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Hartree-Fock states in the thermodynamic limit. III. Low-density clustering effects

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We present an orthonormal set of single-particle orbital functions explicitly satisfying the Hartree-Fock (HF) equations for occupied states, and which give low-density few-particle clustering effects in a many-fermion system. The initial study given here is in one and three dimensions for attractive δ pair interactions. In one dimension the numerical results compare favorably with the exact (lowest) HF state at zero density. The results turn out to be superior to the classic Overhauser HF orbitals, as well as to some more recent ones.

I. INTRODUCTION

In a previous set of two papers [hereafter referred to as I (Ref. 1) and II (Ref. 2)] we presented detailed discussions of several sets of nonplane-wave (i.e., nontrivial) solutions, for occupied states, to the (matrix) Hartree-Fock (HF) equations of a many-fermion system, interacting via repulsive or attractive δ pair interactions. One of the main emphases in both papers was the appearance of bifurcation points, in the energy versus coupling constant plane, beyond which the nontrivial state was stabler than the plane-wave (PW) one, as well as on the nature of the transition (first or second order). II dealt with a generalization of the orbitals in I, in the sense that an arbitrary number of harmonics could be incorporated in each orbital—and hence into the local density—without, however, introducing arbitrarily many variational parameters. The (determinantal) HF states thus constructed gave considerably lower energy than the PW states, particularly for small coupling and/or low density, but without ever reaching zero coupling and/or density as, e.g., in the Cooper pair problem.[3]

In the present paper we report a new set of orbitals which, in addition to being orthonormal, and explicitly satisfying the (matrix) HF equations for occupied states, allow for analytical HF energy expressions (in terms of modified Bessel functions in the variational order parameter). These new orbitals give lower energy than all previous ones known from the literature, for an attractive δ pair interaction, *starting at zero coupling and/or density*. Section II contains a study of the one-dimensional problem, where exact results are known both for the HF (Ref. 4) as well as the Schrödinger[5] equations, for zero density. Section III deals with the three-dimensional case which is qualitatively very different. Throughout, we compare with the classical Overhauser results[6] and, for the three-dimensional case, with the results from the generalized Overhauser orbitals introduced in paper II.

Although a one-dimensional many-fermion system with attractive δ interactions is *stable*,[5] the three-dimensional one is *not*, i.e., it collapses to an infinite density, negatively infinite energy per particle state, as has been known for a long time, just from the Rayleigh-Ritz variational principle (which makes the PW results a rigorous upper bound). The PW result collapses at *high* density; our orbitals give collapse at *all* densities. They further yield interesting results in noncollapsing problems, as seen in the one-dimensional case where 95% of the exact (lowest) HF energy is reproduced at zero density.

The main purpose of this paper, however, is not to report on the study of any particular *real* physical system, but to *initiate* such studies by examining first, as thoroughly as possible, the (admittedly academic) attractive δ-function fermion gas. This was done by comparing a rather wide variety of Hartree-Fock states for that N-body system. An eventual objective of our work is not only in (i) finding out how *reasonable* the Hartree-Fock approximation is in describing a given phenomenon *quantitatively*, but also (and this is perhaps much more important) in (ii) deciding, on the basis of whether the HF scheme provides a reasonable *qualitative* description, if the best HF state can serve as an (unperturbed) vacuum state for a meaningful diagrammatic perturbation theory of correlation effects.

The motivation for the present study, of course, is in eventual applications to finite-range interactions for the study of several physical problems, e.g., (i) α-clustering effects in nuclear matter,[7] (ii) neutron matter crystallization,[8] (iii) ground-state fluid-to-crystal transitions in ^{3}He and ^{4}He systems,[9] (iv) the electron (Wigner) lattice and its magnetic properties.[10] Unfortunately, most of these goals are as yet somewhat remote since one must yet solve several technical problems involved in evaluating the pertinent matrix elements.

II. ONE-DIMENSIONAL ATTRACTIVE δ FERMION GAS

We consider a system of $N \gg 1$ fermions (spin $\frac{1}{2}$) in one dimension, placed inside a "box" of length L, and apply periodic boundary conditions. We shall examine the problem successively with plane wave, Overhauser, so-called "exponential," and finally, a mixture, Overhauser-exponential single-particle, orthonormal orbitals, all of which satisfy the (matrix) HF equations for orbitals occupied in the determinant

$$\Phi_0 = (N!)^{-1/2} \det[\varphi_{k_i \sigma_i}(x_j, \sigma_{3j})] . \tag{1}$$

Except for the PW case, long-range order, i.e., perfect periodicity, appears, in the single-particle local density

$$\rho(x) = \sum_{k_i \sigma_i (\text{occ})} |\varphi_{k_i \sigma_i}(x, \sigma_{3i})|^2 \tag{2}$$

which, for the PW case just gives

$$\rho(x) = \rho_0 = N/L = 2k_0/\pi , \tag{3}$$

where $2k_0$ is the length of the Fermi sea.

A. Plane-wave and Overhauser orbitals

Overhauser[6] introduced the ortnonormal orbitals

$$\begin{aligned} &\varphi_{k,\sigma}(x,\sigma_3) = L^{-1/2}(u_k + v_u e^{-iqx}) e^{ikx} \chi_\sigma(\sigma_3) , \\ &u_k^2 + v_k^2 = 1, \quad -k_0 < k < k_0 , \\ &|q| \geq 2k_0 , \quad \operatorname{sgn} k = \operatorname{sgn} q \end{aligned} \tag{4}$$

which give, by Eq. (2),

$$\rho(x) = \rho_0(1 + \Delta \cos qx) , \tag{5a}$$

$$\Delta \equiv \frac{4}{N} \sum_{k (\text{occ})} u_k v_k , \quad (0 \leq \Delta \leq 1) . \tag{5b}$$

The HF kinetic energy [expectation value with Eq. (1)] then becomes

$$\begin{aligned} &\frac{\langle T \rangle}{N} = \tfrac{1}{3}\epsilon_{k_0} + \epsilon_q \frac{2}{N} \sum_k \left(1 - \frac{2k}{q}\right) v_k^2 , \\ &\epsilon_k \equiv \hbar^2 k^2 / 2m . \end{aligned} \tag{6}$$

For the interaction of present interest

$$\begin{aligned} &v = \sum_{i<j}^{N} v_{ij} , \\ &v_{ij} = -v_0 \delta(x_i - x_j), \quad v_0 > 0 \end{aligned} \tag{7}$$

the HF potential energy is

$$\frac{\langle v \rangle}{N} = -\frac{v_0}{4} \int_{-L/2}^{L/2} \rho^2(x)\, dx = -\frac{v_0 \rho_0}{4}(1 + \tfrac{1}{2}\Delta^2) . \tag{8}$$

The HF energy

$$E/N = (\langle T \rangle + \langle V \rangle)/N \tag{9}$$

from Eqs. (6) and (8) is clearly minimum in q, subject to the restrictions in Eq. (4), for $|q| = 2k_0$. The only remaining variation is that in v_k, namely,

$$\frac{\partial(E/N)}{\partial v_e} = 0 , \tag{10}$$

which leads to a quadratic equation in v_k^2, with solution

$$v_k^2 = \frac{1}{2}\left(1 - \frac{(1 - |k|/k_0)}{[(1 - |k|/k_0)^2 + \gamma^2 \Delta^2]^{1/2}}\right) , \tag{11}$$

$$\gamma \equiv m v_0 / 2\pi \hbar^2 k_0 , \tag{12}$$

and a similar expression for u_k^2, but with the first negative sign replaced by a plus. Using Eq. (5b), converting the sum

$$\sum_{k (\text{occ})}$$

to

$$\frac{L}{2\pi} \int_{-k_0}^{k_0} dk$$

(valid for large L), and doing a simple integral, leads to

$$\Delta = (\gamma \sinh 1/\gamma)^{-1} , \tag{13}$$

which is the "order parameter," tending to zero for $\gamma \to 0$, and to unity for $\gamma \to \infty$. After another simple integral, and some manipulations, the energy gain per particle becomes the nonpositive (for all γ) quantity

$$(E - E_{\text{PW}})/N = \epsilon_{k_0}[1 - \coth(1/\gamma)] \leq 0 , \tag{14}$$

with equality only for $\gamma = 0$, and where E_{PW} is obtained from the above equations by simply putting $v_k = 0$. Note the striking similarity of Eqs. (5b), (11), and (13) to the corresponding ones in the BCS problem.[3] However, the interaction here is of a much more general character than the Cooper pairing force, which only scatters particles into very special states.

B. Exponential orbitals

We now introduce the following HF orthonormal orbitals

$$\begin{aligned} &\varphi_{k\sigma}(x, \sigma_3) = [L I_0(2\alpha)]^{-1/2} e^{ikx} e^{\alpha \cos qx} \chi_\sigma(\sigma_3) ; \\ &-k_0 \leq k \leq k_0 , \quad q \geq 2k_0 , \quad \alpha \geq 0 , \end{aligned} \tag{15}$$

$$\begin{aligned} I_n(y) &\equiv \sum_{s=0}^{\infty} [s!(s+n)!]^{-1} (\tfrac{1}{2}y)^{2s+n} \\ &= (-)^n I_n(-y) \\ &\underset{y \to 0}{\sim} (n!)^{-1} (\tfrac{1}{2}y)^n \\ &\underset{y \to \infty}{\sim} [2\pi y]^{-1/2} e^y , \end{aligned} \tag{16}$$

where the latter are the modified Bessel functions.[11] The single-particle local density is

$$\rho(x) = \rho_0 I_0^{-1}(2\alpha)\, e^{2\alpha \cos qx} \tag{17}$$

and tends, as $\alpha \to \infty$, to a "classical static lattice," i.e., to a lattice of Dirac δ functions each centered on each lattice point. The HF energy per particle, for Eq. (7), is then, as again the minimum in q is for $q = 2k_0$, just

$$\frac{E}{N} = \tfrac{1}{3}\epsilon_{k_0}\left(1 + 6\alpha \frac{I_1(2\alpha)}{I_0(2\alpha)}\right) - \frac{v_0\rho_0}{4}\,\frac{I_0(4\alpha)}{I_0^2(2\alpha)}\,. \tag{18}$$

C. Mixed exponential-Overhauser orbitals

It is of some interest to consider the "mixed" HF orthornormal orbitals

$$\varphi_{k\sigma}(x,\sigma_3) = [L I_0(2\alpha)]^{-1/2} e^{ikx} e^{\alpha \cos 2qx} \times (u_k + v_k e^{-iqx})\,;$$

$$u_k^2 + v_k^2 = 1, \quad \operatorname{sgn} k = \operatorname{sgn} q, \quad |q| \geq 2k_0, \quad \alpha \geq 0 \tag{19}$$

which lead to a local density given by

$$\rho(x) = \rho_0 I_0^{-1}(2\alpha)\, e^{2\alpha \cos 2qx}(1 + \Delta \cos qx)\,, \tag{20}$$

with Δ given by Eq. (5b). Note that the periods of the two original orbitals differ by a factor of 2. This is needed in order to obtain analytical expressions, at the cost of considerably increasing the kinetic energy. Using the recurrence relation[11]

$$\frac{dI_1(y)}{dy} = I_0(y) - \frac{1}{y} I_1(y) \tag{21}$$

and recognizing that, again, the HF energy is minimum in q at $|q| = 2k_0$, one has

$$\frac{E}{N} = \tfrac{1}{3}\epsilon_{k_0}\left[1 + 24\alpha \frac{I_1(2\alpha)}{I_0(2\alpha)} + \frac{12}{N}\sum_{k\,(\mathrm{occ})}\left(1 - \frac{k}{k_0}\right) v_k^2\right] - \frac{v_0\rho_0}{4}\left(\frac{I_0(4\alpha)}{I_0^2(2\alpha)} + \tfrac{1}{4}F(4\alpha)\Delta^2\right), \tag{22}$$

$$F(y) = 2[I_0(y) + I_1(y)] I_0^{-1}\left(\frac{y}{2}\right) \xrightarrow[y \to 0]{} 2\,. \tag{23}$$

For fixed α (to be varied later, numerically) the variation in v_k, Eq. (10), leads to [compare Eq. (13)]

$$\Delta = \{\tfrac{1}{2}\gamma F(4\alpha) \sinh[2/\gamma F(4\alpha)]\}^{-1}. \tag{24}$$

After another integral, the HF energy becomes

$$\frac{E}{N} = \tfrac{1}{3}\epsilon_{k_0}\left\{1 + 24\alpha \frac{I_1(2\alpha)}{I_0(2\alpha)} - 6\gamma \frac{I_0(4\alpha)}{I_0^2(2\alpha)} + 3\left[1 - \coth\left(\frac{2}{\gamma F(4\alpha)}\right)\right]\right\}. \tag{25}$$

D. Exact results

Lieb and one of us (M. de Ll.) have recently established[4] that the lowest HF state for the problem with N even (not necessarily infinite) in an infinite volume, i.e., at *zero* density, consists of a gas of (soliton) "dimers" of spin-up and spin-down particles, and that the energy per particle is

$$E^{\mathrm{HF}}/N = -m v_0^2/24\hbar^2 \quad (\rho_0 = 0)\,. \tag{26}$$

This is *just the character of the exact solution* to the Schrödinger equation as found by Yang,[5] who obtained an energy per particle, also in an infinite volume, of

$$E^{\mathrm{exact}}/N = -m v_0^2/8\hbar^2 \quad (\rho_0 = 0)\,. \tag{27}$$

Note that $E^{\mathrm{HF}}/E^{\mathrm{exact}} = \frac{1}{3}$; subtracting off the spurious center-of-mass effect from the total Hamiltonian, however, *improves* this to $\frac{2}{3}$.

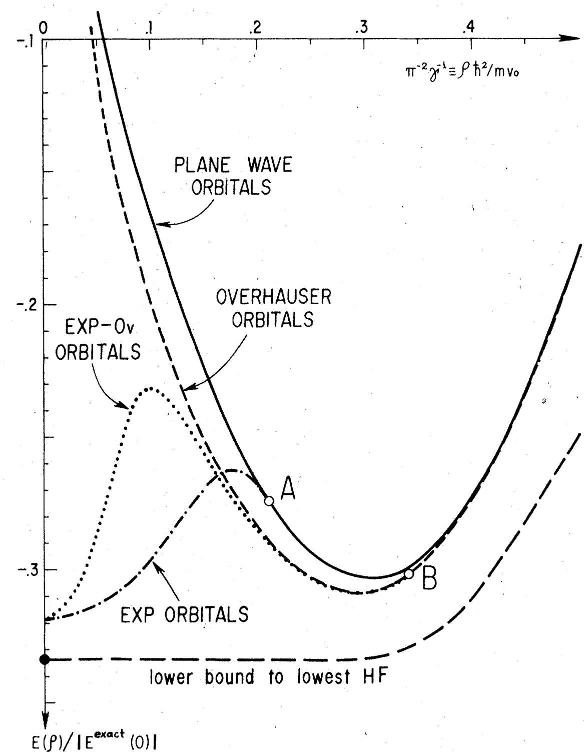

FIG. 1. Energies vs inverse coupling $\pi^{-2}\gamma^{-1} \equiv \rho_0\hbar^2/mv_0$ in units of the exact ground-state energy Eq. (27) at zero density ρ_0, for the various states discussed in this paper for the one-dimensional two-component (spin-$\frac{1}{2}$) N-fermion system with attractive pair δ interactions of strength v_0. Lowest curve (long dashes), a lower bound, Eq. (28), to the lowest HF energy for all densities. Full curve, plane-wave result; short dashed curve, the classic Overhauser result implied by Eq. (14). A and B are bifurcation points, respectively, for the "exponential orbitals" HF energy, and for the "mixed exponential-Overhauser" orbitals HF energy, obtained by minimizing in Eqs. (18) and (25), respectively for each value of the coupling.

E. Lower bound to HF for all ρ_0

Also in Ref. 4 a simple, but rigorous, lower bound to the lowest HF energy *for all densities* is deduced. It is

$$\frac{E^{\mathrm{HF}}(\rho_0)}{N} \geq \frac{\pi^2\hbar^2\rho_0^2}{24m} - \frac{\pi v_0\rho_0}{12} \quad \left(\frac{\rho_0\hbar^2}{mv_0} \geq \frac{1}{\pi}\right)$$

$$\geq -\frac{mv_0^2}{24\hbar^2} \quad \left(\frac{\rho_0\hbar^2}{mv_0} \leq \frac{1}{\pi}\right). \tag{28}$$

F. Results

In Fig. 1 are shown the HF energies for the different orbitals, in units of the exact energy, Eq. (27), as functions of inverse coupling. The HF energies corresponding to the "exponential" and the "exponential-Overhauser" orbitals, Eqs. (18) and (25), were minimized in $\alpha > 0$ numerically. Both energies bifurcate from points A and B, respectively, in Fig. 1, and approach, as density decreases and/or coupling increases ($\gamma \to 0$), the value 0.317, i.e., 95% of the lowest HF energy. Furthermore, a log-log plot of α vs $\pi^{-2}\gamma^{-1} \equiv \rho_0\hbar^2/mv_0 < 0.1$, shows $\rho_0\hbar^2/mv_0 \to |\mathrm{const}|\alpha^{-0.496}$. Thus, as $\rho_0/v_0 \to 0$, α must diverge and, because from Eq. (17),

$$\rho(0) = \rho_0 e^{2\alpha} I_0^{-1}(2\alpha) \underset{\alpha \gg 1}{\longrightarrow} \rho_0\sqrt{2\pi\alpha}$$

$$\to |\mathrm{const}|\alpha^{0.004},$$

one tends to the "classical static lattice" alluded to before.

III. THREE-DIMENSIONAL ATTRACTIVE δ FERMION GAS

We now consider a three-dimensional *four*-species N-fermion system (in order to compare with previous results[2] dealing with nuclear matter) with attractive δ pair interactions. Each HF orbital is now a product of x, y, and z orbitals of Eqs. (4) and (15) of Sec. II [except that previous $\chi_\sigma(\sigma_3)$ is now $\chi_\sigma^{1/3}(\sigma_3)$, for obvious reasons].

A. Plane-wave and Overhauser orbitals

The single-particle density is now

$$\rho(x) = \rho_0(1 + \Delta\cos qx)(1 + \Delta\cos qy) \times (1 + \Delta\cos qz), \tag{29}$$

where the plane-wave local density is

$$\rho_0 \equiv 4(k_0/\pi)^3, \tag{30}$$

and the "order parameter"

$$\Delta \equiv 2\left(\frac{4}{N}\right)^{1/3} \sum_{k\,(\mathrm{occ})}{}' u_k v_k \quad (0 \leq \Delta \leq 1). \tag{31}$$

Defining dimensionless energies and coupling constant by

$$\epsilon \equiv 2mE/N\hbar^2 k_0^2, \tag{32}$$

$$\lambda \equiv 3mv_0k_0/\hbar^2\pi^3, \tag{33}$$

the energy difference between the Overhauser and the PW HF states is then

$$\Delta\epsilon \equiv \epsilon - \epsilon_{\mathrm{PW}} = 3\left(\frac{q}{k_0}\right)^2 \left(\frac{4}{N}\right)^{1/3} \sum_{k\,(\mathrm{occ})} \left(1 - \frac{2k}{q}\right) v_k^2 - \lambda\left[\left(1 + \frac{1}{2}\Delta^2\right)^3 - 1\right], \tag{34}$$

which is minimum in $|q| \geq 2k_0$ at $|q| = 2k_0$. The variation Eq. (10) leads to

$$v_k^2 = \frac{1}{2}\left(1 - \frac{(1 - |k|/k_0)}{[(1 - |k|/k_0)^2 + \frac{1}{4}\lambda^2\Delta^2(1 + \frac{1}{2}\Delta^2)^4]^{1/2}}\right) \tag{35}$$

and, for u_k^2, a similar expression but with the first minus sign replaced by a plus. Substituting into Eq. (31), converting the sum to an integral as before, one finally gets the *implicit* equation

$$\Delta = \{\tfrac{1}{2}\lambda(1 + \tfrac{1}{2}\Delta^2)^2 \times \sinh[(\tfrac{1}{2}\lambda)^{-1}(1 + \tfrac{1}{2}\Delta^2)^{-2}]\}^{-1}, \tag{36}$$

which again vanishes as $\lambda \to 0$ and tends to unity as $\lambda \to \infty$. After another integral, the energy difference Eq. (34) becomes

$$\Delta\epsilon = 3\{1 - \coth[(\tfrac{1}{2}\lambda)^{-1}(1 + \tfrac{1}{2}\Delta^2)^{-2}] + \tfrac{1}{2}\lambda(1 + \tfrac{1}{2}\Delta^2)^2\Delta^2\} - \lambda[(1 + \tfrac{1}{2}\Delta^2)^3 - 1]$$

$$\underset{\lambda \to 0}{\sim} -6e^{-4/\lambda}$$

$$\underset{\lambda \gg 1}{\sim} -\tfrac{5}{4}\lambda \tag{37}$$

and is, in fact, nonpositive for all $\lambda \geq 0$. [Note: for repulsive δ interactions, the antiferromagnetic des Cloiseaux[12] state, i.e., Overhauser orbitals with the v_k coefficients multiplied by a factor $s_\sigma = +1$ (spin up) and $= -1$ (spin down), leads directly to Eqs. (36) and (37), but with the signs of $\frac{1}{2}\Delta^2$ changed and the $-\lambda$ in the potential energy difference replaced by $+\lambda$.]

B. Exponential orbitals

The generalization to three dimensions here is almost trivial, and leads to the energy difference, with respect to the plane-wave value,

$$\Delta\epsilon \equiv \epsilon(\alpha^2) - \epsilon(0) = 6\alpha\frac{I_1(2\alpha)}{I_0(2\alpha)} - \lambda\left(\frac{I_0^3(4\alpha)}{I_0^6(2\alpha)} - 1\right). \tag{38}$$

A numerical minimization in α of this expression was attempted and we found that *no finite, minim-*

izing α *exists*, i.e. $\Delta\epsilon \to -\infty$ for $\alpha \to \infty$, suggesting an "infimum" in α. In fact, from Eqs. (16) and (38),

$$\Delta\epsilon \underset{\alpha \gg 1}{\sim} 6\alpha - \lambda\left[(2\pi\alpha)^{3/2} - 1\right], \tag{39}$$

so that $\Delta\epsilon < 0$ if

$$\lambda > \left[6/(2\pi)^{3/2}\right](1/\sqrt{\alpha}\,) \underset{\alpha \gg 1}{\sim} 0^{+}. \tag{40}$$

Furthermore, also for $\alpha \to \infty$,

$$\begin{aligned} \frac{\partial \epsilon(\alpha^2)}{\partial(\alpha^2)} &\sim -\tfrac{3}{4}\lambda(2\pi)^{3/2}\frac{1}{\sqrt{\alpha}} \to 0^{-}, \\ \frac{\partial^2 \epsilon(\alpha^2)}{\partial(\alpha^2)^2} &\sim \tfrac{3}{16}\lambda(2\pi)^{3/2}\alpha^{-5/2} \to 0^{+}, \end{aligned} \tag{41}$$

as occurs, say, for $f(\alpha) = e^{-\alpha}$ (whose infimum is zero). In other words, the HF energy for the exponential orbitals is arbitrarily large and negative *for all positive values of the coupling constant* as defined by Eq. (33). The significance of this fact will be appreciated, of course, when employing pair interactions which do *not* lead to collapse as the three-dimensional attractive δ used here.

C. Results

In Fig. 2 we compare the energy difference $\Delta\epsilon$ with respect to PW versus coupling λ, for several non-PW HF orbitals. The collapse for all λ for the exponential orbitals is indicated by the vertical (dot-dash) line coinciding with the negative ordinate axis. The (full) curves marked $DW\text{-}n$ ($n = 1, 2,$ 3, 10, 50) refer to orbitals, discussed in Ref. 2, where the function modulating the PW is $(1 + \alpha e^{-iqx})^n$, where α = const *independent* of k. Those marked $DSW\text{-}n$ refer to a factor $(1 + \alpha \cos qx)^n$ instead. On extrapolating[2] to $n \to \infty$, however, the bifurcation points (open circles on the abscissa) tend to accumulate at a *finite* nonzero value of λ [which turned out to be -1.26 for the first ($DW\text{-}n$) and -0.08 for the second ($DSW\text{-}n$), family]. The Overhauser energy difference (dashed curve), like the exponential orbitals HF state, also has its bifurcation point at $\lambda = 0$, and was obtained by numerically eliminating Δ between Eqs. (36) and (37). Its energy is always lower than for the DW-1 state, consistent with the fact that the Overhauser orbitals effectively allow α to depend on k. Its energy gain over plane wave, however, is quite modest compared with other states, except for weak coupling and/or density, where the energy gain is exponentially small anyway—exception made of the "exponential orbitals."

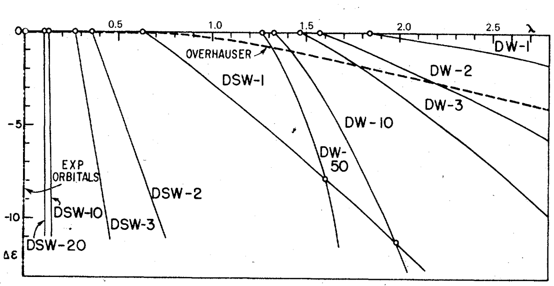

FIG. 2. Energy difference $\Delta\epsilon$ with respect to the plane-wave HF state vs coupling λ as defined by Eq. (33), of the various HF states discussed herein for the three-dimensional four-component (spin-$\frac{1}{2}$, isospin-$\frac{1}{2}$) N-fermion system with attractive pair δ interactions. Full lines refer to states dealt with in paper II, which, however, *all* have a finite, nonzero, value of λ at their bifurcation points (open circles). Dashed line, the Overhauser HF energy, Eqs. (36) and (37), with its bifurcation point at $\lambda = 0$. The dot-dashed line, coinciding with the negative ordinate axis, is the result obtained for the "exponential orbitals."

IV. DISCUSSION

We have introduced a set of orthonormal Hartree-Fock orbitals, called "exponential orbitals," which when analyzed in both one- and three dimensional N-fermion system with attractive δ pair interactions, have considerably lower energy than previous non-PW HF orbitals, including the classic Overhauser ones. In one dimension, for which exact Schrödinger results, as well as exact (lowest) HF results are known, *but only for zero particle density*, the "exponential orbitals," applicable for *all* densities, give 95% of the lowest HF energy at zero density. In three dimensions, although the system collapses, the new orbitals collapse "infinitely faster" (in coupling and/or density) than the plane-wave HF state, indicating a tendency to form condensed dimers, tetramers, etc., according to the number of intrinsic degrees of freedom associated with each fermion.

As immediate possible applications, though, of course, with more realistic and noncollapsing finite-range interactions, we mention a possible "dimerization" of spin-up and spin-down electrons in the Wigner lattice, as suggested, e.g., in Ref. 13, and the "tetramerization" associated with α-clustering effects[7] in nuclear matter. Both these phenomena occur, if at all, at low densities.

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