PHYSICAL REVIEW A VOLUME 17, NUMBER 2 FEBRUARY 1978

d-dimensional turbulence

Jean-Daniel Fournier and Uriel Frisch
Observatoire de Nice, 06-Nice, France
(Received 8 September 1977)

d-dimensional homogeneous isotropic incompressible turbulence is defined, for arbitrary nonintegral d, by analytically continuing the Taylor expansion in time of the energy spectrum $E_k(t)$, assuming Gaussian initial conditions. If $d < 2$, the positivity of the energy spectrum is not necessarily preserved in time. For $d \geq 2$ all steady-state and initial-value calculations have been made with a realizable second-order closure, the eddy-damped quasinormal Markovian approximation. Near two dimensions the enstrophy (mean square vorticity) conservation law is weakly broken, enough to allow ultraviolet singularities to develop in a finite time but not enough to prevent energy from cascading in the infrared direction. A systematic investigation is made of zero-transfer (inertial) steady-state scaling solutions $E_k \propto k^{-m}$ and of their stability. Energy-inertial solutions with $m = 5/3$ exist for arbitrary d; the direction of the energy cascade reverses at $d = d_c \simeq 2.05$. For $d < d'_c \simeq 2.06$ there are in addition, as in the cascade model studied by Bell and Nelkin, inertial solutions with zero energy flux; their exponents $m(d)$ are given by a roughly parabolic curve in the (m,d) plane, linking enstrophy cascade ($m = 3$, $d = 2$) to enstrophy equipartition ($m = 1$, $d = 2$). For any point in the (m,d) plane such that the transfer integral is finite and negative, a steady-state scaling solution $E_k \propto k^{-m}$ is obtained when the fluid is subject to random forces with spectrum $F_k \propto k^{3(m-1)/2}$. A special case is the "model B" $[m = -1+\frac{2}{3}\epsilon + O(\epsilon^2),\ d = 4-\epsilon]$ obtained by Forster, Nelson, and Stephen using a dynamical renormalization-group procedure. Forced steady-state solutions are actually not resticted to the neighborhood of $m = -1$, $d = 4$; they are amenable to renormalization-group calculations on the primitive equations for arbitrary $d > 2$ when m is close to the crossover -1 and, perhaps, also near the crossover $+3$.

I. INTRODUCTION

The dynamics of fully developed two-dimensional (2-d) and three-dimensional (3-d) turbulence differ in an essential way at both the infrared (ir) and the ultraviolet (uv) ends of the spectrum.[1-5] This stems from an additional conservation law, vorticity conservation, which holds only in two dimensions. What happens when this conservation law is weakly broken? For example, large-scale flow in Earth's atmosphere and oceans is known to be mostly 2-d but, of course, not exactly so.[6] Is it safe to apply the theory of 2-d turbulence to such geophysical problems or would some kind of "2.5-dimensional turbulence" be more appropriate?[7]

There are more fundamental reasons to investigate d-dimensional (d-d) turbulence for nonintegral d.[8] The statistical theory of homogeneous isotropic turbulence can be set up in a framework with (mostly superficial) similarities to field theory and statistical mechanics.[9-12] In the latter fields, continuation of the space dimension to nonintegral values has produced interesting results, including for integral values of d. Particularly important has been the concept of *crossover* dimension in critical phenomena, which may be defined as a dimension beyond which the statistics become essentially Gaussian. Just below such a crossover, it is usually possible to calculate perturbatively.[13] Attempts have been made to carry over some of the ideas of critical phenomena to fully developed turbulence.[14] In particular, there have been speculations about a possible crossover dimension above (or below) which Kolmogorov's 1941 theory (K41) would become exact.[15,16] The existence of such a crossover has been questioned in an earlier Letter.[17] Note, however, that the possibility that K41 would become exact in infinite dimensions is still open.[18]

At a more technical level, Forster, Nelson, and Stephen have been recently able to solve several ir problems for d-d turbulence near certain crossover dimensions, by using dynamic renormalization group methods.[19,20] An attempt has also been made to investigate uv properties by such methods.[21] Finally, an important motivation for the present work comes from the cascade model studied by Bell and Nelkin[22]; although not explicitly a d-d problem, the cascade model suggested to us that qualitative changes of both the ir and uv properties of turbulence can take place at dimensions somewhere between two and three.

In an earlier letter some preliminary closure-based results for fully developed d-d turbulence were reported.[17] The present paper is devoted to a detailed exposition of these and more recent results. In Sec. II we define d-d turbulence and study questions of realizability. In Sec. III we extend to d-d turbulence a closure technique which has been frequently used in two and three dimensions and which is known to be K41 compatible. Sec. IV contains technical preparatory material for Sec. V which is devoted to steady-state scaling solutions with or without forcing. In Sec. VI we study time-dependent solutions (we also recall some basic facts about 2-d and 3-d turbulence). In Sec. VII we summarize the principal results and discuss questions which are beyond closure.

The reader will be helped by some acquaintance with recent reviews of the analytic theory of turbulence.[23-25] No knowledge of renormalization-group or field-theoretic methods will be assumed, except in Sec. VIIC.

II. ANALYTIC CONTINUATION OF THE ENERGY SPECTRUM

We start with the d-dimensional (d integer $\geqslant 2$) Navier-Stokes equation for viscous incompressible flow without boundaries,

$$\partial_t \vec{v}(\vec{x},t)+\vec{v}\cdot\nabla\vec{v}=-\nabla p+\nu\nabla^2\vec{v}+\vec{f}(\vec{x},t),\quad \nabla\cdot\vec{v}=0. \tag{2.1}$$

The initial condition $\vec{v}(\vec{x},0)$ and the external forces $\vec{f}(\vec{x},t)$ are prescribed Gaussian[26] homogeneous isotropic random functions with zero mean value; the forces are restricted to have white-noise dependence on time.

The *energy spectrum* $E_k(t)$ is related to the spatial Fourier transform $U_{ij}(\vec{k},t)$ of the (one-time) covariance $\langle \vec{v}_i(\vec{x},t)\vec{v}_j(\vec{x}',t)\rangle$ by

$$S_d k^{d-1}U_{ij}(\vec{k},t)=(d-1)^{-1}P_{ij}(\vec{k})E_k(t). \tag{2.2}$$

In (2.2), $P_{ij}(\vec{k})$ is define as $\delta_{ij}-k_ik_j/k^2$ and S_d is the surface of the d-d unit sphere, related to the Γ function by

$$S_d=2\pi^{d/2}/\Gamma(\tfrac{1}{2}d). \tag{2.3}$$

Equation (2.2) ensures that in any dimension the mean kinetic energy per mass and the energy spectrum are related by

$$\tfrac{1}{2}\langle v^2(t)\rangle=\int_0^\infty E_k(t)\,dk. \tag{2.4}$$

Although the primitive Navier-Stokes equation (2.1) is meaningful only for integral d, in the statistical case we can continue its *solution* analytically to nonintegral d. A formal solution of the statistical Navier-Stokes equation may be obtained by expanding $\vec{v}(\vec{x},t)$ in powers of t or of the Reynolds number. Various moments are then obtained by averaging term by term and using the Gaussian property of initial conditions and forces.[9,10,12] The resulting expressions, which may for convenience be represented by diagrams, can be continued analytically, term by term, as functions of the dimension in more or less the same way as one continues Feynman diagrams in field theory and statistical mechanics.[27]

A. Second-order Taylor expansion

Let us illustrate the procedure by making a Taylor expansion of the energy spectrum to second order in time; for simplicity viscosity and forces will be dropped. We rewrite the Navier-Stokes equation symbolically as

$$\partial_t v=\gamma vv,\quad v(0)=v_0, \tag{2.5}$$

where γvv stands for all the quadratic terms (γ is the "bare vertex"). Taking successive time derivatives of (2.5) we obtain the Taylor coefficients at $t=0$; to order t^2 we then have, still symbolically,

$$v(t)=v_0+t\gamma v_0v_0+t^2\gamma\gamma v_0v_0v_0+O(t^3). \tag{2.6}$$

Then we average over the initial conditions and use the Gaussian property to obtain

$$\langle v(t)v(t)\rangle=\langle v_0v_0\rangle+t^2\gamma\gamma\langle v_0v_0\rangle\langle v_0v_0\rangle+O(t^4). \tag{2.7}$$

For the explicit calculation, we write the Navier-Stokes equation in Fourier space as[24]

$$\partial_t v_i(\vec{k},t)=-\frac{i}{2}\int_{\vec{p}+\vec{q}=\vec{k}}P_{ijl}(\vec{k})v_j(\vec{p})v_l(\vec{q})\,d^dp, \tag{2.8}$$

with

$$P_{ijl}(\vec{k})=k_jP_{il}(\vec{k})+k_lP_{ij}(\vec{k}). \tag{2.9}$$

We than carry out a number of now classical algebraic operations which are the same as in deriving the quasinormal approximation (The 3-d case is treated in Ref. 24, Sec. 4.4.) The final result reads

$$E_k(t)=E_k(0)+t^2C_d\int_{\Delta_k}dp\,dq\left(\frac{\sin\alpha}{k}\right)^{d-3}\frac{k}{pq}\times[a_{kpq}^{(d)}k^{d-1}E_p(0)E_q(0)-b_{kpq}^{(d)}p^{d-1}E_q(0)E_k(0)]+O(t^4),$$
$$C_d=S_{d-1}/[(d-1)^2S_d]. \tag{2.10}$$

The following notation has been used in (2.10). $E_k(t)$ is the energy spectrum defined by Eq. (2.2). S_d is the surface of the d-d unit sphere (2.3). Δ_k is the strip in the (p,q) plane limited by the triangular inequalities $|p-q|<k<p+q$. In the (k,p,q) triangle the angles are denoted by α, β, and γ and their cosines (to be used below) by x, y, and z. The geometric coefficients $a^{(d)}$ and $b^{(d)}$ are given by

$$a_{kpq}^{(d)}=\tfrac{1}{4}[2(d-2)+(3-d)(y^2+z^2)-4y^2z^2-2xyz], \tag{2.11}$$

$$b_{kpq}^{(d)}=\tfrac{1}{2}pk^{-1}[(d-3)z+(d-1)xy+2z^3]. \tag{2.12}$$

For the reader who wants to rederive Eq. (2.10) we mention the two major differences with the 3-d case (see also Appendix 2 of Ref. 25). We have used the following expression of the d-d volume element in bipolar coordinates:

$$d^d p = S_{d-1}(pq/k)^{d-2}(\sin\alpha)^{d-3}\, dp\, dq. \tag{2.13}$$

The $a^{(d)}$ and $b^{(d)}$ coefficients arise as contractions of products of $P_{ij}(\cdot)$ and $P_{ijl}(\cdot)$ operators. It is easily seen that such contractions are linear functions of the dimension; hence, it suffices to check Eqs. (2.11) and (2.12) for $d=2$ and $d=3$.

The analytic continuation of Eq. (2.10) into nonintegral dimensions is straightforward since d appears linearly in $a^{(d)}$ and $b^{(d)}$, and S_d is defined for arbitrary d by the Γ function.

B. Convergence

Higher-order terms can be obtained similarly with, of course, increasing amounts of algebra. We mention also that the analytic continuation can be done on the so-called renormalized expansions, which to each order sum infinite classes of terms from the primitive expansion.[11,28] To lowest order one obtains the direct interaction approximation (DIA) of Kraichnan.[29] The d-dimensional DIA equations may be found in Ref. 18. What do we know about the convergence properties of formal, primitive, or renormalized expansions? Recall first that for the infinite-Reynolds-number problem one must carefully take the limit $\nu \to 0$, which is certainly not the same as putting $\nu = 0$ from the start.[24,25] *A priori* there is no reason to believe that the formal Taylor series has more than zero radius of convergence.[30,31] Indeed, individual realizations of the inviscid Navier-Stokes equation (Euler equation) in any dimension $d>2$ are likely to blow up at a finite time which, by the Gaussian assumption, can be arbitrarily close to $t=0$. We have recently investigated this question on Burgers's equation which is known to produce singularities at a finite time. We have shown that the formal Taylor series in powers of t of the energy spectrum has for any fixed wave number an *infinite* radius of convergence.[32] There are also strong indications that the formal solution differs from the true ($\nu \to 0$) solution by a nonanalytic function with an identically vanishing Taylor series, something like $\exp(-1/t^2)$.

For the Navier-Stokes equation the convergence properties of the formal expansion are unknown. Still, we shall assume that such an expansion can be used to define d-d turbulence.

C. Lack of realizability for $d < 2$

In integral ($d \geqslant 2$) dimensions the energy spectrum is, by definition, non-negative because it is *realized* as the mean square of the Fourier components of the velocity. A realizability problem can occur only by making some approximation, say a closure. This is not so any more in nonintegral dimensions since the analytic continuation of a positive function need not be positive. One way of proving realizability of the d-d energy spectrum would be to exhibit a set of amplitude equations (not necessarily exactly soluble) for some random field Ψ having $E_k(t)$ as its spectrum. This appears difficult although perhaps not impossible in view of some recent results in statistical mechanics: certain lattices can be shown to have an effectively nonintegral dimensionality.[33]

We have not so far succeeded in proving realizability for $d>2$, but at least we can easily show that $d=2$ constitutes a crossover: for $d<2$, if realizability holds at $t=0$, it may (but need not) be lost for arbitrarily small positive times. This is shown in Appendix A by constructing an explicit counterexample. The proof is based on the observation that the coefficient $a_{kpq}^{(d)}$ can become negative for suitable choices of k,p,q when $d<2$. Realizability is discussed further in Sec. VII A.

III. EDDY-DAMPED QUASINORMAL MARKOVIAN EQUATION

The Taylor expansion (2.10) allows us to calculate the energy spectrum only for short times; it does not seem to tell very much about stationary turbulence. In integral dimensions the exact energy spectrum has so far been calculated only in special situations such as absolute equilibrium (Sec. IV A). Otherwise, one has to use closure. Fortunately, there exist now several closures which can be realized by model amplitude equations.[34] Such closures are usually chosen so as to preserve certain structural properties of the primitive equations considered as "essential."[35] For example, it is possible to impose agreement with the true spectrum to order t^2 (in the initial-value problem) plus compatibility with the Kolmogorov 1941 theory. The simplest such closure is the eddy-damped quasinormal Markovian (EDQNM)[36] one which is now briefly outlined (see Refs. 24 and 25 for details).

Starting from the symbolic expression (2.7) giving the exact spectrum to order t^2, we time differentiate and revert the expansion to express the time derivative of the spectrum at time t in terms of the spectrum at time t itself

$$\partial_t \langle vv\rangle = t\gamma\gamma\langle vv\rangle\langle vv\rangle + O(t^3), \tag{3.1}$$

where $\langle vv\rangle$ stands for $\langle v(t)v(t)\rangle$. Dropping the $O(t^3)$ correction and using Eq. (3.1) for all times, we obtain a closure which may be proved realizable (for $d \geqslant 2$) but which is not compatible with K41. This comes from the absence of any mechanism to prevent indefinite buildup of triple correlations. K41 compatibility is achieved by changing the factor t in the right-hand side into a triad relaxation operator θ to obtain the EDQNM equation, which reads in explicit notation

$$\partial_t E_k(t) + 2\nu k^2 E_k = T_k + F_k, \tag{3.2}$$

where the *transfer* T_k is given by (all spectra taken at time t)

$$T_k = 2C_d \int_{\Delta_k} dp\, dq\, \theta_{kpq}(t) \left(\frac{\sin\alpha}{k}\right)^{d-3} \left(\frac{k}{pq}\right) \times (a_{kpq}^{(d)} k^{d-1} E_p E_q - b_{kpq}^{(d)} p^{d-1} E_q E_k). \tag{3.3}$$

In Eq. (3.2) we have reintroduced the viscous term $2\nu k^2 E_k$ and the forcing term F_k (=spatial spectrum of external forces) which were dropped so far to bring out more clearly the essential steps in the closure. Such terms do not pose any closure problem. The notation in Eq. (3.3) is the same as in Eq. (2.10). Note that the integrand in (3.3) may be obtained from the integrand in (2.10) by changing $E_k(0)$ into $E_k(t)$ and inserting the triad relaxation time $\theta_{kpq}(t)$. In the EDQNM this time is expressed in terms of the spectrum by

$$\theta_{kpq}(t) = [1 - \exp - t\mu_{kpq}(t)]/\mu_{kpq}(t), \tag{3.4}$$

with

$$\mu_{kpq}(t) = \mu_k(t) + \mu_p(t) + \mu_q(t),$$
$$\mu_k(t) = \nu k^2 + \lambda_d \left(\int_0^k r^2 E_r(t)\, dr\right)^{1/2}, \tag{3.5}$$

where λ_d is a purely numerical positive constant depending on dimension (see below). Note that for wave numbers k such that the viscous term is negligible, $\mu_k(t)$ is essentially the root-mean-square strain on wave number k due to motions of wave number smaller than k; in other words, μ_k^{-1} is the local eddy turnover time.

It has been shown that in high dimensions all characteristic dynamical times scale with $d^{1/2}$.[18] Therefore λ_d, which appears by its inverse in the eddy turnover time, should be taken as $\propto d^{-1/2}$ as $d \to \infty$. We shall not elaborate on this question, since most of the subsequent results of this paper are independent of the choice of λ_d (see Ref. 37 for the optimal choice of λ_d in relation with the Kolmogorov constant).

Remark. Kraichnan has introduced a systematic procedure, the test field model (TFM) for calculating the triad relaxation time in both two and three dimensions.[34,38] This is easily extended to arbitrary dimensions. The TFM produces the same steady-state scaling solutions as the EDQNM (except for possible multiplicative constants) and it differs only slightly for time-dependent solutions. We have found that the TFM, in its present formulation, becomes inadequate in high dimensions because it gives a triad relaxation time proportional to d instead of to $d^{1/2}$. The following explanation has been proposed by R. H. Kraichnan (personal communication): in the TFM, characteristic dynamical times are obtained by studying the interplay of solenoidal and compressive components of a fictitious advected test field. As $d \to \infty$ the number of solenoidal components $(d-1)$ becomes too large for its interaction with the single compressive component to be representative of the actual dynamics.

In the remainder of this paper we shall make a detailed study of the solutions of the EDQNM equation. It may be of interest to recall the main properties of this closure: (i) EDQNM and true spectra agree to order t^2 (even t^3 at zero viscosity). (ii) EDQNM is realizable for $d \geqslant 2$ (see Appendix 2 of Ref. 25 for a direct proof; this can also be shown by constructing a model Langevin equation as in Ref. 34). (iii) EDQNM is compatible with K41 in three dimensions (see also Sec. V B).

IV. (m,d) PLANE (m = SPECTRAL EXPONENT; d = DIMENSION)

A. Transfer integral

As preparatory material for the next sections, we study the transfer integral (3.3) when the energy spectrum is a power law

$$E_k = k^{-m}. \tag{4.1}$$

The viscosity is set equal to zero and stationarity is assumed, so that the triad relaxation time becomes

$$\theta_{kpq} = (\mu_k + \mu_p + \mu_q)^{-1},$$
$$\mu_k = \lambda_d \left(\int_0^k r^{2-m}\, dr\right)^{1/2}. \tag{4.2}$$

Assuming convergence (see below), we obtain

$$\mu_k = \lambda_d (3-m)^{-1/2} k^{(3-m)/2}. \tag{4.3}$$

Using Eqs. (4.2) and (4.3) in Eq. (3.3), we obtain

$$T_k = \frac{2C_d (3-m)^{1/2}}{\lambda_d} \int_{\Delta_k} T_{kpq}^{(d)}\, dp\, dq, \tag{4.4}$$

$$T_{kpq}^{(d)} = [k^{(3-m)/2} + p^{(3-m)/2} + q^{(3-m)/2}]^{-1} (\sin\alpha/k)^{d-3} \times (k/pq)[a_{kpq}^{(d)} k^{d-1} (pq)^{-m} - b_{kpq}^{(d)} p^{d-1} (kq)^{-m}]. \tag{4.5}$$

The integrand $T_{kpq}^{(d)}$ is homogeneous

$$T_{\lambda k, \lambda p, \lambda q}^{(d)} = \lambda^{-(1+3m)/2} T_{kpq}^{(d)}; \tag{4.6}$$

therefore, provided the integral (4.4) converges, we have

$$T_k = T_{md} k^{3(1-m)/2}, \tag{4.7}$$

where T_{md} is given by Eq. (4.4) for $k=1$.

We have found that Eq. (4.4) converges in the strip

$$-1 < m < 3. \tag{4.8}$$

For $m \geq 3$ the mean-square-velocity gradient in the large scales is infinite, giving an infinite strain on wave numbers $O(1)$; this shows up as an ir divergence of Eq. (4.4). For $m \leq -1$ there is an uv divergence stemming from triads such that $k \ll p \sim q$; this may be interpreted as a divergence of the eddy viscosity due to small-scale motion (see Sec. VID).

B. Sign of transfer: analytical results

In subsequent sections we shall see that inertial steady-state solutions correspond to zero transfer and that their stability is determined by the sign of transfer in the neighborhood. Although the transfer integral (4.4) must in general be calculated numerically, some of its zeros can be obtained analytically, as we shall now explain.

We introduce the symmetrized transfer integrand

$$2\overline{T}^{(d)}_{kpq} = T^{(d)}_{kpq} + T^{(d)}_{kqp}, \tag{4.9}$$

which may be used instead of T_{kpq} since the integration domain is symmetric in p and q. $\overline{T}^{(d)}_{kpq}$ satisfies the following relations:

$$\overline{T}^{(d)}_{kpq} + \overline{T}^{(d)}_{pkq} + \overline{T}^{(d)}_{qkp} = 0 \quad \text{(any spectrum, any } d), \tag{4.10}$$

$$\overline{T}^{(d)}_{kpq} = 0 \quad (m = 1-d, \text{ any } d), \tag{4.11}$$

$$k^2\overline{T}^{(2)}_{kpq} + p^2\overline{T}^{(2)}_{pkq} + q^2\overline{T}^{(2)}_{qkp} = 0 \quad \text{(any spectrum, } d=2), \tag{4.12}$$

$$\overline{T}^{(2)}_{kpq} = 0 \quad (m = +1, d=2). \tag{4.13}$$

Such relations are standard[24]; they are derived from

$$2a^{(d)}_{kpq} = b^{(d)}_{kpq} + b^{(d)}_{kqp}, \tag{4.14}$$

$$2k^2 a^{(2)}_{kpq} = p^2 b^{(2)}_{kpq} + q^2 b^{(2)}_{kqp}, \tag{4.15}$$

which are necessary to ensure energy conservation in arbitrary dimensions and enstrophy conservation in two dimensions (see Sec. V A). One also has to use the invariance under permutations of kpq of θ_{kpq} (by construction) and of $(\sin\alpha)/k$ (from the law of sines).

From Eq. (4.13) the transfer vanishes for $m = 1, d=2$. From Eq. (4.11) it vanishes for $m = 1-d$ and arbitrary d. Note that for $d \geq 2$ this lies outside of the convergence strip (4.8); but that does not matter since the integrand vanishes identically.

1. Vanishing of transfer for $m = \frac{5}{3}$

The proof that the transfer vanishes for $m = \frac{5}{3}$ is exactly the same as in 3-d. It makes use of the homogeneity relation (4.6) and of (4.10). At first sight it appears difficult to use (4.10) since it involves permutation of a fixed variable (k) and of integration variables (p and q). There exist however nonlinear changes of variables which, for homogeneous integrands, are essentially equivalent to such permutations. Details will be found in Ref. 24, p. 317.

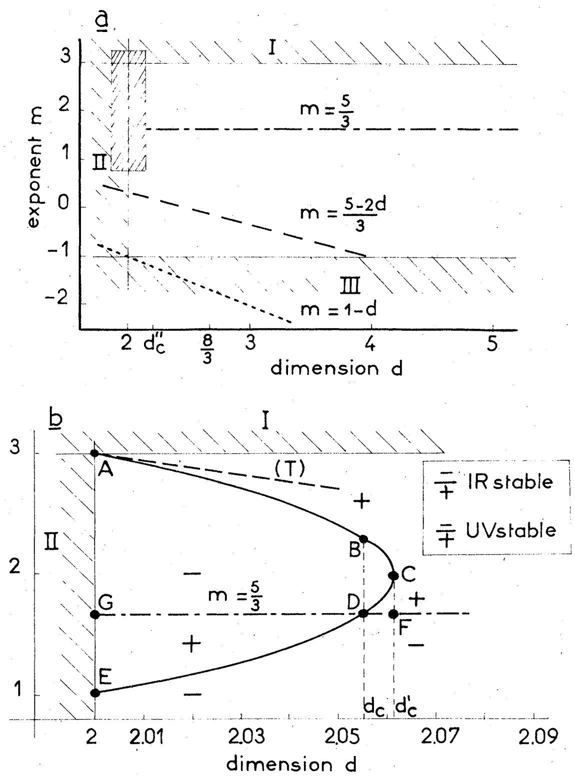

FIG. 1. (a) (m,d) plane: m, spectral exponent; d, dimension. Region I ($m \geq 3$) gives an ir divergence of the transfer integral; region III ($m \leq -1$) gives an uv divergence. Region II($d<2$): lack of realizability. Shaded area: see (b). Short-dashed curve, energy equipartition solutions; dot-dashed curve, energy-inertial solutions with $m = \frac{5}{3}$; long-dashed curve, "model B" of Forster, Nelson, and Stephen, steady-state solutions with k^{d-1} forcing. d_c'' crossover for sign of eddy viscosity. (b) The (m,d) plane near $d=2$. Regions I and II as in (a). A, ($m=3$, $d=2$) enstrophy-inertial solution; E, ($m=1$, $d=2$) enstrophy-equipartition solution. Solid curve, fluxless-inertial solutions; tangent (T) at A has equation $3-m=\frac{16}{3}(d-2)$; ABC branch has equation $m=m_1(d)$; EDC branch, $m=m_2(d)$. Dot-dashed curve, energy-inertial solutions with $m=\frac{5}{3}$. Sign of transfer integral as indicated. d_c crossover for direction of energy cascade; d_c' crossover for existence of fluxless solutions.

2. Zero-transfer branch near $m = 3$, $d = 2$

The same transformations which yield the $m=\frac{5}{3}$ solution, when applied to (4.12) (valid only for $d=2$) instead of (4.10), give $m=3$. However, for $m=3$ there is a logarithmic ir divergence so that this solution is not acceptable. (see, however, Sec. VB2). Now, assuming $d=2+\epsilon$ and $m=3-\eta$ ($\epsilon, \eta > 0$), although Eq. (4.12) does not hold any more, the transformed integrand will, by continuity, be small. Also, since we are close to the $m=3$ borderline, the transfer integrand will converge but mostly arise from triads with $p \ll k \sim q$ or $q \ll k \sim p$. The integrand may then be expanded in powers of p/k or q/k and the transfer integral evaluated analytically to leading order in ϵ and η. In this way we proved that the transfer has zeros near $m=3$ and $d=2$, given by

$$m = 3 - 16(d-2)/3 + O(d-2)^2. \qquad (4.16)$$

C. Sign of transfer: numerical results

We have done a search of all branches of zero transfer in the strip

$$d \geq 2, \quad -1 < m < 3 \qquad (4.17)$$

of the (m, d) plane. For this we have calculated numerically the integral (4.4) for $k=1$, using a standard method for integration of closure equations described, for example, in Ref. 24. We took the integration variables in the range $k_{\min}=2^{-5}$, $k_{\max}=2^5$ and $F=16$ points per wave number octave. Figures 1(a) and 1(b) show the curve of zero transfer and the sign of transfer. The branch which was calculated perturbatively near A ($m=3, d=2$) continues with a roughly parabolic shape, linking A to E ($m=1, d=2$) with a summit at C ($m \simeq 1.9$, $d=d_c' \simeq 2.06$) and intersecting $m=\frac{5}{3}$ at D ($d=d_c \simeq 2.05$).[39] We shall denote by $m_1(d)$ and $m_2(d)$ the ABC and CDE branches, respectively.

We have also checked that there are no zero-transfer branches, besides $m=\frac{5}{3}$, beyond $d=5$, the region not represented in Fig. 1(a).

V. STEADY-STATE SCALING SOLUTIONS

In this section we study stationary scaling (power-law solutions) of the d-d EDQNM equation with zero viscosity,

$$\partial_t E_k = F_k + T_k, \qquad (5.1)$$

where the transfer T_k is given by Eq. (3.3). The forcing spectrum F_k will be zero in Secs. VA and VB and a power law in Sec. VC.

A. Invariants and absolute equilibria

It has been noted by Burgers and others that the Navier-Stokes equation admits absolute equilibrium solutions similar to the thermal equilibrium solutions of a classical conservative many-body system.[40] For this, viscosity and forcing are to be removed; the equations are Fourier transformed in the space variables and then conservatively "truncated." This means that one keeps only the nonlinear triad interactions such that all three wave numbers lie in a finite truncation interval

$$k_{\min} < k < k_{\max}. \qquad (5.2)$$

Such a truncation conserves the energy

$$E(t) = \int |\hat{v}_k|^2 \, d^d k . \qquad (5.3)$$

More generally, any quadratic invariant of the primitive Navier-Stokes equation of the form

$$\Omega_s(t) = \int |k|^{2s} |\hat{v}_k|^2 \, d^d k \qquad (5.4)$$

will also survive truncation. The only known example is enstrophy ($s=1, d=2$).

Remark. It is easily shown that the search for isotropic reflection-invariant quadratic invariants of the form

$$I = \int F(|k|) |\hat{v}_k|^2 \, d^d k , \qquad (5.5)$$

where $F(\cdot)$ is ah even analytic function, reduces to the search of invariants of the form (5.4) with integer s (expand F in a Taylor series and use the invariance of the Euler equation under $x \to \lambda x$, $t \to \lambda t$). There are examples of isotropic invariants which are not reflection-invariant, such as the kinetic helicity[41]

$$H = \frac{1}{2} \int \vec{v} \cdot \operatorname{curl} \vec{v} \, d^3 x . \qquad (5.6)$$

The helicity invariant can probably be generalized to higher than three integral dimensions,[42] but hardly to nonintegral ones. There also remains the possibility of quadratic invariants of the form

$$I = \int B_{ij}(\vec{x}, \vec{x}') v_i(\vec{x}) v_j(\vec{x}') \, d^d x \, d^d x' , \qquad (5.7)$$

where $B_{ij}(\vec{x}, \vec{x}')$ is not a function of $\vec{x} - \vec{x}'$. In spite of the translation invariance of the Euler equation, such invariants cannot be ruled out *a priori*; they will not in general survive truncation.[43]

To the energy conservation (5.3) there correspond energy-equipartition absolute-equilibrium solutions. The spectrum is then just proportional to the volume within a d-d sphere of radius k:

$$E_k \propto k^{d-1}. \qquad (5.8)$$

Since $m = 1-d$ gives vanishing transfer (Sec. IV), we see that energy-equipartition solutions are found in nonintegral dimensions too.

In two dimensions there is a greater variety of solutions because of the simultaneous conservation of energy and enstrophy. Their general form is[1,3]

$$E_k \propto k/(\alpha+\beta k^2). \tag{5.9}$$

They comprise two power-law solutions: energy equipartition ($\beta=0, m=-1$) and enstrophy equipartition ($\alpha=0, m=+1$). An important question for d-d turbulence is what happens to enstrophy conservation and to enstrophy-equipartition solutions near $d=2$. We have checked that enstrophy conservation does not go over continuously into another conservation law of the form (5.3) near $d=2$. We actually proved the more general result that energy conservation and enstrophy conservation ($d=2$) exhaust the list of quadratic invariants of the form (5.3) in any integral or nonintegral dimensions (see Appendix B). Note that it suffices to obtain this negative result for the EDQNM equation, since its solution agrees with the true spectrum to order t^2 and an invariant must, of course, also survive to order t^2. As for the enstrophy-equipartition solution, we know from the numerical results of Sec. IV that it goes over into a zero-transfer branch [see Fig. 1(b)], which is however not made of absolute-equilibrium solutions.

B. Inertial solutions

We study now zero-transfer power-law solutions $E_k=k^{-m}$ of the EDQNM equation (3.2). Such solutions are called *inertial* because only the nonlinear (inertial) terms of the Navier-Stokes equation are used. The investigation of Sec. IV shows that we have two different classes of inertial power-law solutions: The first class has $m=\frac{5}{3}$ and is a d-d version of the K41 solution. The second class corresponds to the ABCDE (roughly parabolic) branch in the (m,d) plane [Fig. 1(b)], and therefore exists only for $d<d_c'$. The main difference between the two is that the former have a nonvanishing energy flux and will be called "energy inertial," whereas the latter have a zero energy flux and will be called "fluxless inertial."

The *energy flux* π_K through wave number K is usually defined as the amount of energy flowing per unit time from $k<K$ to $k>K$ minus the reverse flow.[29] Let the transfer integral (3.3) be written

$$T_k=\int T_{kpq}\,dp\,dq\,, \tag{5.10}$$

where T_{kpq} is understood to be zero when the triangular inequalities between kpq are not satisfied. We then have[25]

$$\pi_K=\int_K^\infty dk\int_0^K dp\int_0^K dq\,T_{kpq}-\int_0^K dk\int_K^\infty dp\int_K^\infty dq\,T_{kpq}. \tag{5.11}$$

Differentiating, we recover the transfer

$$\frac{\partial\pi_K}{\partial K}=-T_K. \tag{5.12}$$

When a power-law spectrum $E_k=k^{-m}$ is used, the flux integrals (5.11) are found convergent for $-1<m<3$ (same condition as for the transfer integral). Because of the homogeneity of the integrands, π_K must be a power law. From Eqs. (4.7) and (5.12), we obtain

$$\pi_K=2(3m-5)^{-1}T_{md}K^{(5-3m)/2}\quad(E_k=k^{-m}). \tag{5.13}$$

For fluxless-inertial solutions, $T_{md}=0$ and $m\neq\frac{5}{3}$; so that the energy flux indeed vanishes. For energy-inertial solutions, T_{md} is still zero but, since $m=\frac{5}{3}$, we must carefully take the limit in Eq. (5.13); we then obtain a wave-number-independent, generally nonvanishing, energy flux

$$\pi_K\equiv\pi(d)=\frac{2}{3}\frac{\partial}{\partial m}T_{md}\Big|_{m=5/3}\quad(E_k=k^{-5/3}). \tag{5.14}$$

1. Energy-inertial solutions ($m=\frac{5}{3}$)

From Eq. (5.14) and Fig. 1(b) we conclude that the energy flux is positive for $d>d_c$ and negative for $d<d_c$. For $d=3$ we recover the K41 solution with an energy cascade to high wave numbers (uv)[15]; for $d=2$ we have the 2-d inverse (ir) energy cascade.[1,5] The 2-d inverse cascade is usually explained by invoking the enstrophy conservation which prevents energy from cascading to high wave numbers.[44] The absence of an enstrophylike conservation law near $d=2$ makes the existence of an inverse cascade for $2<d<d_c\simeq2.05$ somewhat puzzling. We come back to this in Sec. VID.

It is customary to write energy-inertial solutions in the form

$$E_k=C_{\rm Kol}^{(d)}|\epsilon|^{2/3}k^{-5/3}, \tag{5.15}$$

where ϵ is the energy flux and $C_{\rm Kol}^{(d)}$ the Kolmogorov constant.[15] The energy transfer and energy flux being homogeneous to $E^{3/2}$ for stationary solutions [see Eqs. (3.3)–(3.5)], we obtain

$$C_{\rm Kol}^{(d)}=|\pi(d)|^{-2/3}\propto|d-d_c|^{-2/3}, \tag{5.16}$$

where the latter relation holds only near d_c. For the behavior of the Kolmogorov constant as $d\to\infty$, see Ref. 18.

2. Fluxless inertial solutions ($m\neq\frac{5}{3}$)

Consider the ABCDE branch in the (m,d) plane [Fig. 1(b)]. The corresponding inertial solutions exist only for $d<d_c'$, have a vanishing energy flux, and are not associated to any quadratic invariant. Still, these novel solutions go over continuously into known solutions as $d\to2$. For $d\to2$ and $m\to1$, we obtain the enstrophy-equipartition absolute-

equilibrium solution (Sec. V A). For $d \to 2$ and $m \to 3$, we obtain the two-dimensional k^{-3} enstrophy-inertial solution which has a zero energy flux but a nonvanishing enstrophy flux.[1-5] To be accurate, the situation is slightly more complicated: $m=3$ gives a divergent transfer; the actual enstrophy inertial solution is obtained only at the uv end of the spectrum and requires a logarithmic correction.[2] Such trouble can be avoided by cutting off nonlocal interactions, that is, by removing all nonlinear interactions between triads kpq such that $\min(k,p,q)/\max(k,p,q)<a$, where a is a cutoff parameter. One then has an exactly k^{-3} enstrophy-inertial solution. The time-dependent aspects of inertial solutions (stability, etc.) are further discussed in Sec. VI.

C. Solutions with power-law forcing

In most 3-d homogeneous turbulence problems a steady state is obtained by balancing energy injection (at low wave numbers) and energy dissipation (at high wave numbers), the process being mediated by nonlinear transfer (at intermediate inertial wave numbers). The particular form of the forcing spectrum is then irrelevent for inertial-range dynamics.

Suppose, however, that the fluid is subject to power-law forcing,

$$F_k \propto k^{-r}. \tag{5.17}$$

We shall now show that, with certain restrictions on the exponent r, a steady state is possible in which energy injection is balanced directly by transfer and not anymore indirectly by dissipation. Indeed, let the energy spectrum be a prescribed power law $E_k = k^{-m}$ with m in the convergence strip (4.8). The transfer is then given by Eq. (4.7); if T_{md} is negative, we can balance the transfer with a positive forcing,

$$F_k = -T_k > 0. \tag{5.18}$$

From Eqs. (4.7), (5.17), and (5.18), the exponent r is given by

$$r = \tfrac{3}{2}(m-1). \tag{5.19}$$

Convergence of transfer and negativity put some constraints on r which depend on the dimension. From Figs. 1(a) and 1(b) we see that there are two allowed regions in the (m,d) plane. (i) The region limited above by the ED curve and the $m=\frac{5}{3}$ line beyond D and limited below by the $m=-1$ line. We must then have

$$\begin{aligned} &2<d<d_c, \quad -3<r<\tfrac{3}{2}[m_2(d)-1], \\ &d>d_c, \quad -3<r<1. \end{aligned} \tag{5.20}$$

Note that such solutions exist in arbitrary dimensions. A special case was obtained by Forster, Nelson and Stephen[20]; it corresponds to ($\epsilon>0$)

$$d=4-\epsilon, \quad r=-3+\epsilon, \quad m=\tfrac{1}{3}(5-2d), \tag{5.21}$$

and is plotted as a dashed line in Fig. 1(a). This solution was obtained by a renormalization group calculation, not by closure, and will be discussed further in Sec. VII C. (ii) The region limited by the ABCDG curve, for which

$$\begin{aligned} &2<d<d_c, \quad 1<r<\tfrac{3}{2}[m_1(d)-1] \\ &d_c<d<d_c', \quad \tfrac{3}{2}[m_2(d)-1]<r<\tfrac{3}{2}[m_1(d)-1] \\ &d>d_c', \quad \text{no solution.} \end{aligned} \tag{5.22}$$

Finally, we note that with suitable choice of r we can obtain forced steady states arbitrarily close to energy- or fluxless-inertial solutions.

VI. TIME-DEPENDENT SOLUTIONS

Turbulence is in an essential way a nonequilibrium problem because the Navier-Stokes equation is dissipative. Although a steady state can be obtained by balancing the dissipation with an energy source, much insight is gained by looking at the initial-value (Cauchy) problem. Let us briefly recall some of the closure-based results for 3-d and 2-d turbulence at infinite Reynolds number (see Ref. 25 for details).

In three dimensions the free decay of an initial spectrum with finite energy and enstrophy produces a singularity in the enstrophy after a finite time t_*.[45] Up to that time there is no dissipation, but after t_* the spectrum has an uv energy-inertial range and there is a finite rate of dissipation.[37] Eventually, the whole spectrum decays to zero in a self-similar way.[46] With energy injection in a narrow wave-number band, nothing essential is changed, except that the energy grows linearly before t_* and that a steady state is eventually reached.

In two dimensions energy and enstrophy are conserved for all times and the "palinstrophy" (mean-square Laplacian of velocity) grows at most exponentially.[47] In the unforced decay problem, the spectrum approaches very slowly a k^{-3} enstrophy-inertial range extending in the uv direction. With narrow-band forcing one observes, in addition, an inverse cascade which progressively fills a $k^{-5/3}$ energy-inertial range extending in the ir direction. "Local" stationarity holds, in the sense that the spectrum at any fixed wave number tends to a finite limit as $t\to\infty$. The total energy has however an indefinite linear growth in time and no global steady state is obtained.[5]

We have investigated similar questions for d-d turbulence, using mostly numerical integration of the EDQNM equation (3.2). We took initial conditions of the form

$$E_k(0) \propto k^{d+1} \exp(-k^2) \tag{6.1}$$

normalized to give unit total energy. This ensures that initial velocities, length scales, and turnover times are of order unity. For the forced calculations, energy was injected in a narrow wavenumber band near $k=1$. The numerical method is described for example in Ref. 24. Integrations up to several thousand large eddy turnover times were made; in order to save computer time, we took only $F=4$ points per octave instead of the 16 used in the steady state calculations. This slightly modifies the numerical values of the dimensions d_c and d_c' but without changing the overall aspect of Fig. 1(b). Reynolds numbers (based on integral scale) up to 3×10^6 have been used and the results reported hereafter are simply extrapolated to $R=\infty$.

A. Singularities and dissipation

Numerical integration of the unforced EDQNM equation with initial conditions given by Eq. (6.1) shows that the enstrophy Ω has a singularity at a finite time t_* in any $d>2$ (Fig. 2). For d near two we found that $t_* \propto (d-2)^{-1}$, in agreement with its infinite value in 2-d. For high dimensions we have $t_* \propto \sqrt{d}$.[18] The result on singularities can be obtained analytically when the triad relaxation time is chosen constant (this is the so-called Markovian random-coupling model[48]). One can then prove (see Appendix 4 of Ref. 25)

$$\frac{d\Omega}{dt} = 2(d-2)d^{-1}(d-1)^{-1}\theta_0\Omega^2\,, \tag{6.2}$$

where θ_0 is the constant value of the triad relaxa-

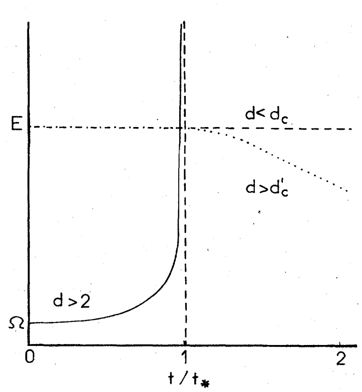

FIG. 2. Evolution of energy E and enstrophy Ω without forcing at infinite Reynolds number. For any $d>2$, enstrophy becomes infinite at t_* (proportional to $(d-2)^{-1}$ near $d=2$). No energy dissipation occurs for $2<d_c\simeq2.05$. For $d>d_c'\simeq2.06$ there is an energy catastrophe at t_*.

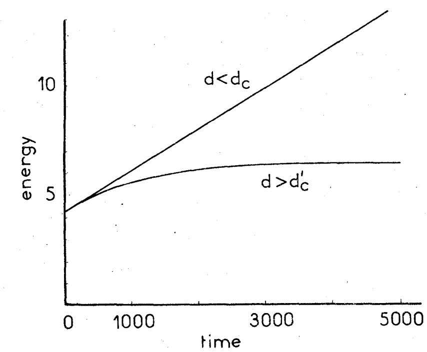

FIG. 3. Evolution of energy with narrow-band forcing: $d<d_c$, indefinite linear growth; $d>d_c'$, saturation.

tion time. It follows from Eq. (6.2) that the enstrophy becomes infinite at a time $\propto (d-2)^{-1}$ near $d=2$.

Let us now consider the dissipation of the total energy $E(t)$. In the absence of forcing, because the nonlinear terms conserve energy, we have, from Eq. (3.2),

$$\frac{dE(t)}{dt} = -2\nu\Omega(t)\,. \tag{6.3}$$

For $0<t<t_*$ we have $\lim 2\nu\Omega=0$, since $\Omega(t)$ is finite. But after t_* this limit need not vanish anymore. Figure 2 shows the evolution of $E(t)$ for $d<d_c$ and $d>d_c'$: a finite dissipation rate after t_* is observed only when $d>d_c'$. The case $d_c<d<d_c'$, which represents a range of less than 0.01 in the dimension variable, will be discussed in Sec. VI C.

Finally, in Fig. 3, we have plotted the total energy in the forced case. For $d<d_c$ we obtain an indefinite linear growth of $E(t)$, consistent with a vanishing dissipation and a constant injection rate. For $d>d_c'$ we find that the total energy saturates; this is consistent with a finite dissipation and indicates that a global steady state is reached.

B. Evolution of spectra

In the unforced case, the initial spectrum (6.1) develops after t_* an uv power-law range with an exponent given by

$$m = \begin{cases} \frac{5}{3} & \text{if } d>d_c' \\ m_1(d) & \text{if } 2<d<d_c\,, \end{cases} \tag{6.4}$$

where $m_1(d)$ represents the ABC branch of the zero-transfer curve in the (m,d) plane [Fig. 1(b)].

In the forced case the same power-law range is obtained at the uv end but at the ir end we observe, for $d<d_c$, an ir energy-inertial range with the exponent $\frac{5}{3}$ (Fig. 4). This range extends from ap-

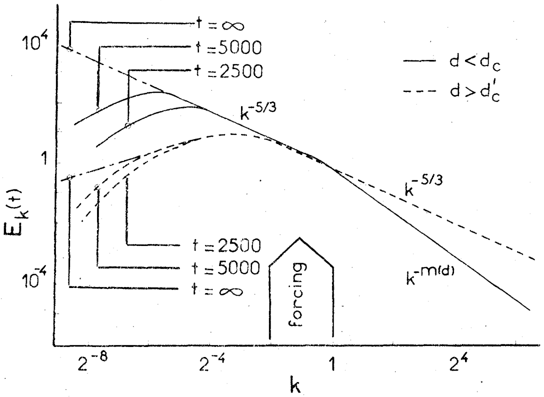

FIG. 4. Evolution of the energy spectrum with narrow-band forcing near $k=1$ at infinite Reynolds number. For $d<d_c$ there is an uv fluxless-inertial range with exponent $m=m_1(d)$ and an ir energy-inertial range with $m=\frac{5}{3}$. For $d>d_c'$ there is only an uv energy-inertial range with $m=\frac{5}{3}$.

proximately $k=1$ (forcing wave number) to a wave number $K(t)\propto t^{-3/2}$; for any fixed k, a local steady state is obtained in a time $\propto k^{-2/3}$. This is, at the ir end, indistinguishable from the 2-d case.[5] For $d>d_c'$ no inverse cascade is observed. For $d_c<d<d_c'$, see Sec. VI C.

C. Relation to steady-state solutions

Suppose that in some wave-number range

$$k_1<k<k_2, \quad k_2/k_1 \gg 1 \tag{6.5}$$

we obtain a solution $E_k \propto k^{-m}$ with m in the convergence strip (4.8). The transfer integral T_k is then *locally* determined, that is, it depends on wave numbers in the neighborhood of k (about a decade on each side[2]), so that the finiteness of the range hardly matters. In particular, inertial steady-state solutions may be found in semi-infinite uv or ir ranges in the Cauchy problem. Whether or not such a solution is actually observed depends mostly on its *stability*: When an inertial solution is subject to a small perturbation, the resulting transfer may or may not tend to restore the initial state. Arbitrary perturbations cannot be studied analytically, but we can restrict ourselves to perturbations which just change the value of the exponent and obtain necessary conditions for stability. By Eq. (4.7), the sign of transfer is that of T_{md}. Hence, for uv (ir) stability, T_{md} must be an increasing (decreasing) function of m. From Fig. 1(b), the stable uv branches are AB, BC, DE, and the $m=\frac{5}{3}$ line above d_c; the stable ir branches are CD and the $m=\frac{5}{3}$ line below d_c. Let us now consider the various cases in detail.

1. $d>d_c'$

This is the simplest case. There is a single inertial solution, the $m=\frac{5}{3}$ energy cascade with positive energy flux and, therefore, a finite dissipation. It is uv stable and is, indeed, observed after t_*.

2. $2<d<d_c$

The inverse cascade observed in Fig. 4 corresponds to the ir-stable energy-inertial solution with $m=\frac{5}{3}$. The uv $k^{-m_1(d)}$ range generated after t_* corresponds to the uv-stable AB branch of Fig. 1(b). This solution is fluxless; hence, no dissipation is obtained at infinite Reynolds number.

We have never been able to observe the uv-stable DE branch, and this requires some explanation. First, there is a question of "basin of attraction": when we start with a very steep initial spectrum, we are attracted by the uv-stable AB branch which is met first. Similarly, if we start with an initial uv power law spectrum with $\frac{5}{3}<m<m_1(d)$, the solution will again be attracted by the AB branch, since the $\frac{5}{3}$ branch, being uv-unstable, acts repulsively. The trouble is that, even with an initial $m<\frac{5}{3}$, the solution was not attracted by the DE branch. Now, we make a second observation. From Eq. (5.13) we know that the energy flux is zero on the DE branch but becomes infinite positive (negative) as $k\to\infty$ when m is just below (above) $m_2(d)$. To ensure stability of the $m=m_2(d)$ solution in a calculation with a finite upper cutoff in wave numbers, we need a mechanism capable of acting either as an energy sink or an energy source near the cutoff, and viscosity is capable only of the former. It remains possible that the DE branch is a stable solution of the inviscid untruncated equation.

3. $d_c<d<d_c'$

There are now two uv-stable branches, DF and BC, and an ir-stable branch, CD. The DF branch corresponds to the usual energy-inertial solution with a positive energy flux. The BC and CD branches are fluxless. If energy is injected by narrow-band random forces, it cannot cascade to small wave numbers and it seems therefore that the DF ($m=\frac{5}{3}$) branch is favored, unless energy accumulates. In the absence of forcing, the situation is not so clear anymore because a finite energy dissipation is not needed. Which of the fluxless BC branch and of the dissipative DF branch will be favored? Is the BC branch metastable? We have not been able to unequivocally answer such questions by numerical integration because the change in the dimension from d_c to d_c' is less than one-half of one percent.

There remains also open the case $d=d_c$ (with forcing). A uniform energy flux can be established neither in the uv nor in the ir direction, and the Kolmogorov constant is infinite [Eq. (5.16)]. We believe that no steady state is obtained, not even a local one.

D. Mechanism of the inverse cascade near $d = 2$

1. uv vs ir transfer

When energy is initially restricted to a wave-number band $a<k<b$, it can be transferred both to $k'<a$ and to $k'>b$ (except in infinite dimensions where only the latter is possible[18]). The sign of the energy flux which obtains in an energy-inertial range depends on which kind of transfer is favored. A simple argument suggests that direct transfer is favored as the dimension increases: when two independent isotropic vectors $\vec{p}$ and $\vec{q}$ of the same length are added, the probability that $|\vec{k}|=|\vec{p}+\vec{q}|>|\vec{p}|$ increases with dimension (because of the multidimensional solid angle involved). However, this argument, taken too literally, would seem to imply a direct cascade in two dimensions! Actually, the probability of various wave-vector combinations is constrained by the conservation laws. In the 2-d case it is the enstrophy conservation which makes a direct cascade impossible.[44] How does the situation change near $d=2$? The enstrophy conservation does not hold anymore and does not, as already noted, go over into another conservation law. Still, the various terms in the EDQNM equation (3.2) depend continuously on d; a change in d of say, ϵ, should have a small effect on the evolution over times of the order of ϵ^{-1} (when all other parameters are of order unity). This explains why there is an initial tendency for energy to cascade to small wave numbers. However, for $d=2.02$ ($\epsilon=0.02$), the inverse cascade was found to persist at least to $t=5000$, which is much more than ϵ^{-1} (Fig. 4). We believe that, once energy has cascaded in the ir direction something of the order of one decade (the approximate range of the transfer integral), the continuity argument can be started all over. But as we shall now see there is a simpler mechanism for explaining the inverse cascade near $d=2$.

2. Negative eddy-viscosity

Following Kraichnan's analysis of the 2-d inverse cascade,[3,49] let us consider the effect on scales $\sim k^{-1}$ of much smaller scales with wave numbers

$$p \text{ and/or } q > k_m \gg k . \tag{6.6}$$

The corresponding contributions to transfer are

$$T_{>k_m} = \int' T_{kpq}\, dp\, dq , \tag{6.7}$$

where the primed integral means that the domain is restricted by (6.6). By expanding the integrand in powers of k/k_m we obtain to lowest order

$$T_{>k_m} = -2\nu(d)k^2E_k , \tag{6.8}$$

$$\nu(d) = C_d' \int_{k_m}^{\infty} dq\, \theta_{qqk}\left((d^3-2d^2+1)E_q + (d-1)q\frac{\partial E_q}{\partial q}\right), \tag{6.9}$$

where C_d' is a finite positive numerical constant. We see that the effect of the small scales is just to modify (renormalize) the molecular viscosity by an eddy viscosity $\nu(d)$. Evaluating the eddy viscosity in the energy-inertial range, we obtain

$$\nu(d) = C_d'(d-1)(d^2-d-\tfrac{8}{3}) \int_{k_m}^{\infty} dq\, \theta_{qqk} q^{-5/3} . \tag{6.10}$$

From Eq. (6.10) we conclude that the eddy viscosity is positive in three dimensions, negative in two, and changes sign a single time in the realizable ($d \geq 2$) domain at

$$d_c'' = [1+(\tfrac{35}{3})^{1/2}]/2 \simeq 2.208 . \tag{6.11}$$

So, for $d<d_c''$, the small inertial scales will enhance the large inertial scales instead of depleting them as in 3-d; this clearly favors inverse transfer. Of course, we cannot expect to obtain the exact value of the crossover dimension by a calculation involving only distant interactions (in Fourier space). The only correct way is to calculate the energy flux $\pi(d)$ which changes sign at $d_c \simeq 2.05$ (Secs. IV and V).

E. Relation to the cascade model of Bell and Nelkin

Desnyansky and Novikov have introduced a phenomenological cascade model based on discrete variables u_n such that $u_n^2 \sim kE_k$ is the mean energy in an octave band (shell) around $k_n=2^n k_0$. Their equations, which have only nearest-neighbor shell interactions, read ($n=1,2,\ldots$)

$$\frac{du_n}{dt} = -\nu k^2 u_n + f_n(t) + k_n(u_{n-1}^2 - 2u_n u_{n+1}) - 2^{1/3}Ck_n(u_{n-1}u_n - 2u_{n+1}^2) . \tag{6.12}$$

In the special case $C=0$, Desnyansky and Novikov obtained an uv energy-inertial solution with $E_k \propto k^{-5/3}$ (Ref. 50). Bell and Nelkin[22] studied the case $C \neq 0$ and found that (i) the energy cascade reverses when $C>1$, and (ii) the model admits fluxless inertial solutions of the form

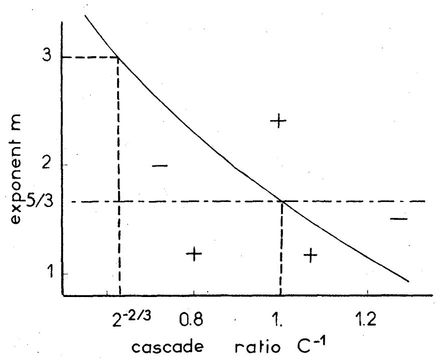

FIG. 5. Cascade model studied by Bell and Nelkin. The adjustable parameter C^{-1} plays the role of the dimension; m is the spectral exponent. Solid curve, $m=\frac{5}{3}-2\ln C^{-1}/\ln 2$, fluxless-inertial solutions; dot-dashed curves, energy-inertial solutions with $m=\frac{5}{3}$. Sign of transfer as indicated. Crossover for direction of energy cascade $C^{-1}=1$.

$$E_k \propto k^{-(5/3+\zeta)},\qquad \zeta = 2\ln C/\ln 2\,. \tag{6.13}$$

Such solutions are uv stable for $C>1$ and ir stable for $C<1$. In particular a k^{-3} uv fluxless solution obtains when $C=2^{2/3}$, but enstrophy conservation never holds.

To bring out more clearly analogies and differences with d-d turbulence, we have represented their results in a kind of (m,d)-plane diagram where we used C^{-1}, the cascade ratio, instead of the dimension (Fig. 5). The comparison of Figs. 1(b) and 5 shows an important difference: In the cascade model, fluxless solutions are present for arbitrary high values of the cascade ratio, whereas they disappear in d-d turbulence when $d>d_c'$. It would be of interest to modify the cascade model to make its zero-transfer diagram geometrically more similar to the (m,d)-plane diagram. Perhaps this can be achieved by introducing additional couplings to next-nearest-neighbor shells. It could also help to resolve some of the open questions of Sec. VI C.

Finally, we mention that Bell and Nelkin[22] have studied the self-similar decay of an unforced spectrum for $t\to\infty$ and calculated the exponent of the energy decay. Self-similar decay can be studied also with the d-d EDQNM equation. However, only the 3-d case has been worked out so far because the calculations require a nontrivial modification of existing numerical schemes to account for interactions between widely separated scales which are very important for the dynamics of the largest eddies.[46] Such "nonlocal" interactions are not present in the cascade model.

VII. BEYOND CLOSURE

While summarizing our principal results, we shall now discuss several questions which lie beyond closure.

A. Realizability

d-dimensional (d-d) turbulence was defined in this paper by analytically continuing formal expansions term by term. We proved that (true) d-d turbulence with $d<2$ can lose realizability. For nonintegral $d>2$, realizability has been shown only for closure. We therefore feel somewhat uncomfortable when we make conjectures about true d-d turbulence. This problem is, however, not limited to turbulence. We are, for example, not aware of any proof that the specific heat of the d-d Ginzburg-Landau model of a ferromagnet is non-negative when d is not an integer. We also stress that the lack of realizability in $d<2$ does not mean that all calculations become meaningless. For instance, Forster, Nelson, and Stephen have investigated the temporal fluctuations of the energy-equipartition absolute-equilibrium solution and found a nontrivial fixed point by a renormalization-group ϵ expansion below two dimensions;[19,20] their solution does not seem to suffer from lack of realizability, at least to lowest order in ϵ.

B. Inertial solutions

1. Energy cascades, intermittency, and crossover dimensions

The energy-inertial solutions obtained in Sec. V B are the usual Kolmogorov 1941 energy cascades; their novel feature is the reversal of the direction of the cascade below a crossover dimension $d_c \simeq 2.05$. It seems that close to two dimensions the enstrophy conservation law is too weakly broken to allow energy to leak out in the ultraviolet direction. We conjecture that a similar crossover takes place in the true problem.

We now remind the reader that true 3-d turbulence is probably intermittent: as the cascade proceeds to high wave numbers, fluctuations in the rate of energy transfer build up, the statistics of the flow become increasingly non-Gaussian,[51] the small-scale motion becomes less and less space filling, and the $\frac{5}{3}$ exponent could be slightly modified.[52-54] Phenomenological arguments suggest that intermittency corrections to the spectral exponent should be positive for a direct cascade and negative for an inverse cascade.[3,54] It would, however, be premature to conclude that intermittency corrections vanish at the (true) crossover dimension d_c, since steady-state inertial solutions may well not exist in that case (with the closure calculation the Kolmogorov constant becomes infinite).

It has been speculated that intermittency disappears below $d=\frac{8}{3}$ (Ref. 16). This is, however, inconsistent with the idea that inertial-range properties should not depend on the precise form of the dissipative term.[17] We also mention that Mandelbrot has shown that for $d>4$, if intermittency takes a rather extreme form, the Navier-Stokes equations could lose global regularity in time: the viscous term cannot anymore prevent singularities.[54,55] Finally, the possibility that intermittency disappears as $d\rightarrow\infty$ remains still open.[18]

2. Fluxless solutions

The fluxless solutions of Sec. V B have a number of unusual features. They constitute a sort of d-d version of the k^{-3} enstrophy cascade into which they go over as $d\rightarrow 2$. They exist only for $d<d_c'\simeq 2.06$ and have actually been obtained with certainty in initial-value calculations only for $d<d_c$ (as uv-stable solutions). The fluxless solutions are singular (uv divergence of the enstrophy) but do not produce any dissipation in the limit of infinite Reynolds number. Their spectral exponent $m=m_1(d)$ cannot be determined *a priori* by a conservation law and a dimensional argument; it has to be really calculated, for example, perturbatively for $d=2+\epsilon$ (Sec. IV B 2).

3. Persistence of initial conditions

The closure-based results of Secs. IV and V on inertial steady-state solutions have an interesting counterpart for the true problem, concerning the persistence of the energy spectrum for Gaussian initial conditions (with zero viscosity and zero forcing). An initial power-law spectrum $E_k(0)\propto k^{-m}$ will generally change after a time $O(t^2)$; but for certain values of the spectral exponent m, it can persist up to $O(t^4)$; this happens when the integral in (2.10) vanishes. Comparison with the EDQNM transfer integral (3.3) shows that these values can be calculated just as in Sec. IV, provided the triad relaxation time θ_{kpq} is removed. Energy-inertial and fluxless-inertial (pseudo-) solutions are obtained as before. However, the spectral exponent for energy-inertial solutions is now $m=2$ instead of $\frac{5}{3}$. The numerical values of d_c and d_c' are increased to about 2.09 and become too close to be distinguished with certainty. Otherwise, the overall aspect of Fig. 1(b) is unchanged.

C. Power-law forcing and renormalization group calculations

1. Closure vs renormalization group

There is presently no systematic method for calculating fully developed turbulence in any dimension. Thus the bulk of the theoretical work on turbulence relies on closure and/or phenomenological models. An interesting exception is provided by the recent work of Forster, Nelson, and Stephen (FNS).[20] Their "model B" is concerned with forced steady-state scaling solutions of the kind introduced in Sec. V C. The forcing is white noise in time with a flat modal spectrum, so that, in our notation,

$$F_k\propto k^{d-1}. \qquad (7.1)$$

Using methods borrowed from dynamical critical phenomena,[56] FNS calculate the ir properties of the solution for $d=4-\epsilon$. For the spectrum, they obtain

$$E_k\propto k^{-m},\quad m=\tfrac{1}{3}(5-2d)+O(\epsilon^2). \qquad (7.2)$$

It must be stressed that their renormalization-group (RG) calculation involves a systematic expansion of the primitive equations near the crossover, not a more or less *ad hoc* closure. However, at the technical level, we shall now show that their method is strongly related to more traditional closure methods which lead to the same result (7.2) but without $O(\epsilon^2)$ corrections [see Eq. (5.21)].[57] Indeed, FNS calculate an approximate recursion relation valid to order ϵ by a diagrammatic perturbation method. Only second-order diagrams contribute, so that the calculation is equivalent to using the lowest-order "mass-renormalized" equations, namely Kraichnan's DIA (see Ref. 11 for a field-theoretic viewpoint of the DIA and higher-order approximations). The EDQNM, like the test-field-model, can be viewed as a Markovianized version of the DIA.[34] Still, it is known that the EDQNM and the DIA do not produce the same exponents for the energy-inertial range: because of a spurious ir divergence due to lack of random Galilean invariance, the latter gives $m=\frac{3}{2}$ instead of $\frac{5}{3}$.[58] It is however easily checked that the ir divergence of the DIA disappears when the total energy itself has no ir divergence, as is the case for model B. The EDQNM and DIA thus give identical exponents although they may differ by the numerical values of constants in front of the power laws. We mention also that, at precisely four dimensions, we found a discrepancy in the exponents of the logarithmic correction between the EDQNM and the FNS result; possibly this comes from the Markovianization of the DIA. We do not propose to have this resolved experimentally!

FNS speculate that their result (7.2) is actually valid to all orders in ϵ, as suggested by the EDQNM result. This can also be supported by a simple argument with a Kolmogorov 1941 flavor. We can construct only one dimensionally consistent expression for the transfer T_k in terms of the wave number k and the energy spectrum E_k, namely,

$$T_k \propto (kE_k)^{3/2} . \tag{7.3}$$

When we equate this to minus the forcing spectrum (7.1), we recover the EDQNM result (5.21). It is hard to see how this argument could be ruined by intermittency in a fluid subject to random forcing at all places and all scales. This does not, of course, mean that second-order closure, say the DIA, is exact for this problem. We would rather conjecture that, when vertex corrections of arbitrarily high order are included, only numerical factors will change but not the exponents.

2. *Generalizations of model B of Forster, Nelson, and Stephen*

Model B is just a special case of the class (i) forced steady-state scaling solutions of Sec. V C. It corresponds to the dashed line in Fig. 1(a). Note that the FNS crossover dimension $d=4$ lies at the intersection with the lower boundary $m=-1$ of the convergence strip of the transfer integral. We believe that a more objective crossover parameter is provided by the spectral exponent m itself. For $m<-1$ the transfer integral has an uv divergence. When a cutoff $k_{\max}$ is used, the transfer at the ir end reduces to an eddy-viscosity term $-2\nu_e k^2 E_k$, where ν_e depends on $k_{\max}$ but not on k [cf. Eq. (6.9)]. Hence, the ir problem is governed by an essentially linear Langevin equation (if we assume that the fluctuations in the eddy viscosity are negligible); this leads to a Gaussian fixed point in the RG formalism. For $m=-1+\epsilon$ and *arbitrary* $d>2$, the fixed point is no longer Gaussian but can be calculated perturbatively with essentially the procedure used by FNS, the result being to lowest order identical to the closure result (5.20). Working in fixed space dimension and varying the spectral exponent (or, equivalently, the forcing spectral exponent) has an advantage besides being a physically more transparent procedure: it allows the consideration of problems which are not easily continued to nonintegral dimensions such as helical turbulence.[43]

In Sec. V C we found a second class of forced steady-state scaling solutions which corresponds to the inside of the ABCDG curve [Fig. 1(b)]. It is possible that such solutions are also amenable to RG calculations on the primitive equations near the crossover value $m=+3$ of the spectral exponent which gives an ir divergence of the transfer integral. For $m>3$ the dynamics at the uv end are determined mostly by the quasiuniform straining action of the largest eddies. This is again an essentially linear problem, but contrary to the $m<-1$ case, not an easy one because it involves an equation of motion with stochastic coefficients. The somewhat simpler question of the quasiuniform straining of a passive scalar in arbitrary dimensions d has been studied in Ref. 59; the vector problem (complicated by pressure effects) has been investigated only in the case where the large scale motion is a deterministic uniform shear.[60] A satisfactory solution of this linear stochastic problem seems a prerequisite for perturbative calculations at $m=3-\epsilon$.

ACKNOWLEDGMENTS

We have greatly benefited from discussions with many of the participants of the 1977 Turbulence Session held at the Aspen Institute of Physics, Colorado. We also wish to thank H. A. Rose for suggesting the negative eddy-viscosity interpretation of the inverse cascade near $d=2$.

APPENDIX A: LACK OF REALIZABILITY FOR $d<2$

We start from (2.10) giving the Taylor expansion to order t^2 of the energy spectrum. We wish to show that for suitable choice of a non-negative initial spectrum, we can have $E_k(t)<0$ for some k and small t. Let the initial spectrum vanish beyond some K. For $k>K$, noting that only the so-called emission term $a^{(d)}_{kpq}E_pE_p$ contributes, we obtain

$$E_k(t)=t^2 C_d \int_{\Delta_k} \frac{k^d}{pq}(\sin\alpha)^{d-3} a^{(d)}_{kpq} \times E_p(0)E_q(0)\,dp\,dq + O(t^4) . \tag{A1}$$

Trigonometric transformations enable us to rewrite the $a^{(d)}_{kpq}$ coefficient, given by (2.11), as

$$a^{(d)}_{kpq}=\tfrac{1}{4}\left[(y^2+z^2-2yz\cos(\beta-\gamma)+(d-2)(2-y^2-z^2)\right] . \tag{A2}$$

Hence, for $d<2$,

$$a^{(d)}_{k,p_0,p_0}=\tfrac{1}{2}(d-2)(1-y^2)<0 . \tag{A3}$$

We choose k between K and $2K$, and p_0 between $\frac{1}{2}k$ and K, and take the initial spectrum equal to one in the interval $(p_0-\epsilon, p_0+\epsilon)$ and equal to zero otherwise. By continuity, we can take ϵ sufficiently small to ensure that $a^{(d)}_{kpq}<0$ for all nonvanishing $E_p(0)E_q(0)$, so that the integral in (A1) is negative. We then take t small enough to ensure that the sign of the right-hand side of (A1) is that of the integral. This completes the construction of the counterexample to realizability.

APPENDIX B: SEARCH FOR QUADRATIC INVARIANTS

By substitution in the EDQNM equation (3.2) (with zero viscosity and zero forcing), it is easily checked that a necessary and sufficient condition for the invariance of

$$\Omega_s = \int_0^\infty |k|^{2s} E_k \, dk \tag{B1}$$

is that, for any k, p, and q satisfying the triangular inequalities,

$$2k^{2s} a^{(d)}_{kpq} \equiv p^{2s} b^{(d)}_{kpq} + q^{2s} b^{(d)}_{kqp} . \tag{B2}$$

This relation is satisfied for $s=1, d=2$ [Eq. (4.15)] and for $s=0$ and arbitrary d [Eq. (4.14)]. We wish to show that it cannot hold for any other values. Using (4.14) in (B2), we obtain

$$(k^{2s} - p^{2s}) b^{(d)}_{kpq} + (k^{2s} - q^{2s}) b^{(d)}_{kqp} \equiv 0 . \tag{B3}$$

$b^{(d)}_{kpq}$ is given by (2.12) and may be expressed as a rational function of k, p, and q. By homogeneity, it suffices to test (B3) for $k=1$. After some algebra, we obtain

$$(1-p^{2s})[(d-1)p^2 - q^2(p^2+1-q^2)] + (1-q^{2s})[(d-1)q^2 - p^2(q^2+1-p^2)] \equiv 0 . \tag{B4}$$

Specializing to $p=q$, we have

$$2(1-p^{2s})(d-2)p^2 \equiv 0 \quad \text{for any } p > \tfrac{1}{2} . \tag{B5}$$

For $d \neq 0$, this requires $s=0$ (energy conservation). For $d=2$, (B4) becomes, assuming now $p \neq q$ and dividing by $p^2 - q^2$,

$$(1-p^{2s})(1-q^2) - (1-q^{2s})(1-p^2) \equiv 0 . \tag{B6}$$

Assuming $p \neq 1$, $q \neq 1$ we have, for $|p-q| < 1 < p+q$,

$$(1-p^{2s})/(1-p^2) \equiv (1-q^{2s})/(1-q^2) , \tag{B7}$$

which requires $s=1$ (enstrophy conservation).

---

[1]R. H. Kraichnan, Phys. Fluids 10, 1417 (1967).
[2]R. H. Kraichnan, J. Fluid Mech. 47, 525 (1971).
[3]R. H. Kraichnan, J. Fluid Mech. 67, 15 (1975).
[4]G. K. Batchelor, Phys. Fluids (Suppl. 2) 12, 233 (1969).
[5]A. Pouquet, M. Lesieur, J. C. André, and C. Basdevant, J. Fluid Mech. 72, 305 (1975).
[6]O. Talagrand, *Fluid Dynamics*, Les Houches, 1973, edited by R. Balian and J. L. Peube (Gordon and Breach, New York, 1977), pp. 640–656; M. Coantic, J. Mec. (Paris) 12, 197 (1973).
[7]The expression "2.5-dimensional turbulence" belongs to the jargon of atmospheric physicists. Maybe they did not expect to be taken literally!
[8]*d*-dimensional turbulence for arbitrary (but integral) *d* seems to have been considered first by S. Corrsin (1951, unpublished) who generalized some of the current ideas about 3-d turbulence (Loitsiansky "invariant," law of self-similar decay at infinite Reynolds number, etc.). This work is cited in H. Tsuji, J. Phys. Soc. (Jpn.) 10, 278 (1955).
[9]H. W. Wyld, Ann. Phys. (New York) 14, 143 (1961).
[10]R. H. Kraichnan, J. Math. Phys. 2, 124 (1961).
[11]P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A 8, 423 (1973).
[12]A. Monin and A. M. Yaglom, *Statistical Fluid Mechanics of Turbulence* (English edition), edited by J. L. Lumley (MIT, Cambridge, Mass., 1975), Vol. 2.
[13]K. G. Wilson and J. Kogut, Phys. Rev. 126, 75 (1974); S. Ma, *Modern Theory of Critical Phenomena, Frontiers in Physics* (Benjamin, New York, 1976).
[14]M. Nelkin, Phys. Rev. A 9, 388 (1974); M. Nelkin, Proceedings of STATPHYS 13, the IUPAP Conference on Statistical Physics, Haifa, August, 1977 (unpublished).
[15]A.N. Kolmogorov, C. (Dokl.) R. Acad. Sci. URSS 30, 301 (1941).
[16]M. Nelkin, Phys. Rev. A 11, 1737 (1975); P. G. de Gennes, in *Proceedings of the NATO Advanced Study Institute on Fluctuations, Instabilities and Phase Transition, Jeilo, 1974*, edited by T. Riste (Noordhoff, Leiden, 1975), Series B, p. 1; P. C. Martin, J. Phys. (Paris) Colloq. 37, C1-57 (1976).
[17]U. Frisch, M. Lesieur, and P.-L. Sulem, Phys. Rev. Lett. 37, 895 (1976).
[18]J.-D. Fournier, U. Frisch, and H. A. Rose, J. Phys. A 11, No. 1 (1978); P.-L. Sulem, J.-D. Fournier, U. Frisch, and H. A. Rose, Proceedings of STATPHYS 13, the IUPAP Conference on Statistical Physics, Haifa, August, 1977 (unpublished).
[19]D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. Lett. 36, 867 (1976).
[20]D. Forster, D. R. Nelson, and M. J. Stephen, Phys. Rev. A 16, 732 (1977).
[21]S. Grossmann and E. Schnedler, Z. Phys. B 26, 307 (1977). In this reference the starting point is not the Navier-Stokes equation, but a somewhat *ad hoc* stationary probability functional with strong similarities to the Ginzburg-Landau model. Standard static RG techniques are applied but the iterations are carried out in the uv not the ir direction; some of the approximations made therefore become hard to justify.
[22]T. L. Bell and M. Nelkin, Phys. Fluids 20, 345 (1977); and unpublished.
[23]D. C. Leslie, *Developments in the Theory of Turbulence* (Clarendon, Oxford, 1973).
[24]S. A. Orszag in *Fluid Dynamics*, Les Houches, 1973, edited by R. Balian and J.-L. Peube (Gordon and Breach, New York, 1977), pp. 235–374.
[25]H. A. Rose and P.-L. Sulem, J. Phys. (Paris) (to be published).
[26]Non-Gaussian random functions must be prescribed by an infinite set of cumulants or by their characteristic functional; the continuation to nonintegral dimensions

becomes problematic.
[27]K. G. Wilson, Phys. Rev. Lett. 28, 548 (1972).
[28]R. H. Kraichnan, J. Fluid Mech. 83, 349 (1977).
[29]R. H. Kraichnan, J. Fluid Mech. 5, 497 (1959).
[30]R. H. Kraichnan, in *Dynamics of Fluids and Plasmas*, edited by S. I. Pai (Academic, New York, 1966), pp. 239–255.
[31]R. H. Kraichnan, J. Fluid Mech. 41, 189 (1970).
[32]J.-D. Fournier and U. Frisch (unpublished).
[33]D. Dhar, J. Math. Phys. 18, 577 (1977). The nonintegral dimension discussed in this paper is not related to the *fractal* dimension of Mandelbrot (see Refs. 54 and 55).
[34]R. H. Kraichnan, J. Fluid Mech. 47, 513 (1971).
[35]The intermittency of the small scales is lost by closure; although intermittency does not very much affect the energetics, from many viewpoints it is an "essential" aspect of fully developed turbulence (see Sec. VII B).
[36]S. A. Orszag, J. Fluid Mech. 41, 363 (1970).
[37]J.-C. André and M. Lesieur, J. Fluid Mech. 81, 187 (1977).
[38]P.-L. Sulem, M. Lesieur, and U. Frisch, Ann. Geophys. (Paris) 31, 487 (1975).
[39]In Ref. 17, a value of $d_c \simeq 2.03$ was reported which was based on an $F=4$ calculation.
[40]J. M. Burgers, Verh. K. Akad. Wet. Amsterdam Afd. Natuurkd. 32, 643 (1929).
[41]R. Betchov, Phys. Fluids 4, 925 (1961); H. K. Moffatt, J. Fluid Mech. 35, 117 (1969); A. Brissaud, U. Frisch, J. Léorat, M. Lesieur, and A. Mazure, Phys. Fluids 16, 1366 (1973).
[42]B. Carter (private communication).
[43]J.-D. Fournier, thesis (Observatoire de Nice, 1977) (unpublished).
[44]R. Fjortoft, Tellus 5, 225 (1953); P. E. Merilees and H. Warn, J. Fluid Mech. 69, 625 (1975).
[45]In the true problem (no closure), one obtains $t_* = 0$ (see Ref. 32).
[46]M. Lesieur and D. Schertzer, J. Mec. (Paris) (to be published).
[47]Similar results can in fact be proved for the true 2-d Euler equation: see Ref. 25 and W. Wolibner, Math. Z. 37, 668 (1933).
[48]U. Frisch, M. Lesieur, and A. Brissaud, J. Fluid Mech. 65, 145 (1974).
[49]R. H. Kraichnan, J. Atmos. Sci. 33, 1521 (1976).
[50]V.N. Desnyansky and E. A. Novikov, Prikl. Mat. Mekh. 38, 507 (1974) [J. Appl. Math. Mech. (1974)].
[51]As already stated, intermittency is suppressed by closure which can handle only near-Gaussian situations.
[52]A. N. Kolmogorov, J. Fluid Mech. 13, 82 (1962).
[53]R. H. Kraichnan, J. Fluid Mech. 62, 305 (1974).
[54]U. Frisch, P.-L. Sulem, and M. Nelkin, J. Fluid Mech. (to be published).
[55]B. B. Mandelbrot, in *Turbulence and Navier-Stokes Equation*, edited by R. Temam, Vol. 565 of *Lecture Notes in Mathematics* (Springer, New York, 1977), p. 121.
[56]S. Ma and G. F. Mazenko, Phys. Rev. B 11, 4077 (1975). This reference contains what we believe is a somewhat misleading statement for applications to problems without a fluctuation-dissipation theorem, such as forced steady-state scaling solutions of the Navier-Stokes equation. On p. 4085 it is stated that after elimination of the high-wave-number modes, the equations should be "averaged over the ζ_q", the high-wave-number forces. Actually the equations should not be averaged but may, for convenience, have their coefficients separated into mean and fluctuations. Otherwise, a zero-mean-value term, which may contribute to the renormalization of the force-strength, is lost. For systems with a fluctuation-dissipation theorem this does not matter since it is enough to calculate the renormalized viscosity which is obtained correctly. This was noted independently by D. Nelson.
[57]Similar observations were made by J. D. Gunton and K. Kawaski [J. Phys. A 8, L9 (1975)]; the approximation they use for the self-energy in their mode-mode coupling equations is equivalent to Kraichnan's DIA.
[58]R. H. Kraichnan, Phys. Fluids 7, 1723 (1964).
[59]R. H. Kraichnan, J. Fluid Mech. 64, 737 (1974).
[60]A. Craya, Publ. Sci. Tech. Minist. Air (Fr.), 345a, 1 (1967).