac{d\tilde{C}}{dz'}\right), \tag{3.9a}$$

$$\tfrac{1}{2}(1+\sigma+\overline{S}z')\tilde{P}(z')=\left(\tilde{\overline{C}}(z')-\frac{\overline{S}}{2}z'\frac{d\tilde{\overline{C}}}{dz'}\right), \tag{3.9b}$$

$$2[Z\tilde{C}(z')+Z\tilde{\overline{C}}(z')]=(H-z')\tilde{P}(z'). \tag{3.9c}$$

The formal solution of Eq. (3.9a) can be written as

$$\tilde{C}(z')=\frac{1}{2}\left(1-\frac{S}{2}z'\frac{d}{dz'}\right)^{-1}(1+\sigma+Sz')\tilde{P}(z') \tag{3.10}$$

and a similar equation for $\tilde{\overline{C}}(z')$ follows from Eq. (3.9b). Substituting these expressions for $\tilde{C}(z')$ and $\tilde{\overline{C}}(z')$ into Eq. (3.9c) one finds

$$\left[\left(1-\frac{S}{2}z'\frac{d}{dz'}\right)^{-1}\left(h+\frac{z'}{\sigma_T}\right)+\left(1-\frac{\bar{S}}{2}z'\frac{d}{dz'}\right)^{-1}\left(\bar{h}+\frac{z'}{\bar{\sigma}_T}\right)\right]\tilde{P}(z')=(H-z')\tilde{P}(z'), \tag{3.11}$$

where definition (3.3) has been used. Finally multiplying both sides of this equation on the left by $(1-\frac{1}{2}Sz'\,d/dz')(1-\frac{1}{2}\bar{S}z'd/dz')$ one finds for $\tilde{P}(z')$ the second-order differential equation

$$z'(H-z')\frac{d^2\tilde{P}}{dz'^2}+\{h(1-2S^{-1})+\bar{h}(1-2\bar{S}^{-1})-z'[3-2Z(1+\sigma_T)-2\bar{Z}(1+\bar{\sigma}_T)]\}\frac{d\tilde{P}}{dz'}$$
$$-[4Z\bar{Z}(1+\sigma_T)(1+\bar{\sigma}_T)+1-2Z(1+\sigma_T)-2\bar{Z}(1+\bar{\sigma}_T)-4Z\bar{Z}]\tilde{P}=0\ . \tag{3.12}$$

Introducing the new variable

$$x=z'/H \tag{3.13}$$

and using the notations

$$\alpha_\pm=1-Z(1+\sigma_T)-\bar{Z}(1+\bar{\sigma}_T)\pm\sqrt{\Delta}\ ,$$
$$\Delta=[Z(1+\sigma_T)-\bar{Z}(1+\bar{\sigma}_T)]^2+4Z\bar{Z}>0\ , \tag{3.14}$$
$$\gamma=1-2(h/H)\bar{Z}\bar{\sigma}_T-2(\bar{h}/H)Z\sigma_T<1\ ,$$

Eq. (3.12) takes the form

$$x(1-x)\frac{d^2\tilde{P}}{dx^2}+[\gamma-x(\alpha_++\alpha_-+1)]\frac{d\tilde{P}}{dx}-\alpha_+\alpha_-\tilde{P}=0\ . \tag{3.15}$$

This is the hypergeometric equation. For $0\le x<1$ the general solution can be written[3]

$$\tilde{P}(x)=C_1F(\alpha_+,\alpha_-,\gamma,x)+C_2x^{1-\gamma}\times F(\alpha_++1-\gamma,\alpha_-+1-\gamma,2-\gamma,x)\ , \tag{3.16}$$

where F is the hypergeometric function and we have taken into account that in our case, as one easily verifies,

$$\alpha_++\alpha_--\gamma<1\ .$$

More precisely for the values of the parameters in the laser-absorber system one has that $\alpha_++\alpha_--\gamma<0$ and $|\alpha_++\alpha_--\gamma|\gg1$, so that (i) the solution (3.16) converges also for $x=1$ and (ii) a suitable large number of derivates of (3.16) converges for $x=1$.

The admissible solution $\tilde{P}(x)$ must be integrable in the following sense:

$$\int_0^\infty e^{2Hx}\tilde{P}(x)\,dx<\infty \tag{3.17}$$

as follows from the normalization condition of the quasiprobability distribution P. No one of the solutions obtained by analytic continuation of (3.16) provides convergence of the integral (3.17). Hence, the only possibility to fulfill condition (3.17) is to look for a solution which vanishes identically for x larger than a suitable value $\bar{x}$ [cf. Eq. (3.7)]. Since (apart from the boundaries $x=0$ and $x=\infty$) $x=1$ is the only possible point of nonanalyticity of the solutions of Eq. (3.15), one has necessarily $\bar{x}=1$. For the sake of continuity of P at $x=1$ one has to impose $\tilde{P}(x=1)=0$ and this can be achieved by a suitable choice of constants C_1 and C_2 when $\gamma\ne0,-1,-2,\ldots$. This leads to the following form of $\tilde{P}$:

$$\tilde{P}(x)=\begin{cases}\mathfrak{N}\left(\dfrac{F(\alpha_+,\alpha_-,\gamma,x)}{\Gamma(\gamma)\Gamma(1-\alpha_+)\Gamma(1-\alpha_-)}-x^{1-\gamma}\dfrac{F(\alpha_++1-\gamma,\alpha_-+1-\gamma,2-\gamma,x)}{\Gamma(2-\gamma)\Gamma(\gamma-\alpha_+)\Gamma(\gamma-\alpha_-)}\right) & \text{for } 0\le x\le1\ ,\\ 0 \text{ for } x\ge1\ . \end{cases} \tag{3.18}$$

One can verify that also $d\tilde{P}/dx$ and $d^2\tilde{P}/dx^2$ are continuous at $x=1$. Finally, the stationary solution is

$$P(z')=e^{2z'}\tilde{P}(z'/H)\ , \tag{3.19}$$

where $\mathfrak{N}$ is determined by the normalization condition

$$\pi\int_0^\infty P(z')dz'=1\ .$$

We notice that when $S=\bar{S}$ from (3.14) one has

$$\alpha_+=1-2S^{-1}=\gamma\ ,$$
$$\alpha_-=1-2S^{-1}-2(Z+\bar{Z})\ .$$

Hence, taking into account that $|\Gamma(0)|=\infty$ and that $F(\alpha_+,\alpha_-,\alpha_+,x)=(1-x)^{-\alpha_-}$ one easily verifies that (3.19) reduces to the previously found distribution (3.7).

IV. STATIONARY SOLUTION: APPROXIMATE TREATMENT FOR $\bar{S}\gg S$

Unfortunately, the solution (3.18), (3.19) does not seem suitable for the numerical computation of the mean values. Therefore, here, we give an approximate treatment which is valid for $\bar{S}\gg S$. To illustrate it, let us come back to the case of the usual laser with active atoms only. In Ref. 2 it is shown that the derivative terms in Eq. (3.1b) is irrelevant in the threshold region. This can be easily understood because dC/dz' is multiplied by the factor Sz'. In fact, this factor is very small for the relevant values of z' in the thresold region. Note that other terms in Eq. (3.1b) contain the factor Sz', but they cannot be neglected because they ensure the saturation of the gain for large z', which is essential for the laser above threshold.[4] On the other hand, when the laser is well above threshold, the derivative term appreciably influences the width of the distribution.

Let us now come back to the case of the laser with saturable absorber. On the basis of the previous discussion, we can say that the derivative terms in Eqs. (3.1b), (3.1c) are negligible when $Sz_+ \ll 1$, $\bar{S}z_+ \ll 1$, respectively, where z_+ is the positive stable semiclassical value of the intensity (which corresponds to the normalized value I_+ of paper I). Now in I we have shown that the bistable situation, which is the interesting one, occurs for $a > C/C - 1$, which in the present notations [see (2.8)] gives

$$\bar{S} > S(1 + \bar{\sigma}_T / |\bar{\sigma}|) . \tag{4.1}$$

Let us consider the case $\bar{S} \gg S$. Then there is a range of values of the pump parameter σ such that $\bar{S}z_+ \gtrsim 1$ but $Sz_+ \ll 1$. In this situation we can safely neglect the term with dC/dz' in Eq. (3.1b). In this section, we shall analyze the problem within this approximation. Clearly, this procedure does not allow us to treat the very high-intensity situation. Hence, this treatment is in a sense analogous to Risken's analysis of the usual laser, which is limited to the threshold region.[5] The interesting feature in the present problem is that even in this region the absorber is saturated, so that one needs a high-intensity treatment for it.

Neglecting the derivative term in Eq. (3.1b) we get

$$C(z') = \frac{1}{2}\,\frac{1+\sigma+Sz'}{1+Sz'}\,P(z') . \tag{4.2}$$

By inserting Eq. (4.2) into Eq. (3.2) we find

$$\bar{C}(z') = \left(\frac{H-z'}{2\bar{Z}} - \frac{h+z'/\sigma_T}{2\bar{Z}(1+Sz')}\right) P(z') . \tag{4.3}$$

Let us now substitute Eqs. (4.2), (4.3) into Eq. (3.1a). Assuming that $S \ll 1$, $H \gg 1$, $h \gg \sigma_T^{-1}$ (as it occurs in the usual lasers) one has

$$\left.\begin{array}{l} \left|\dfrac{d}{dz'}\,\dfrac{1+\sigma+Sz'}{1+Sz'}\right| \\[2ex] \left|\dfrac{d}{dz'}\,\dfrac{H-z'}{2\bar{Z}}\right| \\[2ex] \left|\dfrac{d}{dz'}\,\dfrac{h+z'/\sigma_T}{2\bar{Z}(1+Sz')}\right| \end{array}\right\} \ll \frac{1+\sigma+Sz'}{1+Sz'},\ \left|\frac{H-z'}{2\bar{Z}}\right|$$

so that we neglect the terms containing these derivates and obtain

$$\frac{dP}{dz'}\left(\frac{1}{2\sigma_T}\,\frac{1+\sigma+Sz'}{1+Sz'} - \frac{\bar{S}}{2}\,\frac{h+z'/\sigma_T}{(1+Sz')} + \frac{\bar{S}}{2}(H-z')\right) + P\left(1+\frac{1}{\sigma_T}+\frac{1}{\bar{\sigma}_T} - \frac{1}{\sigma_T}\,\frac{1+\sigma+Sz'}{1+Sz'} - \bar{S}(H-z') + \bar{S}\,\frac{h+z'/\sigma_T}{1+Sz'}\right) . \tag{4.4}$$

Finally, by performing some trivial but lengthy algebraic manipulations one gets

$$\begin{aligned} y_2(z')\,\frac{dP}{dz'} &= y_1(z')P(z') , \\ y_1(z') &= z'^2 - z'[Z(\sigma-\sigma_T) + \bar{Z}(\bar{\sigma}-\bar{\sigma}_T)] + (S\bar{S})^{-1}\left(1 - \frac{\sigma}{\sigma_T} - \frac{\bar{\sigma}}{\bar{\sigma}_T}\right) , \\ y_2(z') &= z'^2 - z'\left[Z(\sigma-\sigma_T) + \bar{Z}\left(1+\bar{\sigma}+\frac{\sigma}{\sigma_T}\right)\right] - (S\bar{S})^{-1}\left(\frac{1+\sigma}{\sigma_T} + \frac{1+\bar{\sigma}}{\bar{\sigma}_T}\right) . \end{aligned} \tag{4.5}$$

Equation $y_2(z') = 0$ has two real solutions $z' = n_1$, $z' = -n_2$ with $n_1, n_2 > 0$. Hence, the normalizable solution of Eq. (4.5) is

$$P(z') = \begin{cases} \mathfrak{N} e^{2z'}(n_1 - z')^{2L}(n_2 + z')^{2M} & \text{for } 0 \le z' \le n_1 , \\ 0 & \text{for } z' \ge n_1 , \end{cases} \tag{4.6}$$

where $\mathfrak{N}$ is the normalization constant and

$$\begin{aligned} L &= \left[(S\bar{S})^{-1}\left(1+\frac{1}{\sigma_T}+\frac{1}{\bar{\sigma}_T}\right) + n_1\bar{Z}\left(1+\bar{\sigma}_T+\frac{\bar{\sigma}_T}{\sigma_T}\right)\right](n_1+n_2)^{-1} , \\ M &= \left[n_2\bar{Z}\left(1+\bar{\sigma}_T+\frac{\sigma}{\sigma_T}\right) - (S\bar{S})^{-1}\left(1+\frac{1}{\sigma_T}+\frac{1}{\bar{\sigma}_T}\right)\right](n_1+n_2)^{-1} . \end{aligned} \tag{4.7}$$

Contrary to (2.7), the distribution (4.6) vanishes from $z'=n_1$ onwards. Distributions (2.7), (4.6) have the same behavior in the neighborhood of the origin $z'=0$, because they give

$$\left(\frac{dP/dz}{P}\right)_{z'=0} = -2\,\frac{1-\sigma/\sigma_T-\bar{\sigma}/\bar{\sigma}_T}{(1+\sigma)/\sigma_T+(1+\bar{\sigma})/\bar{\sigma}_T}\,. \quad (4.8)$$

The (nonzero) extrema of $P(z)$ are given by the equation $y_1(z')=0$. This equation can be rewritten as follows:

$$1-\frac{\sigma}{\sigma_T}\,\frac{1}{1+Sz'}-\frac{\bar{\sigma}}{\bar{\sigma}_T}\,\frac{1}{1+\bar{S}z'}=0\,, \quad (4.9)$$

which is the semiclassical stationary equation for the intensity [see Eq. (3.18) of I and (2.8)]. Hence, both distributions (2.7), (4.6) have a minimum at $z'=z_-$ and a maximum at $z'=z_+$, where $z_\pm$ are the nonzero semiclassical stationary values of the intensity. Note in this connection that $n_1>z_+,z_-$ because one easily verifies that $y_1(z')>y_2(z')$ always holds. Clearly, the relative heights and the widths of the two peaks at $z'=0$, $z'=z_+$ in the bistable region is in general different in the two distributions (2.7), (4.6). A numerical comparison of the two distributions is shown in the next section.

V. COMPARISON

By means of Eq. (4.6) one can evaluate the moments of the photon distribution; e.g., the first two moments are given by

$$\langle n\rangle=\pi\int_0^\infty dz'\,P(z')z'\,,$$
$$\langle n^2\rangle=\pi\int_0^\infty dz'\,P(z')(z'^2+z')\,, \quad (5.1)$$

where n is the number-of-photons operator. We have numerically evaluated the quantities (5.1) as functions of σ for the values of the parameters

$$S=10^{-6},\quad \bar{S}=10^{-1},\quad \sigma_T=\bar{\sigma}_T=10^{-3}\,,\quad \bar{\sigma}=-0.5 \quad (5.2)$$

and we have compared the results with the corresponding quantities calculated (numerically) with distribution (2.7) obtained in the Fokker-Planck approximation.

From Eq. (2.8) we see that the parameters (5.2) correspond to

$$a=10^5,\quad C\simeq 500,\quad \epsilon\sim 10^3\ .$$

Hence, the present situation is drastically different from that considered in I, in which we had $a=\frac{4}{3}$, $C=20$, $\epsilon=100$. Firstly, in the present case the hysteresis cycle is much wider, since it ranges from $A\simeq 1.146$ to $A\simeq 500$ [see Eqs. (3.33) and (3.34) of I]. Secondly, the value of the parameter a is now much larger. As we shall see, these features have the effect that the transition region turns out to be very near the left-hand boundary of the hysteresis cycle (i.e., it corresponds to values of A only slightly larger than 1.146). Table I shows the mean photon number $\langle n\rangle$ and the mean-square fluctuation $\langle n^2\rangle-\langle n\rangle^2$ as a function of A in the transition region, where $\langle n\rangle$ and $\langle n^2\rangle$ are calculated by means of distribution (4.6). Table II shows the same quantities, but $\langle n\rangle$ and $\langle n^2\rangle$ are calculated by means of distribution (2.7). Note that, in Table I $S\langle n\rangle$ remains small for all the values of σ considered in the table, whereas $\bar{S}\langle n\rangle$ is large for $\bar{\sigma}\gtrsim 1.172\,39\sigma_T$. The most striking feature that appears from the comparison of the two tables is that the transition is much sharper in the case of distribution (4.6). Hence this sharpening is the main effect of the terms with derivatives in β,β^* of order higher than second in Eq. (2.5). A minor effect is that the threshold region is slightly shifted towards

TABLE I. Mean photon number and intensity fluctuation for distribution (4.6).

σ/σ_T	$\langle n\rangle$	$\langle n^2\rangle-\langle n\rangle^2$
1.172 365	1.5019	4.06
1.172 37	1.5020	1.35×10^1
1.172 375	1.5044	3.21×10^2
1.172 38	1.5824	1.03×10^4
1.172 385	4.1250	3.37×10^5
1.172 39	8.6962×10	1.10×10^7
1.172 395	2.7309×10^3	3.43×10^8
1.172 4	5.3260×10^4	4.01×10^9
1.172 405	1.2316×10^5	6.57×10^8
1.172 41	1.2833×10^5	2.21×10^7
1.172 415	1.2850×10^5	9.25×10^5
1.172 42	1.2851×10^5	2.75×10^5
1.172 425	1.2852×10^5	2.55×10^5

TABLE II. Mean photon number and intensity fluctuation for distribution (2.7).

σ/σ_T	$\langle n\rangle$	$\langle n^2\rangle-\langle n\rangle^2$
1.275	7.40	1.33×10^6
1.28	2.62×10^1	6.24×10^6
1.285	1.15×10^2	3.00×10^7
1.29	5.50×10^2	1.48×10^8
1.295	2.71×10^3	7.41×10^8
1.3	1.34×10^4	3.58×10^9
1.305	5.87×10^4	1.34×10^{10}
1.31	1.68×10^5	2.08×10^{10}
1.315	2.59×10^5	9.75×10^9
1.32	2.92×10^5	3.00×10^9

smaller values of A in the case of distribution (4.6).

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