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Growth of fluctuations in quenched time-dependent Ginzburg-Landau model systems

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A dynamical theory is presented for the enhanced fluctuations that occur in a time-dependent Ginzburg-Landau model system with the order parameter not conserved which is quenched from a thermodynamically stable to an unstable state. In a certain weak-coupling, long-time, and long-distance limit, diffusion and saturation effects can be treated separately. As a result explicit expressions are found for the probability distribution functional, the two-point reduced distribution function, and the pair correlation function of the fluctuations, which evolve from an arbitrary initial probability distribution functional. The behavior of the latter two functions is also displayed graphically. A central role is played by the time-independent nonlinear transformation of the order parameter which takes care of the saturation effects. The nature of such a transformation is discussed in a general context. If the problem is viewed as a nonequilibrium critical phenomenon, the theory corresponds to the Landau mean-field theory. An expansion in $\epsilon = 4 - d$ is suggested to improve our treatment, where d is the dimensionality of space.

I. INTRODUCTION

In recent years, the time evolution of the fluctuations of a system suddenly brought (quenched) into a thermodynamically unstable state has received increasing attention from theoretical[1] and experimental physicists.[2,3] Major efforts have been directed towards constructing methods capable of handling the strongly nonlinear character of the problem. The subject is of fundamental importance because of its relevance to the old problem of obtaining a correct statistical-mechanical description[4] of metastable and unstable states. The problem is also of interest because it involves a second length (the "domain" size) in addition to the correlation length associated with phase transitions.

The present status of our theoretical understanding is somewhat mixed. On the one hand considerable success has been achieved in the development of approximate calculational schemes[1] which adequately explain for example the fluctuation spectrum of the spin-exchange kinetic Ising model[2] as given by computer simulation studies. On the other hand, recent spinodal decomposition experiments in fluids[3,4] seem to indicate inadequacies of the existing theories. In any case the present theory is characterized by the absence of a controlled approximation scheme in which one has a clear idea of the error involved. In this regard the recent work of Suzuki[5] on simple stochastic models with small nonlinear couplings seems particularly interesting. In essence he has shown that the nonlinear terms in the stochastic equation can be treated as a singular perturbation[6] in the interesting domain of the long-time behavior of the system. Indeed, a simple change of variables reduces his nonlinear stochastic equation for this domain into a readily solvable form.

The range of applicability of the class of models treated by Suzuki is, however, restricted to systems such as lasers in which only a finite number of degrees of freedom are important. One of us, therefore, has attempted in recent works[7] to extend Suzuki's method to treat models with an infinite number of degrees of freedom (field theories) which should have a wider range of applicability. Before describing this work we digress for a moment to present a simplified version of Suzuki's theory in order to facilitate a better understanding of our study of the time-dependent Ginzburg-Landau (TDGL) model. For this purpose we consider a simple "laser" model which is described by the following equation for the probability distribution function $P(S,t)$ of a single variable S:

$$\frac{\partial P(S,t)}{\partial t} = L\frac{\partial}{\partial S}\left(\frac{\partial}{\partial S} - \tau S + \frac{g}{6}S^3\right)P(S,t)\,, \tag{1.1}$$

where L, τ, and g are positive parameters of the model. This equation has the well-known equilibrium solution describing coexisting phases, with peaks around the mean-field values $S = \pm S_m$, with $S_m = (6\tau/g)^{1/2}$. If we start initially with a Gaussian distribution centered around $S=0$, $\exp(-S^2/2\chi)$, with χ being the initial variance, instability leads to a rapidly growing fluctuation. Hence, even for small values of g, a simple perturbation solution of (1.1) in powers of the nonlinear coupling breaks down for times of the order of $(2L\tau)^{-1}\ln[6\tau/g(\chi+\tau^{-1})]$. This is an example of a singular perturbation[6] whose solution in this case requires retaining all orders of g in the perturbation theory which appear in the combination

$$\tilde{y}(t) \equiv (g/6\tau)(\chi+\tau^{-1})e^{2L\tau t}$$

(i.e., one sums the most "dangerous" diagrams). This corresponds to taking the limits $g\to 0$, $t\to\infty$ such that $\tilde{y}(t)$ remains finite. This is somewhat reminiscent of the irreversible statistical-mechanical theory of Van Hove and Prigogine.[8] Suzuki[5] has shown that in this limit it is legitimate to drop the second derivative in (1.1) for long times such that the probability distribution is sufficiently broad and to match the solution of the modified stochastic equation at the initial time. Since the stochastic equation is first order, just like the Liouville equation, its solution can be obtained by first solving the characteristic equation, which in this case is the following deterministic equation of motion for $S(t)$:

$$\frac{dS(t)}{dt} = L\tau S(t) - \frac{gL}{6}S(t)^3 . \tag{1.2}$$

The asymptotic solution for the long-time behavior is

$$S(t) = S^0(t)\left(1+\frac{g}{6\tau}S^0(t)^2\right)^{-1/2} \tag{1.3}$$

with $S^0(t)\equiv S(0)e^{L\tau t}$, which we regard as the transformation $S(0)\to S(t)$. Now, since the time development of the probability distribution goes backward in time as in the Liouville equation, we obtain $P(S,t)$ by substituting the inverse transformation

$$S \to e^{-2L\tau t}S/[1-(g/6\tau)S^2]^{1/2} \tag{1.4}$$

into the following modified initial distribution function obtained by matching:

$$\tilde{P}_0(S) = (2\pi\tilde{\chi})^{-1/2}e^{-S^2/2\tilde{\chi}} \tag{1.5}$$

with $\tilde{\chi} \doteq \chi + L/\tau$. Thus, after including the Jacobian of the transformation, we obtain the following result for the probability distribution function at long times:

$$P(S,t) = [2\pi\tilde{\chi}(t)]^{-1/2}[1-(g/6\tau)S^2]^{-3/2} \times \exp\{-S^2/\tilde{\chi}(t)[1-(g/6\tau)S^2]\}, \tag{1.6}$$

with $|S|<S_m$ and

$$\tilde{\chi}(t) = \tilde{\chi}e^{2L\tau t} . \tag{1.7}$$

Equation (1.6) reduces to a Gaussian distribution function in terms of the variable S^0 given by

$$S^0 = S/[1-(g/6\tau)S^2]^{1/2} , \tag{1.8}$$

which is in fact the same transformation as (1.3).

Eventually, the variance $\tilde{\chi}(t)$ of S^0 increases indefinitely. However, S never exceeds its saturation value $(6\tau/g)^{1/2}$. In fact, the transformation (1.8) squeezes the broadened Gaussian distribution of S^0 at long times into the double-peak distribution of S. In this sense (1.6) may be called the squeezed-in Gaussian distribution function. It should be noted that the *time-independent* transformation (1.8) plays the central role in this method.

The time enters only in the variance of the Gaussian distribution of S^0. At $t\to\infty$ (1.6) reduces to the sum of two δ functions peaked at $S=\pm S_m$, which is incorrect. The distribution function which reduces to the correct equilibrium solution in this limit can be obtained by matching the solution of the modified stochastic equation to the solution of the full stochastic equation near the final state.[7]

As noted above we have attempted elsewhere to extend the above ideas to systems with an infinite number of degrees of freedom, and in particular to TDGL stochastic models which are suddenly quenched from a thermodynamically stable state to an unstable state. For such models there exist a set of variables with long wavelengths which are unstable in the initial regime and are somewhat analogous to the variable S above. However, one encounters a problem in that there exist other variables with short wavelengths which are stable and for which the method described above is not applicable. However, we have been able to show[9] that the dynamics of these long- and short-wavelength fluctuations become asymptotically decoupled for the long-time domain (the "turbulent" regime[10]) and that the effect of the short-wavelength fluctuations is only to renormalize various parameters in the model. This result enables us to limit our consideration just to the dynamics of the unstable long-wavelength fluctuations for which the idea of Suzuki can be applied. The system can then be described by the following model stochastic equation for the probability distribution functional $P(\{S\},t)$ for the local order parameter $\{S\}$ valid at large times:

$$\frac{\partial P(\{S\},t)}{\partial t} = -\int_{\vec{k}}\frac{\delta}{\partial S_{\vec{k}}}c_{\vec{k}}\{S\}P(\{S\},t) , \tag{1.9}$$

where $\{S\}=\{S_{\vec{k}};k<\kappa\}$ with κ defined below and

$$c_{\vec{q}}\{S\} \equiv \gamma_{\vec{q}}S_{\vec{q}} - \frac{gL}{6} \times \int_{\vec{k}_1}\int_{\vec{k}_2}\int_{\vec{k}_3}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\vec{k}_3)S_{\vec{k}_1}S_{\vec{k}_2}S_{\vec{k}_3} . \tag{1.10}$$

Our notation is the following:

$$\int_{\vec{k}} \equiv (2\pi)^{-d}\int_{|\vec{k}|<\kappa} d\vec{k} ,$$

d is the spatial dimensionality, $\delta/\delta S_{\vec{k}}$ is $(2\pi)^d$ times the functional derivative with respect to $S_{\vec{k}}$, δ is $(2\pi)^d$ times the δ function, and L is the kinetic coefficient. The positive quantities g and κ^2 [with $\gamma_{\vec{q}} = L(\kappa^2 - q^2)$] are the parameters which appear in the Ginzburg-Landau-Wilson free energy Φ,

$$\Phi = \frac{1}{2}\int_{\vec{k}} (k^2 - \kappa^2)|S_{\vec{k}}|^2 + \frac{g}{4!}\int_{\vec{k}_1}\int_{\vec{k}_2}\int_{\vec{k}_3}\int_{\vec{k}_4} \delta(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \times S_{\vec{k}_1} S_{\vec{k}_2} S_{\vec{k}_3} S_{\vec{k}_4} . \tag{1.11}$$

The negative sign in front of κ^2 implies that the system is being driven to an equilibrium ordered state. The absence of the second functional derivative in (1.9) is justified[9] in the turbulent regime provided that we match the solution of (1.9) at short times with the solution of the original stochastic model. The original model[9] includes the addition of the second-derivative term

$$L\int_{\vec{k}} \frac{\delta^2 P(\{S\}, t)}{\delta S_{\vec{k}} \delta S_{-\vec{k}}}$$

to the right-hand side of (1.9). It is worth noting here that the approximation scheme that led to (1.9) involved dropping those terms in the final solution for which the coupling constant is not multiplied by the large factor $\exp(2L\kappa^2 t)$. Hence, as far as the description of phase transitions is concerned, our approximation does not go beyond the Landau mean-field theory.

The present work is devoted to the study of the behavior of the model defined by (1.9) and (1.10) in the weak-coupling long-time long-distance limit (see the end of Sec. III). It does not treat, however, the final limiting behavior in which the final equilibrium state is reached. In this case a correct description necessitates including the term

$$L\int_{\vec{k}} \frac{\delta^2 P(\{S\}, t)}{\delta S_{\vec{k}} \delta S_{-\vec{k}}}$$

to (1.9).

Our model shares the perennial difficulty common to all the nontrivial nonlinear field theories, namely the coexistence of the nonlinearity [the last term of (1.10) and (1.11)] and the gradient term [the wave-number-dependent parts of the first terms of (1.10) or (1.11)], which produce the saturation effects and diffusion of local fluctuations, respectively.

As will be discussed in Sec. II, the surprising fact is that in our case the effects of these types of terms can be separated in the long-time domain, at least in the leading order of our approximation. In fact the nonlinearity of the problem can be taken care of by a simple local time-independent transformation of the variable as given in (2.26). Thus, the explicit form of the probability distribution functional in the turbulent regime can be written. In Sec. III, we derive expressions for the n-point probability distribution and correlation functions. In Sec. IV, we display the behavior of two-point distribution and correlation functions as functions of the time and the distance between the two points. In Sec. V we present a further discussion of the time-independent transformation of the variable in a more general context including the case of TDGL model with conserved order parameter. In Sec. VI, we discuss the theory from the view point of nonequilibrium critical phenomena and suggest a possible improvement of the approximation. We also briefly discuss the limitation of the theory with regard to metastable states.

II. TIME EVOLUTION OF THE PROBABILITY DISTRIBUTION FUNCTIONAL

Since our model stochastic equation (1.9) is first order in the functional derivative, the time evolution of $P(\{S\}, t)$ is known if we know the solution of the following deterministic equation of motion which corresponds to (1.2):

$$\frac{dS_{\vec{q}}(t)}{dt} = c_{\vec{q}}\{S(t)\} \tag{2.1}$$

subject to the initial condition $S_{\vec{q}}(t=0) = S_{\vec{q}}$. If the solution with the initial value $\{S\}$ is denoted simply by $S(t)$, the formal solution of (1.9) with $P(\{S\}, t=0) = \tilde{P}_0\{S\}$ is then obtained as

$$P(\{S\}, t) = \exp\left(-t\int_{\vec{k}} \frac{\delta}{\delta S_{\vec{k}}} c_{\vec{k}}\{S\}\right)\tilde{P}_0\{S\} = \tilde{P}_0\{S(-t)\} J(\{S\}, t), \tag{2.2}$$

where

$$J(\{S\}, t) = \exp\left(-t\int_{\vec{k}} \frac{\delta}{\delta S_{\vec{k}}} c_{\vec{k}}\{S\}\right)\cdot 1 \tag{2.3}$$

is the Jacobian in the function space associated with the transformation $\{S\} \to \{S(-t)\}$.

We now turn to the problem of solving (2.1) and introduce the new variables

$$S^0_{\vec{q}}(t) \equiv e^{\gamma_{\vec{q}} t} S_{\vec{q}}, \tag{2.4}$$

$$\mathbb{S}_{\vec{q}}(t) \equiv e^{-\gamma_{\vec{q}} t} S_{\vec{q}}(t). \tag{2.5}$$

Using first (2.4) and then (2.5), (2.1) can be transformed into the following integral forms:

$$S_{\vec{q}}(t)=S^{0}_{\vec{q}}(t)-\frac{gL}{6}\int_{\vec{k}_1}\int_{\vec{k}_2}\int_{\vec{k}_3}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\int_0^t dt_1 e^{\gamma_{\vec{q}}(t-t_1)}S_{\vec{k}_1}(t_1)S_{\vec{k}_2}(t_1)S_{\vec{k}_3}(t_1)$$

and

$$\mathcal{S}_{\vec{q}}(t)=S_{\vec{q}}-\frac{gL}{6}\int_{\vec{k}_1}\int_{\vec{k}_2}\int_{\vec{k}_3}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\int_0^t dt_1 \exp[(\gamma_{\vec{k}_1}+\gamma_{\vec{k}_2}+\gamma_{\vec{k}_3}-\gamma_{\vec{q}})t_1]\mathcal{S}_{\vec{k}_1}(t_1)\mathcal{S}_{\vec{k}_2}(t_1)\mathcal{S}_{\vec{k}_3}(t_1)\,. \tag{2.6}$$

The first few terms of the iterative solution of (2.6) take the form

$$\begin{aligned}\mathcal{S}_{\vec{q}}(t)=S_{\vec{q}}&-\frac{gL}{6}\int_{\vec{k}_1}\int_{\vec{k}_2}\int_{\vec{k}_3}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\vec{k}_3)\int_0^t dt_1 \exp[(\gamma_{\vec{k}_1}+\gamma_{\vec{k}_2}+\gamma_{\vec{k}_3}-\gamma_{\vec{q}})t_1]S_{\vec{k}_1}S_{\vec{k}_2}S_{\vec{k}_3}\\ &+3\left(\frac{gL}{6}\right)^2\int_{\vec{k}_1}\cdots\int_{\vec{k}_5}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\vec{k}_3-\vec{k}_4-\vec{k}_5)\\ &\qquad\times\int_0^t dt_1\int_0^{t_1}dt_2 \exp[(\gamma_{\vec{q}-\vec{k}_4-\vec{k}_5}+\gamma_{\vec{k}_4}+\gamma_{\vec{k}_5}-\gamma_{\vec{q}})t_1]\\ &\qquad\times\exp[(\gamma_{\vec{k}_1}+\gamma_{\vec{k}_2}+\gamma_{\vec{k}_3}-\gamma_{\vec{q}-\vec{k}_4-\vec{k}_5})t_2]S_{\vec{k}_1}S_{\vec{k}_2}S_{\vec{k}_3}S_{\vec{k}_4}S_{\vec{k}_5}+\cdots\,.\end{aligned} \tag{2.7}$$

The structure of the iterative solution is shown by diagrams in Fig. 1. A typical higher-order term may be represented by a diagram shown in Fig. 2. The contribution from such an nth-order term in (2.7) has the following general structure:

$$\begin{aligned}\left(\frac{-gL}{6}\right)^n\int_{\vec{k}_1}\cdots\int_{\vec{k}_{2n+1}}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\cdots-\vec{k}_{2n+1})\int_0^t dt_1\int_0^{t_1}dt_2\cdots\\ \times\int_0^{t_{n-1}}dt_n \exp(\Gamma_1 t_1+\Gamma_2 t_2+\cdots+\Gamma_n t_n)S_{\vec{k}_1}S_{\vec{k}_2}\cdots S_{\vec{k}_{2n+1}}\end{aligned} \tag{2.8}$$

where Γ_i takes the form

$$\Gamma_i\equiv\gamma_{\vec{K}'_i}+\gamma_{\vec{K}''_i}+\gamma_{\vec{K}'''_i}-\gamma_{\vec{K}_i} \tag{2.9}$$

in which $\vec{K}'_i$, $\vec{K}''_i$, and $\vec{K}'''_i$ are the wave vectors associated with the three lines emerging to the right of the nth vertex and $\vec{K}_i$ is the wave vector of the line emerging to the left. Now, for long times the n-fold time integral of the exponential function in (2.8) can be well approximated with error of the relative order of $(2Lt)^d\kappa^{2d}e^{-2\gamma_0 t}$ by

$$\frac{1}{n!\,(2\gamma_0)^n}\exp\left(t\sum_{i=1}^{n}\Gamma_i\right), \tag{2.10}$$

where in the denominator the $\gamma_{\vec{K}}$'s have been replaced by γ_0 since the K's are of the order of $(2Lt)^{-1/2}$. It is then clear from (2.9) that the $\gamma_{\vec{K}}$'s associated with an internal line cancel in the sum $\sum_i\Gamma_i$ and (2.10) reduces to

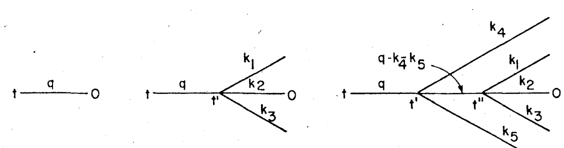

FIG. 1. Diagrams of the zeroth-, first-, and second-order iterative solutions of (2.6).

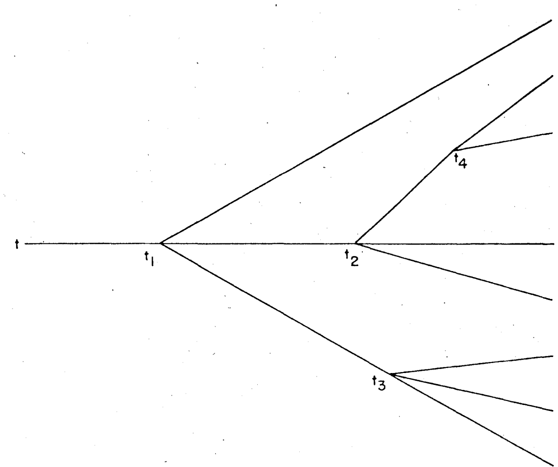

FIG. 2. Typical higher-order diagram.

$$\frac{1}{n!\,(2\gamma_0)^n}\exp\left[t\left(\sum_{i=1}^{2n+1}\gamma_{\vec{k}_i}-\gamma_{\vec{q}}\right)\right]. \tag{2.11}$$

That is, all the terms of the nth order contribute an equal amount (2.11) to (2.7). On the other hand, there are $(2n-1)!!$ terms like (2.8) in the nth order. In other words, there are $(2n-1)!!$ diagrams like the one shown in Fig. 2 in which the times are ordered and all the lines are considered to be distinct. Hence, if we use (2.4) and (2.5), (2.7) reduces to

$$S_{\vec{q}}(t)=S^0_{\vec{q}}(t)+\sum_{n=1}^{\infty}\frac{(-\alpha)^n(2n-1)!!}{2^n n!}\int_{\vec{k}_1}\cdots\int_{\vec{k}_{2n+1}}\delta(\vec{q}-\vec{k}_1-\vec{k}_2-\cdots-\vec{k}_{2n+1})S^0_{\vec{k}_1}(t)S^0_{\vec{k}_2}\cdots S^0_{\vec{k}_{2n+1}}(t)\,, \tag{2.12}$$

where

$$\alpha\equiv gL/6\gamma_0\,. \tag{2.13}$$

The result (2.12) takes a simpler form if the variables are expressed in coordinate space as

$$S(\vec{r},t)=\int_{\vec{k}}S_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{r}}\,, \tag{2.14}$$

$$S^0(\vec{r},t)=\int_{\vec{k}}S^0_{\vec{k}}(t)e^{i\vec{k}\cdot\vec{r}}\,. \tag{2.15}$$

Thus, we obtain

$$S(\vec{r},t)=S^0(\vec{r},t)+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2^n n!}(-\alpha)^n S^0(\vec{r},t)^{2n+1}\,. \tag{2.16}$$

Use of the formula

$$\frac{x}{(1+x^2)^{1/2}}=x+\sum_{n=1}^{\infty}(-1)^n\frac{(2n-1)!!}{2^n n!}x^{2n+1} \tag{2.17}$$

finally reduces (2.16) to

$$S(\vec{r},t)=S^0(\vec{r},t)/[1+\alpha S^0(\vec{r},t)^2]^{1/2}\,. \tag{2.18}$$

This together with

$$S^0(\vec{r},t)=\exp[tL(\kappa^2+\nabla^2)]S(\vec{r})\,, \tag{2.19}$$

which follows from (2.4) and (2.15), provide the transformation $\{S\}\to\{S(t)\}$. In order to obtain the needed transformation $\{S\}\to\{S(-t)\}$ we simply have to replace $S(\vec{r},t)$ by $S(\vec{r})$ and $S(\vec{r})$ by $S(\vec{r},-t)$ in (2.18) and (2.19) to get[11]

$$S(\vec{r},-t)=\frac{e^{-tL(\kappa^2+\nabla^2)}S(\vec{r})}{[1-\alpha S(\vec{r})^2]^{1/2}}\,. \tag{2.20}$$

The Jacobian of the transformation follows directly from (2.20) by noting that the exponential operator merely contributes a constant factor, namely,

$$J(\{S\},t)=\text{const}\exp\left(-\frac{3}{2v_0}\int d\vec{r}\,\ln[1-\alpha S(\vec{r})^2]\right). \tag{2.21}$$

The factor v_0 is the cell volume arising from the division of the system into small cells before taking the continuum limit. Combining (2.2), (2.20), and (2.21) yields the following expression for the probability distribution functional for long times $t\gg 1/2\gamma_0$:

$$P(\{S\},t)=\text{const}\exp\left\{-\frac{3}{2v_0}\int d\vec{r}\,\ln[1-\alpha S(\vec{r})^2]\right\}\times\tilde{P}_0\left(\frac{e^{-tL(\kappa^2+\nabla^2)}S(\vec{r})}{[1-\alpha S(\vec{r})^2]^{1/2}}\right) \tag{2.22}$$

This functional evolves from an arbitrary "initial" probability distribution functional P_0 which is matched to the probability distribution functional of the original stochastic model in the initial regime $t\lesssim 1/2\gamma_0$.

Next, note that if the nonlinear coupling term were absent in the stochastic model ($\alpha=0$) the probability distribution functional would evolve in time according to

$$\tilde{P}^0_t(\{S\})=\tilde{P}_0\{\exp[-tL(\kappa^2+\nabla^2)]S(\vec{r})\}\,. \tag{2.23}$$

Hence (2.22) can be rewritten

$$P(\{S\},t)=\text{const}\exp\left(-\frac{3}{2v_0}\int d\vec{r}\,\ln[1-\alpha S(\vec{r})^2]\right)\times\tilde{P}^0_t\left(\frac{S(\vec{r})}{[1-\alpha S(\vec{r})^2]^{1/2}}\right). \tag{2.24}$$

This result clearly demonstrates the disentangling of the gradient and nonlinear coupling terms in the stochastic equation (1.9). Namely, the gradient term affects the time evolution of the probability distribution functional in the absence of nonlinearity through the time-dependent linear transformation of variables

$$S(\vec{r})\to\exp[-tL(\kappa^2+\nabla^2)]S(\vec{r})\,. \tag{2.25}$$

The effects of the nonlinear term are taken into account through the time-independent local nonlinear transformation of the variables

$$S(\vec{r})\to S(\vec{r})/[1-\alpha S(\vec{r})^2]^{1/2}\,. \tag{2.26}$$

The two transformations (2.25) and (2.26) are very suggestive as to what we generally expect for the behavior of the probability distribution functional (2.24). First, the transformation (2.25) tends to broaden the probability distributions of the variables that are unstable in the absence of nonlinearity (i.e., $S_{\vec{k}}$ with $k<\kappa$). The transformation (2.26) then squeezes in the distribution whenever the width of the distribution of the local variable $S(\vec{r})$ in the absence of nonlinearity exceeds its saturation value $S_m \equiv 1/\alpha^{1/2}$. Hence, as the time approaches infinity, the probability distribution functional peaks at the saturation value $|S(\vec{r})| = S_m$. It should be noted that because of (2.18) the values $S(\vec{r})$ can take are limited to be within the finite interval $(-S_m, S_m)$. $P(\{S\},t)$ must be set equal to zero if $|S(\vec{r})|$ at any $\vec{r}$ exceeds S_m.

What is happening here can be summarized in physical terms as follows. First, the fluctuations everywhere start to grow rapidly and at the same time these fluctuations diffuse over the distance $(2Lt)^{1/2}$, within which fluctuations will be strongly correlated. As the value of $|S(\vec{r})|$ at any point approaches S_m, saturation sets in. Thus, in the language of magnetism, after a while the system will be divided into a number of magnetic domains of the average size $(2Lt)^{1/2}$ with the saturation magnetization S_m or $-S_m$.

In order to actually visualize the picture described here in the following we shall restrict our discussion to the case with the initial probability distribution functional $P^0\{S\}$ of the original stochastic model having the following Gaussian form:

$$P^{(0)}\{S\} = \text{const}\exp\left(-\frac{1}{2}\int_{\vec{k}} (|S_{\vec{k}}|^2/\chi_{\vec{k}})\right). \tag{2.27}$$

The choice

$$\chi_{\vec{k}} = (k^2+\kappa_0^2)^{-1} \tag{2.28}$$

corresponds to an initial disordered state with the inverse correlation range of fluctuations κ_0, since in this case the quartic term in (1.11) has a minor effect and can be dropped in the spirit of our approximation scheme as described in the introduction. Then the "initial" probability distribution functional of (1.9) which is to be matched to (2.27) is given by

$$\tilde{P}_0\{S\} = \text{const}\exp\left(-\frac{1}{2}\int_{\vec{k}} (|S_{\vec{k}}|^2/\tilde{\chi}_{\vec{k}})\right), \tag{2.29}$$

where

$$\tilde{\chi}_{\vec{k}} = \chi_{\vec{k}} + L/\gamma_{\vec{k}}. \tag{2.30}$$

By (2.23) we have

$$\tilde{P}_t^0\{S\} = \text{const}\exp\left(-\frac{1}{2}\int |S_{\vec{k}}|^2/\tilde{\chi}_{\vec{k}}(t)\right), \tag{2.31}$$

with

$$\tilde{\chi}_{\vec{k}}(t) = e^{2\gamma_{\vec{k}} t}\,\tilde{\chi}_{\vec{k}}. \tag{2.32}$$

III. REDUCED DISTRIBUTION AND CORRELATION FUNCTIONS

Since the complete probability distribution functional $P(\{S\},t)$ contains too detailed information for us to visualize, it is useful to introduce the n-point distribution function $\rho_n(\{S\}_n;\{\vec{r}\}_n;t)$ which is the joint probability distribution of the local order parameter at $\{\vec{r}\}_n$, where $\{S\}_n \equiv S_1, S_2, \ldots, S_n$ and $\{r\}_n = \vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$:

$$\rho_n(\{S\}_n;\{\vec{r}\}_n;t) \equiv \int d\{S'\}\prod_{i=1}^{n}\delta[S(\vec{r}_i) - S'(\vec{r}_i)] \times P(\{S'\},t). \tag{3.1}$$

By virtue of (2.24), (3.1) takes the following useful form:

$$\rho_n(\{S\}_n;\{\vec{r}\}_n;t) = \prod_{i=1}^{n}[1-\alpha S(\vec{r}_i)^2]^{-3/2} \times \rho_n^0\left(\left\{\frac{S}{(1-\alpha S^2)^{1/2}}\right\}_n;\{\vec{r}\}_n;t\right), \tag{3.2}$$

where $\rho_n^0(\{S\}_n,\{\vec{r}\}_n;t)$ is the n-point distribution function for the linear case $(\alpha=0)$ and is defined by (3.1) with $P(\{S'\},t)$ replaced by $\tilde{P}_t^0\{S'\}$.

We also introduce the n-point correlation function $C_n(\{\vec{r}\}_n,t)$ through

$$C_n(\{\vec{r}\}_n,t) \equiv \int d\{S\}\prod_{i=1}^{n} S(\vec{r}_i)P(\{S\},t) = \prod_{i=1}^{n}\int_{-S_m}^{S_m} dS_i\, S_i\,\rho_n(\{S\}_n,\{\vec{r}\}_n,t). \tag{3.3}$$

Use of (3.2) then yields the following more useful expression for C_n:

$$C_n(\{S\}_n;\{\vec{r}\}_n;t) = \prod_{i=1}^{n}\int_{-\infty}^{\infty} dS_i\,\frac{S_i}{(1+\alpha S_i^2)^{1/2}} \times \rho_n^0(\{S\}_n;\{\vec{r}\}_n;t). \tag{3.4}$$

For the Gaussian initial distribution (2.31), an explicit form of ρ_n^0 can be easily found with the help of the characteristic function defined by

$$E_n^0(\{\xi\}_n,\{\vec{r}\}_n,t) \equiv \left\langle \exp\left(-i\sum_{j=1}^{n}\xi_j S(\vec{r}_j)\right)\right\rangle_t^0, = \left\langle \exp\left(-i\int_{\vec{k}}\tilde{\xi}_{-\vec{k}}S_{\vec{k}}\right)\right\rangle_t^0, \tag{3.5}$$

where $\{\xi\}_n = \xi_1, \xi_2, \ldots, \xi_n$, $\xi_{\vec{k}} \equiv \sum_{j=1}^{n}\xi_j\exp(-i\vec{k}\cdot\vec{r}_j)$, and $\langle\cdots\rangle_t^0$ is an average over $\tilde{P}_t^0\{S\}$, (2.31). The change of integration variable $S_{\vec{k}} \to S_{\vec{k}} - i\xi_{\vec{k}}\tilde{\chi}_{\vec{k}}(t)$ immediately yields

$$E_n^0(\{\xi\}_n, \{\vec{r}\}_n, t) = \exp\left(-\frac{1}{2}\int_{\vec{k}} \tilde{\chi}_{\vec{k}}(t)\tilde{\xi}_{\vec{k}}\tilde{\xi}_{-\vec{k}}\right). \quad (3.6)$$

The n-point distribution function is then

$$\rho_n^0(\{S\}_n, \{\vec{r}\}_n, t) = \frac{1}{(2\pi)^n}\int_{-\infty}^{\infty} d\xi_1 \cdots \int_{-\infty}^{\infty} d\xi_n \exp\left(i\sum_{j=1}^{n} \xi_j S_j\right) \times E_n^0(\{\xi\}_n, \{\vec{r}\}_n, t), \quad (3.7)$$

which after some straightforward algebra takes the following form:

$$\rho_n^0(\{S\}_n, \{\vec{r}\}_n, t) = (2\pi)^{-n/2}|\det A_n|^{-1/2} \times \exp\left(-\frac{1}{2}\sum_{i,j=1}^{n} (A_n^{-1})_{ij} S_i S_j\right). \quad (3.8)$$

Here $A_n(\{\vec{r}\}_n, t)$ is the $n\times n$ matrix whose ij element is the following function of $\{\vec{r}\}_n$ and t:

$$[A_n(\{\vec{r}\}_n, t)]_{ij} \equiv \beta(t)\cos\theta_{ij}, \quad (3.9)$$

with

$$\beta(t) \equiv \int_{\vec{k}} \tilde{\chi}_{\vec{k}}(t), \quad (3.10)$$

$$\cos\theta_{ij} = \beta(t)^{-1}\int_{\vec{k}} \cos[\vec{k}\cdot(\vec{r}_i - \vec{r}_j)]\tilde{\chi}_{\vec{k}}(t). \quad (3.11)$$

For the long times $t \gg 1/2\gamma_0$ these functions become

$$\beta(t) = [\tilde{\chi}_0/(4\pi)^{d/2} l(t)^d] e^{2\gamma_0 t}, \quad (3.12)$$

$$\cos\theta_{ij} = \cos\theta\left(\frac{|\vec{r}_i - \vec{r}_j|}{l(t)}\right), \quad (3.13)$$

with

$$\cos\theta(x) = e^{-x^2/4}, \quad (3.14)$$

$$l(t) \equiv (2Lt)^{1/2}. \quad (3.15)$$

The fact that a spatial coordinate $\vec{r}_i$ always enters in the combination in $\vec{r}_i/l(t)$ indicates that $l(t)$ is the average size of a fluctuating domain.

If we note that L also plays the role of a diffusion constant, this fluctuating domain size $l(t)$ is in fact equal to the distance over which the growing fluctuation at a point in space diffuses away during the time t. The condition $L\kappa^2 t \gg 1$ implies $\kappa^{-1} \ll l(t)$, and the length κ^{-1} which is the range of correlation of thermal fluctuations in the mean-field theory can also be interpreted as a measure of the surface thickness of a fluctuating domain.

We shall see below that $\cos\theta$ is just the two-point (pair) correlation $C_2^0(\vec{r}, t)$ of the linear system ($\alpha = 0$) normalized to unity at $\vec{r} = 0$. Another point to note at this stage is that no short-distance divergence occurs in the problem except for $t = 0$. Both ρ_n and C_n are well-defined for $t > 0$ when the distance between any two points tends to zero. This arises from the fact that $\tilde{\chi}_{\vec{k}}(t)$ for $t > 0$ tends to zero exponentially fast as k increases, thus providing a natural upper cutoff at $l(t)^{-1}$.

Let us now discuss some particular examples of the above.

(i) n = 1

Here we have no dependence on the spatial coordinates and

$$\rho_1^0(S, t) = [2\pi\beta(t)]^{-1/2}\exp[-S^2/2\beta(t)]. \quad (3.16)$$

By (3.2) this yields the following one-point distribution function:

$$\rho_1(S, t) = [2\pi\beta(t)]^{-1/2}(1 - \alpha S^2)^{-3/2} \times \exp\left(-\frac{1}{2\beta(t)}\frac{S^2}{1 - \alpha S^2}\right). \quad (3.17)$$

This is just the Suzuki distribution[5] for a single variable where Suzuki's scaled time τ which we denote as $y(t)$ is now given by

$$y(t) = \alpha\beta(t) = \frac{g\tilde{\chi}_0}{6(4\pi)^{d/2}\kappa^2 l(t)^d} e^{2\gamma_0 t}. \quad (3.18)$$

For short times such that $\beta(t) \ll \alpha^{-1/2}$ (3.17) is basically identical to (3.16). As the time and hence $\beta(t)$ increases the wing of the Gaussian distribution (3.16) with $|S| \lesssim \alpha^{-1/2}$ gets squeezed into the interval $(-S_m, S_m)$ and (3.17) develops the double peak structure as described in Suzuki's work.[5]

(ii) n = 2

Here, ρ_2^0 and ρ_2 depend on spatial coordinates through $r = |\vec{r}_1 - \vec{r}_2|$ and

$$\rho_2^0(S_1, S_2, r, t) = \frac{1}{2\pi\beta(t)\sin\theta}\exp\left\{-\frac{S_1^2 + S_2^2 - 2\cos\theta S_1 S_2}{2\beta(t)\sin^2\theta}\right\}. \quad (3.19)$$

Thus the two-point distribution function is

$$\rho_2(S_1, S_2, r, t) = \frac{1}{2\pi\beta(t)\sin\theta}(1 - \alpha S_1^2)^{-3/2}(1 - \alpha S_2^2)^{-3/2} \times \exp\left[-\frac{1}{2\beta(t)\sin^2\theta}\left(\frac{S_1^2}{1 - \alpha S_1^2} + \frac{S_2^2}{1 - \alpha S_2^2} - \frac{2\cos\theta S_1 S_2}{(1 - \alpha S_1^2)^{1/2}(1 - \alpha S_2^2)^{1/2}}\right)\right]. \quad (3.20)$$

From (3.19) we immediately find the two-point correlation function when $\alpha = 0$:

$$C_2^0(r,t) = \langle S(\vec{r})S(0)\rangle_t^0 = \beta(t)\cos\theta\,, \tag{3.21}$$

which verifies our earlier remark. The two-point correlation function is

$$C_2(r,t) = \frac{1}{2\pi\beta(t)\sin\theta}\int_{-\infty}^{\infty} dS_1 \int_{-\infty}^{\infty} dS_2\, \frac{S_1}{(1+\alpha S_1^2)^{1/2}}\,\frac{S_2}{(1+\alpha S_2^2)^{1/2}}\,\exp\left(-\frac{S_1^2+S_2^2-2\cos\theta S_1 S_2}{2\beta(t)\sin^2\theta}\right). \tag{3.22}$$

At this point it is clear that in fact we are taking the limit $g\to 0$, $t\to\infty$, $|\vec{r}|\to\infty$ such that $y(t)$, (3.18), as well as $x = |\vec{r}|/l(t)$ remain finite.

IV. BEHAVIOR OF THE TWO-POINT DISTRIBUTION AND CORRELATION FUNCTIONS

In this section, we describe the behavior of the two-point distribution and correlation as functions of time and distance. For numerical studies it is convenient to express these functions in dimensionless forms. Thus we use the (time-dependent) dimensionless distance $x = |\vec{r}|/l(t)$ and the dimensionless time $y(t)$, (3.18), where $l(t)$ is given by (3.15). We also introduce the dimensionless local order parameter z by

$$z(x) = \alpha^{+1/2}S(r) = S(r)/S_m\,. \tag{4.1}$$

The dimensionless two-point distribution and correlation functions (for which the same notations ρ_2 and C_2 will be used) then become

$$\rho_2(z_1,z_2,x,y) = \frac{1}{2\pi y\sin\theta(x)}(1-z_1^2)^{-3/2}(1-z_2^2)^{-3/2}\exp\left[-\frac{1}{2y\sin^2\theta(x)}\left(\frac{z_1^2}{1-z_1^2}+\frac{z_2^2}{1-z_2^2}-\frac{2z_1z_2\cos\theta(x)}{(1-z_1^2)^{1/2}(1-z_2^2)^{1/2}}\right)\right] \tag{4.2}$$

and

$$C_2(x,y) = \frac{1}{2\pi y\sin\theta(x)}\int_{-\infty}^{\infty} dz_1\int_{-\infty}^{\infty} dz_2\,\frac{z_1}{(1+z_1^2)^{1/2}}\,\frac{z_2}{(1+z_2^2)^{1/2}}\exp\left(-\frac{z_1^2+z_2^2-2z_1z_2\cos\theta(x)}{2y\sin^2\theta(x)}\right). \tag{4.3}$$

An alternative form of $C_2(x,y)$ is obtained by changing the variables of integration to

$$z_1 = (2y\omega)^{1/2}\sin(\varphi+\tfrac{1}{2}\theta)\,, \tag{4.4a}$$

$$z_2 = (2y\omega)^{1/2}\sin(\varphi-\tfrac{1}{2}\theta)\,, \tag{4.4b}$$

to yield

$$C_2(x,y) = C_2^0(x,y)F(x,y)\,. \tag{4.5}$$

Here,

$$C_2^0(x,y) = y\cos\theta(x) = ye^{-x^2/4} \tag{4.6}$$

is the two-point correlation function when $\alpha = 0$, and $F(x,y)$ is the correction factor due to the nonlinearity given by

$$F(x,y) = (2\pi)^{-1}\int_0^{\infty} d\omega\,\omega e^{-w}\int_0^{2\pi} d\varphi\left(1-\frac{\cos 2\varphi}{\cos\theta(x)}\right)\{1+2y\omega[1-\cos\theta(x)\cos 2\varphi]+(y\omega)^2[\cos 2\varphi-\cos\theta(x)]^2\}^{-1/2}\,. \tag{4.7}$$

The angular integral in (4.7) can be evaluated by first introducing the new variable $t = \sec^2\varphi$ such that

$$F(x,y) = \int_0^{\infty} d\omega\,\omega e^{-w} f(\omega;x,y)\,, \tag{4.8}$$

with

$$f(\omega;x,y) = \frac{1}{\pi[(1+\gamma)+\gamma\cos\theta]}\int_1^{\infty}\frac{(1+1/\cos\theta)-2/(t\cos\theta)}{\{(t-1)[(t-b_1)^2+a_1^2]\}^{1/2}}\,dt\,. \tag{4.9}$$

The variable $\gamma = \omega y$, with

$$b_1 = \frac{2\gamma[(1+\gamma)\cos\theta+\gamma]}{[(1+\gamma)+\gamma\cos\theta]^2}\,, \tag{4.10}$$

$$a_1^2 = \frac{4\gamma^2\sin^2\theta(1+2\gamma)}{[(1+\gamma)+\gamma\cos\theta]^4}\,. \tag{4.11}$$

The integral in (4.9) can be expressed in terms of

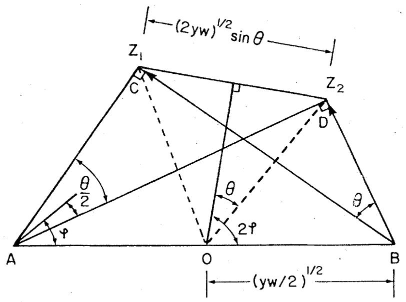

FIG. 3. Geometrical construction of (4.4). The points A, B, C, and D lie on a circle of radius $(\frac{1}{2}yw)^{1/2}$ centered at the point O. The dimensionless order parameters z_1 and z_2 are represented by the vectors $\overrightarrow{BC}$ and $\overrightarrow{BD}$, respectively. The probability distribution of z_1 and z_2 depend only on the size and shape of the triangle OCD but not on its orientation (angle φ).

integrals of Jacobian elliptic functions which can be evaluated to yield[12]

$$f(\omega; x, y) = \frac{2}{\pi[(1+\gamma)^2 - \gamma^2\cos^2\theta]^{1/2}} \times \left\{K(k) + \frac{1+A}{1-A}\,\frac{1}{\cos\theta} \times \left(-K(k) + \Pi[-(1-A)^2/4A, k]\right)\right\}, \quad (4.12)$$

where K and Π are complete elliptic integrals of the first and third kind, respectively, and

$$A = [(1+\gamma) - \gamma\cos\theta]/[(1+\gamma) + \gamma\cos\theta], \quad (4.13)$$

$$k = \gamma\sin\theta/[(1+\gamma)^2 - \gamma^2\cos^2\theta]^{1/2}. \quad (4.14)$$

The remaining integral in (4.8) must in general be evaluated numerically. However, several limiting cases will be discussed first.

In term of the variables ω and φ, the two-point distribution function for $\alpha = 0$ takes a particularly simple form

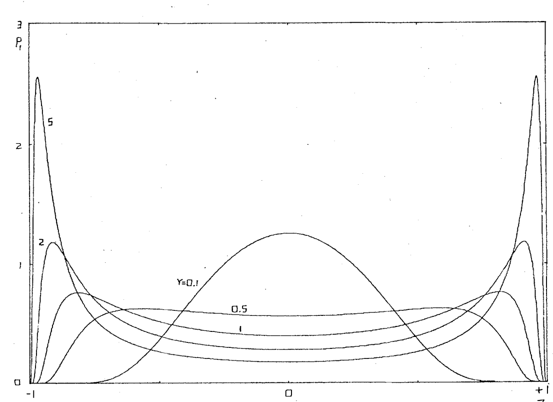

FIG. 4. One-point distribution function $\rho_1(z, y)$ (the Suzuki distribution function).

$$\rho_2^0(\omega, y) = (2\pi)^{-1}e^{-w}. \quad (4.15)$$

A geometrical interpretation of the variables in Eqs. (4.4) is given in Fig. 3, which illuminates the significance of the angle θ as a measure of the correlation. Note that for $\alpha \neq 0$ we obtain the same expression (4.15) for $\rho_2(\omega, \varphi)$ provided that z_i is replaced by $z_i/(1-z_i^2)^{1/2}$ on the left-hand side of (4.4).

The qualitative behavior of ρ_2 and C_2 can be inferred without performing numerical computations.

A. Short-time behavior ($y \ll 1$)

In this case, ρ_2 is peaked in the region $|z_1|$, $|z_2| \lesssim O(y^{1/2})$. Thus, the behavior is almost the same as the linear case ($\alpha = 0$). For short distances ($x \ll 1$), the peak of ρ_2 is further restricted to the region $|z_1 - z_2| \lesssim O(y^{1/2}\sin\theta)$.

B. Long-time long-distance behavior [$y \sim O(1), x \gg 1$]

Here,

$$\rho_2(z_1, z_2, x, y) \cong \rho_1(z_1, y)\rho_1(z_2, y), \quad C_2(x, y) \cong 0, \quad (4.16)$$

where

$$\rho_1(z, y) = (2\pi y)^{-1/2}(1-z^2)^{-3/2} \times \exp[-(1/2y)z^2/(1-z^2)]. \quad (4.17)$$

That is, there is little correlation between z_1 and z_2. As time goes on peaks will develop at the four corners $z_1, z_2 = \pm 1$ and then along the edges $z_1 = \pm 1$ and $z_2 = \pm 1$.

C. Long-time short-distance behavior [$y \sim O(1), x \ll 1$]

Since $\sin\theta \ll 1$, we have

$$\rho_2(z_1, z_2, x, y) \cong [2\pi y\sin\theta(1-\bar{z}^2)^3]^{-1} \times \exp\left(-\frac{(z_1 - z_2)^2}{2y\sin^2\theta(1-\bar{z}^2)^3} - \frac{\bar{z}^2}{2y(1-\bar{z}^2)}\right) \quad (4.18)$$

with $\bar{z} = \frac{1}{2}(z_1 + z_2)$. Here, ρ_2 has large values along the diagonal $z_1 = z_2$. As time increases sharp peaks will develop at the two corners $z_1 = z_2 = \pm 1$. Here $C_2(x, y)$ will be almost equal to the variance of z with respect to $\rho_1(x, y)$, which approaches one as y increases.

D. Long-time limit ($y \gg 1$)

This is the case where the system is divided into domains with $z \cong \pm 1$. It is of some interest to compute $C_2(x, y)$ in this case. Using the expressions (4.5)–(4.7) we see that for $y \gg 1$ the integrand of (4.7) is positive for $\frac{1}{2}\theta < \varphi < \pi - \frac{1}{2}\theta$ and $\frac{1}{2}\pi + \theta < \varphi < 2\pi - \frac{1}{2}\theta$, is negative for $-\frac{1}{2}\theta < \varphi < \frac{1}{2}\theta$ and $\pi - \frac{1}{2}\theta < \varphi < \pi$

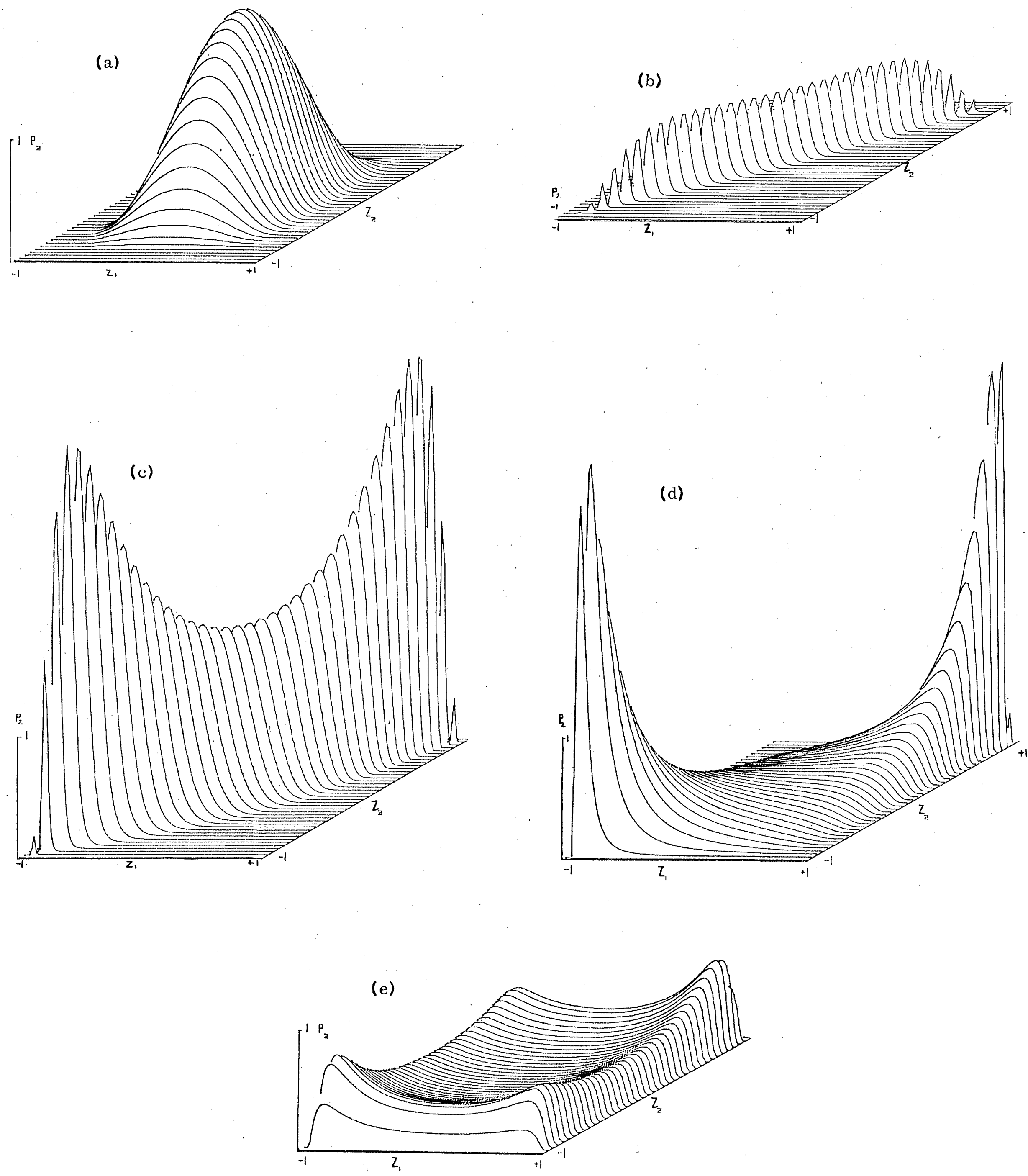

FIG. 5. (a) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,x,y)$ for $x=5$, $y=0.1$. (b) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,0.2,0.2)$. (c) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,0.4,0.4)$. (d) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,1,1)$. (e) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,3,1)$. (f) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,3,3)$. (g) Three-dimensional plot of the two-point distribution function $\rho_2(z_1,z_2,0.5,5)$.

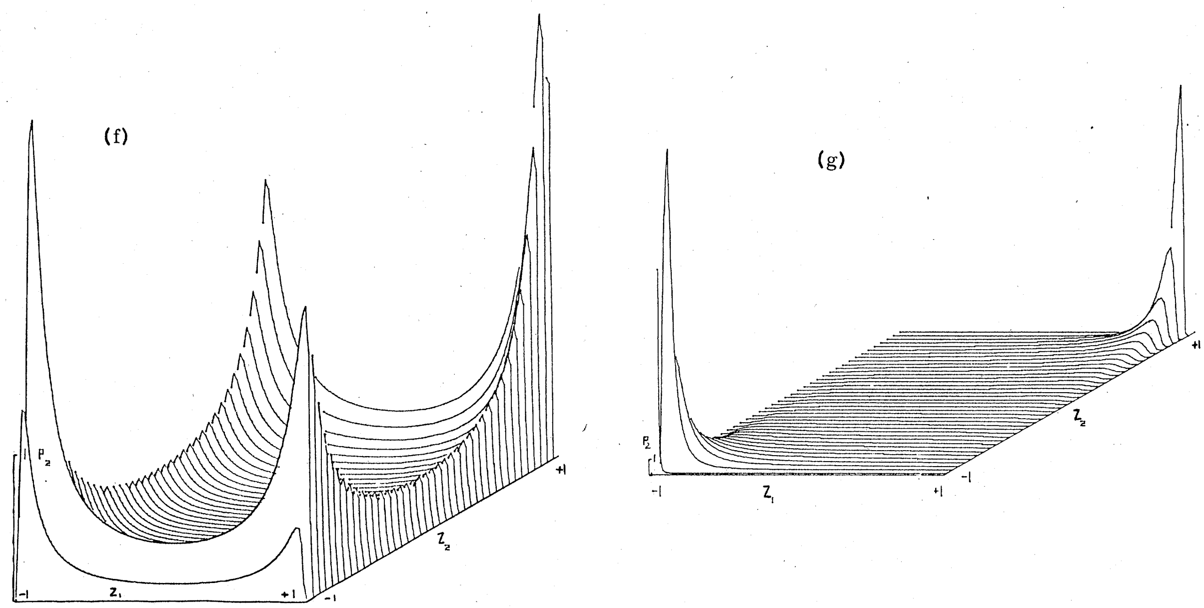

FIG. 5. (*Continued*)

$+\frac{1}{2}\theta$ and is almost zero in the small regions of the width $O(y^{-1/2})$ near $\varphi=\pm\frac{1}{2}\theta$ and $\varphi=\pi+\frac{1}{2}\theta$. Hence, approximate evaluation of the integral leads to[13]

$$C_2(x,y)\cong 1-(2/\pi)\theta(x)\,,\quad y\gg 1\,. \tag{4.19}$$

E. Long-distance limit ($x >> 1$)

Here, we are concerned with the behavior over the distances long compared to a domain size. Since $\cos\theta(x)$ is small we can examine the long-distance tail of $C_2(x,y)$ given by (4.5)–(4.7) by expanding in powers of $\cos\theta(x)$. The leading term is then given by

$$C_2(x,y)\cong B(y)\cos\theta(x)\,, \tag{4.20}$$

where

$$B(y)=\frac{2}{\pi}\int_0^\infty\left(E(k)-\frac{K(k)}{1+\gamma}\right)e^{-w}\,d\omega\,, \tag{4.21}$$

where E is the complete elliptic integral of the second kind and $k\cong\gamma/(1+\gamma)$. For $y\gg 1$, the integral can be approximately evaluated to yield

$$B(y)\cong(2/\pi)\,,\quad y\gg 1\,. \tag{4.22}$$

This result is consistent with (4.19) which reduces for θ near $\frac{1}{2}\pi$ to $(2/\pi)\cos\theta(x)$.

We now display the behavior of the one-point and two-point distribution functions for some typical values of the parameters x and y in Figs. 4–6.

The one-point distribution function has been computed earlier by Suzuki[5] and is shown here for completeness. The behavior of $C_2(x,y)$ is shown in Fig. 7.

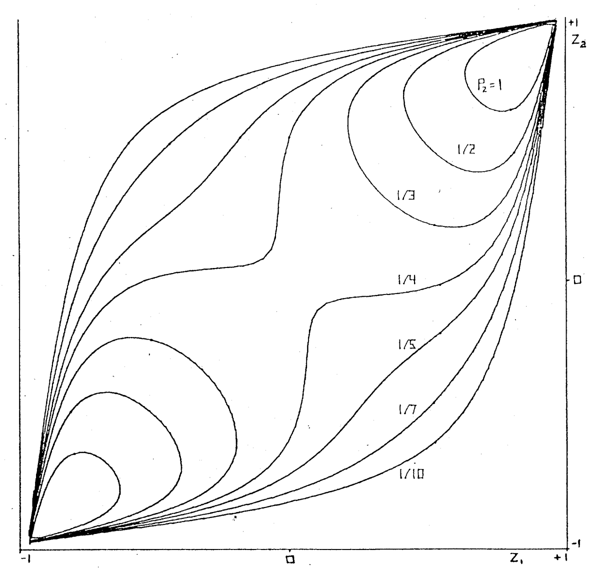

FIG. 6. Contour diagrams of $\rho_2(z_1,z_2,1,1)$ for various values of ρ_2.

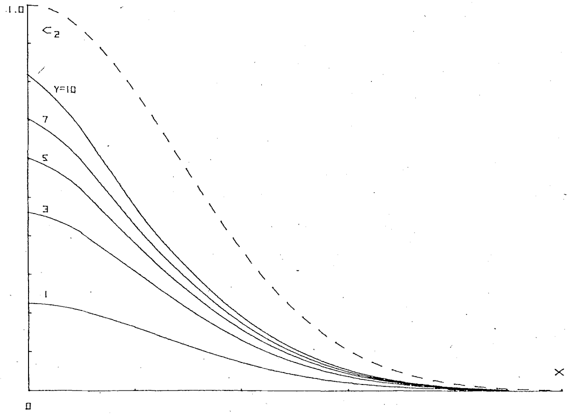

FIG. 7. Graphs of $C_2(x,y)$ for various values of y. The dotted line is $y^{-1}C_2^0(x,y)$. The numbers on the solid lines are the values of y. For large y, $C_2(x,y)$ tends to approach the vertical axis with a finite slope. Indeed, (4.19) yields for $x \ll 1$, $C_2(x,y) \simeq 1-(\sqrt{2}/\pi)x$.

V. GENERALIZED TRANSFORMATION OF VARIABLES

The success of the method presented in this work hinged on the existence of the simple time-independent transformation (2.18). Hence it would be useful to discuss such transformations for more general models. Thus we consider a class of stochastic models of the form (1.9) with

$$c_{\vec q}\{S\} = \gamma_{\vec q} S_{\vec q} + N_{\vec q}\{S\}\,, \tag{5.1}$$

where $\gamma_{\vec q}$ is a function of q which is positive for certain regions of small q in which we are interested and $N_{\vec q}\{S\}$ is nonlinear in $\{S\}$. For instance, one might choose

$$\gamma_{\vec q} = Lq^{2a}(\kappa^2+q^2)\,, \tag{5.2}$$

$$N_{\vec q}\{S\} = -\frac{gL}{6}q^{2a} \times \int_{\vec k_1}\int_{\vec k_2}\int_{\vec k_3} \delta(\vec q-\vec k_1-\vec k_2-\vec k_3) S_{\vec k_1} S_{\vec k_2} S_{\vec k_3}\,, \tag{5.3}$$

where $a=1$ and 0 correspond to the TDGL models with the conserved and nonconserved order parameters, respectively. The deterministic equation of motion $dS_{\vec q}(t)/dt = C_q\{S(t)\}$ with the initial condition $S_q(0)=S_q$ can now be cast into the integral form as

$$S_{\vec q}(t) = S_{\vec q}(0) + \int_0^t dt_1 e^{\gamma_{\vec q}(t-t_1)} N_{\vec q}\{S(t_1)\}\,, \tag{5.4}$$

with

$$S_{\vec q}(t) \equiv e^{t\gamma_{\vec q}} S_{\vec q}\,. \tag{5.5}$$

In the iterative solution of (5.4) we encounter the same type of multiple-time integrals as those that appear in (2.8). Evaluation of these integrals leads to sums of exponentials. The approximation of retaining only the leading terms at long times[14] in each order can be shown to be equivalent to replacing the lower limit of the integral in (5.4) by $-\infty$. In this approximation we have

$$S_{\vec q}(t) = S^0_{\vec q}(t) + \int_0^\infty dt_1 e^{\gamma_{\vec q} t_1} N_{\vec q}\{S(t-t_1)\}\,. \tag{5.6}$$

Viewing this as an equation for $S_{\vec q}(t)$, the explicit time dependence only appears in $S^0_{\vec q}(t)$. Indeed, (5.6) has a solution for $S_{\vec q}(t)=\sigma_{\vec q}$ which is a *time-independent* functional of

$$\sigma^0_{\vec q} = S^0_{\vec q}(t)\,; \quad \sigma_{\vec q} = \sigma_{\vec q}\{\sigma^0_k\}\,.$$

Namely, (5.6) becomes

$$\sigma_{\vec q}\{\sigma^0_{\vec k}\} = \sigma^0_{\vec q} + \int_0^\infty dt_1\, e^{\gamma_{\vec q} t_1} N_{\vec q}\{\sigma(e^{-\gamma_{\vec k} t_1}\sigma^0_{\vec k})\}\,. \tag{5.7}$$

The functional relationship $\sigma\{\sigma^0\}$ obtained by solving (5.7) viewed as a time-independent nonlinear transformation of the variables $\sigma^0 \to \sigma$ is indeed a generalization of (2.26).

The existence of such a time-independent relationship is traced to the fact that (5.6) in fact specifies the initial condition $S_{\vec q}(t) = S^0_{\vec q}(t)$ at $t=-\infty$, and hence, the relationship $\sigma\{\sigma^0\}$ is the result of time evolution from the initial condition during the infinite amount of time starting at $t=-\infty$. However, $\sigma(\vec r,\{\sigma^0\})$ differs from its saturation value $\pm\alpha^{-1/2}$, where

$$\sigma(\vec r,\{\sigma^0\}) = \int_{\vec q} \sigma_{\vec q}\{\sigma^0_{\vec k}\} e^{i\vec q\cdot\vec r}$$

because actually we have specified $S^0_{\vec q}(t)$ to assume a finite value $S_{\vec q}$ at $t=0$. [Indeed, both $S^0_{\vec q}(t)$ and $S_{\vec q}(t)$ become infinitesimal as $t\to-\infty$.] The violation of this true initial condition at $t=0$ is very small in our weak coupling situation as long as $S_{\vec q}$ is not excessively large, and has been ignored.

Now, for the models described by (5.2) and (5.3), we can write (5.7) in terms of the densities

$$\sigma(\vec r,\{\sigma^0\}) = \sigma^0(\vec r) - \frac{gL}{6}(-\nabla^2)^a \times \int_0^\infty dt_1 e^{Dt_1}\sigma(\vec r,\{e^{-Dt_1}\sigma^0\})^3\,, \tag{5.8}$$

where $\sigma^0(\vec r) = \int \sigma^0_{\vec q} e^{i\vec q\cdot\vec r}$ and D is the following differential operator:

$$D \equiv L(-\nabla^2)^a(\kappa^2+\nabla^2)\,.$$

For our models, the linear growth rate has a maximum γ_m at $k=k_m$ where $\gamma_m = L\kappa^2$, $k_m=0$ for $a=0$ and $\gamma_m = \frac{1}{4}L\kappa^4$, $k_m = 2^{-1/2}\kappa$ for $a=1$. We then have $e^{-Dt_1}\sigma^0 \simeq e^{-\gamma_m t_1}\sigma^0$ since $\sigma^0(\vec r) = S^0(\vec r,t)$ has large Fourier components for k near k_m for large enough t. Thus

(5.8) can be further approximated by

$$\sigma(\vec{r},\{\sigma^0\}) = \sigma^0(\vec{r}) - \frac{gL}{6}(-\nabla^2)^a \times \int_0^\infty dt_1 e^{Dt_1}\sigma(\vec{r},\{e^{-\gamma_m t_1}\sigma^0\})^3 . \qquad (5.9)$$

The major contributions to the time integral come from the short times except for the cases where $\sigma^0(\vec{r})$ is excessively large so that $\sigma(\vec{r})$ is nearly equal to its saturation value.

For $a=0$, (5.9) can be further approximated by replacing D by $\gamma_m = \gamma_0$ since the wave numbers involved are small $[\lesssim (2Lt)^{-1/2}]$ and t_1 is small. Thus (5.9) becomes

$$\sigma(\vec{r},\{\sigma^0\}) = \sigma^0(\vec{r}) - \frac{gL}{6} \times \int_0^\infty dt_1 e^{\gamma_0 t_1}\sigma(\vec{r},\{e^{-\gamma_0 t_1}\sigma^0\})^3 . \qquad (5.10)$$

This equation for σ contains $\vec{r}$ only in $\sigma^0(\vec{r})$, and hence $\sigma(\vec{r},\{\sigma^0\})$ can be found in the form of an ordinary function $\sigma(\sigma^0)$. Then (5.10) can be readily converted into the following integral equation for $\sigma(x)$ by changing the variable of integration:

$$\sigma(x) = x\left(1-\alpha\int_0^x dy\,\frac{\sigma(y)^3}{y^2}\right), \qquad (5.11)$$

with α given by (2.13). This equation is easily solved if we note that satisfies the following differential equation:

$$\frac{d}{dx}\frac{\sigma}{x} = -\alpha x\left(\frac{\sigma}{x}\right)^3 \qquad (5.12)$$

with the condition that $\sigma/x \to 1$ as $x\to 0$. Thus we obtain

$$\sigma(x) = x/(1+\alpha x^2)^{1/2}, \qquad (5.13)$$

which is just (2.18).

For $a=1$ (conserved case), we have not succeeded so far in reducing (5.9) further. The difficulty here stems from the fact that the wave vectors at which the initial growth rate reaches maxima form a $d-1$ dimensional spherical shell with $|\vec{k}| = 2^{-1/2}\kappa$ rather than a single point $\vec{k}=0$ as in the case with $a=0$. As a consequence we have to deal with wave vectors which are composed of more than two wave vectors nearly on the shell with $|\vec{k}| = 2^{-1/2}\kappa$ and hence the resultant vectors can be far off the shell.

VI. DISCUSSION

In the preceding sections, we have presented a method to study the enhancement of fluctuations at long times in unstable systems with an infinite number of degrees of freedom. The method is particularly useful when the enhancement of fluctuations occurs in a decreasingly small region around a point in the wave-vector space as in the TDGL model with the order parameter not conserved. This does not mean that the wave-number dependence can be asymptotically dropped (if so, the problem reduces to that of a single degree of freedom such as a laser model). However, the existence of the solution (Suzuki's solution) in the absence of diffusion makes it possible to treat the diffusion effects separately from the nonlinear aspects of the problem.

Although so far in this paper, we have been outside the critical region, it is of some interest to view the problem as an example of nonequilibrium critical phenomena. The first point to note is that even in the mean-field approximation, the problem is far from being trivial. We have here merely provided such a mean-field solution. The next natural step is to try an expansion[15] in $\epsilon = 4-d$ where the corrections to our mean-field results arise from solving (2.1) more precisely and from restoring the second-derivative term $[L\int_{\vec{k}} \delta^2 P(\{S\},t)/\delta S_{\vec{k}}\delta S_{-\vec{k}}]$ to the stochastic equation (1.9).

As an example of nonequilibrium critical phenomena it is interesting to see whether our results conform to the usual dynamical scaling law of critical phenomena.[16] In our model the characteristic length and time are, respectively, κ^{-1} and $\omega^{-1} \equiv (2L\kappa^2)^{-1}$. In Sec. IV the quantities of interest are expressed in terms of the properly scaled order parameter z_i and the dimensionless variables x and $y(t)$, (3.18). In terms of κ and ω we can then write

$$x = \kappa|\vec{r}|/(\omega t)^{1/2}, \qquad (6.1)$$

$$y(t) = \frac{g[1+\kappa^2/\kappa_0^2]}{6(4\pi)^{d/2}\kappa^{4-d}}\frac{e^{\omega t}}{(\omega t)^{1/2}} . \qquad (6.2)$$

If we suppose that the renormalization of the constants in the stochastic model mentioned in Sec. I has been done properly (with the reference wave number chosen to be κ) then the coupling constant g should take the form $g^*\kappa^{4-d}$ where g^* is now the dimensionless coupling constant.[15] Thus, our results conform to dynamical scaling provided that we include the additional length scale κ_0^{-1} which characterizes the correlation range of fluctuations in the initial state.

Now, the fact that we are concerned with the regime $\kappa|\vec{r}|\gg 1$, $\omega t \gg 1$ means that we are well in the hydrodynamic regime where thermal fluctuations are of little importance. The large fluctuations we find are not of a thermal nature although the probability distribution for them is influenced by the thermal fluctuations that existed in the initial state. Thus, here again we see a close parallel between our problem and that of hydrodynamic turbulence[17] where the deterministic equation (2.1) plays the role of the Navier-Stokes equation in turbulence.

Finally, we discuss the range of validity of the present treatment and its possible extension to related problems. There are at least two limitations of our work, namely, the approximations involved in solving the deterministic equation of motion (2.1) and in neglecting the thermal noise. Regarding the former aspect we note that the approximation which leads from (5.8) to (5.10) is valid only when we are far away from complete saturation at time t. Otherwise, $\sigma(\vec{r},\{e^{-Dt_1}\sigma^0\})$ stays almost constant for small $t_1 \lesssim 1/2\gamma_0$. The necessary condition that this does not happen is that the ratio of the quantity $e^{Dt_1}\sigma(\vec{r},\{e^{-Dt_1}\sigma^0\})$ at $t_1=0$ and at $t_1=t$ be much greater than unity. If we replace the value $S^0(\vec{r},t)^2$ that enters by its average value $\beta(t)$ [(3.10)] and use (3.18), this condition takes the following form:

$$e^{2\gamma_0 t}\left(\frac{1+y(0)}{1+y(t)}\right)^{3/2} \gg 1 . \qquad (6.3)$$

Since $y(0)$ is small and of the order of $(g\kappa^{d-4})$, we have[18]

$$e^{2\gamma_0 t} \gg [1+y(t)]^{3/2} . \qquad (6.4)$$

This condition is certainly satisfied over a wide range of t in the weak coupling situation with which we are concerned. In Appendix B, we shall show that essentially a similar constraint is also a sufficient condition.

The second approximation is such that, as was remarked in Sec. I in connection with the laser model, the theory does not correctly describe the final equilibrium state where any deterministic path ends at $\pm S_m$. Here of course thermal fluctuations (which are contained in the second-derivative term dropped in our theory) play a role. The importance of thermal fluctuations in the one-point distribution function can be examined by considering the variance $Y^+(t)$ for the part of the distribution function with positive S which is divided by S_m^2. Thus, from (3.17) we obtain for $y(t) \gg 1$,

$$Y^+(t) \cong [(8/\pi)^{1/2} - (\tfrac{1}{2}\pi)^{1/2}] y(t)^{-1/2} . \qquad (6.5)$$

On the other hand, for the equilibrium distribution function, this variance was obtained in Ref. 9 and is

$$Y_e^+(t) \cong \tfrac{1}{2}\tilde{g}(t) , \qquad (6.6)$$

where

$$\tilde{g}(t) \equiv g/[6(4\pi)^{d/2}\kappa^4 l(t)^d] . \qquad (6.7)$$

Note also that $y(t) = \tilde{g}(t)(1+\kappa^2/\kappa_0^2)e^{2\gamma_0 t}$. Thus our theory will be valid for the times satisfying the condition $Y^+(t) > Y_e^+(t)$. That is,

$$\left(1+\frac{\kappa^2}{\kappa_0^2}\right)^{-1} e^{-2\gamma_0 t}\, y(t)^{3/2} < 2\left[\left(\frac{8}{\pi}\right)^{1/2} - \left(\frac{\pi}{2}\right)^{1/2}\right] . \qquad (6.8)$$

As was noted by Langer[19] and verified by a computer study,[20] thermal fluctuations also become important in those neighborhoods in the function space $\{S(\vec{r})\}$ which satisfy $\delta\Phi/\delta S(\vec{r})=0$, where $\dot{S}$ is zero in the absence of noise. In our problem, this will be the case for the metastable state obtained by adding an external field to the model. Here, it is essential to restore the second-derivative term in the stochastic model equation in order to describe the slow decay of the metastable states towards the equilibrium state. In our approximation scheme, such a process enters as higher-order corrections to our "mean-field" theory. We certainly intend to extend our work to include this interesting case.

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APPENDIX A

Here, we discuss the approximation that led from (2.8) to (2.10), in particular, the following multiple time integral:

$$f_n(\{x\}_n;t) \equiv \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, \exp\left(\sum_{i=1}^n x_i t_i\right), \qquad (A1)$$

where $\{x\}_n = x_1, x_2, \ldots, x_n$ with x_i real and positive. Consider the Laplace transform

$$\hat{f}_n(\{x\}_n;\omega) = \int_0^\infty e^{-\omega t} f_n(\{x\}_n;t)\, dt . \qquad (A2)$$

This is readily evaluated by changing the variables of integration from t_i to $s_i = t_{i-1} - t_i$, $i=1,2,\ldots,n+1$ with $t_0 \equiv t$, $t_{n+1}=0$, and we obtain

$$\hat{f}_n(\{x\}_n;\omega) = \omega^{-1}(\omega - x_1)^{-1}(\omega - x_1 - x_2)^{-1} \times \cdots \left(\omega - \sum_{i=1}^n x_i\right)^{-1} . \qquad (A3)$$

Taking the inverse of the Laplace transformation, we have

$$f_n(\{x\}_n;t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \hat{f}_n(\{x\}_n;\omega) e^{\omega t}\, d\omega \qquad (A4)$$

with $c > \sum_{i=1}^{n} x_i$.

For large t such that $x_i t \gg 1$ for all i, the leading contribution to f_n comes from the largest pole $\sum_{i=1}^{n} x_i$ and we obtain

$$f_n(\{x\}_n; t) \cong \exp\left(t \sum_{i=1}^{n} x_i\right) \Big/ (x_1 + x_2 + \cdots + x_n)(x_2 + x_3 + \cdots + x_n) \cdots (x_{n-1} + x_n)(x_n), \tag{A5}$$

which leads to (2.10).

The error estimate of (2.10) is obtained by noting that the leading correction to (A5) is roughly $\exp(-tx_n)$ times (A5). This implies in (2.8) that three of the wave vectors k's can range up to κ rather than up to $(2Lt)^{-1/2}$. This is however, restricted by the presence of a δ function. Thus the net gain due to the enlarged k space amounts to the factor $\kappa^{2d}(2Lt)^d$. Therefore, the relative correction to (2.10) coming from the correction to (A5) is $\kappa^{2d}(2Lt)^d \exp(-2\gamma_0 t)$.

Also we observe that if the lower limits of the multiple integral (A1) are extended from 0 to $-\infty$, (A1) exactly reduces to the right-hand side of (A5). This fact was used in Sec. V.

Finally we note that an alternative description of our approximation can be obtained by recasting the deterministic differential equation of motion (2.1) in terms of the variable $\tilde{S}^0(\vec{r}, t)$ given by

$$S(\vec{r},t) = \tilde{S}^0(\vec{r},t)/[1 + \alpha \tilde{S}^0(\vec{r},t)^2]^{1/2}. \tag{A6}$$

Then (2.1) becomes

$$\dot{\tilde{S}}^0(\vec{r},t) = L(\kappa^2 + \nabla^2)\tilde{S}^0(\vec{r},t) - \{6L\alpha\tilde{S}^0(\vec{r},t)^2/[1 + 2\alpha\tilde{S}^0(\vec{r},t)^2]\}[\,|\nabla\tilde{S}^0(\vec{r},t)|^2]/\tilde{S}^0(\vec{r},t). \tag{A7}$$

Now since the transformation (A6) is precisely that given in (2.18) and (2.19) with $S^0(\vec{r}, t) = \exp[tL(\kappa^2 + \nabla^2)]S(\vec{r})$ we expect that in some sense the gradient term in (A7) is small. This is obviously the case for small $\alpha\tilde{S}^2{}_0$, since then

$$\frac{6L\alpha\tilde{S}^{02}}{1 + 2\alpha\tilde{S}^{02}} \frac{|\nabla\tilde{S}^0|^2}{\tilde{S}^0} \simeq 6L\alpha\tilde{S}^{02}\frac{|\nabla\tilde{S}^0|^2}{\tilde{S}^0} \ll L\nabla^2\tilde{S}^0 \tag{A8}$$

for $\alpha\tilde{S}^{02} \ll 1$. On the other hand, for very long times such that $\alpha\tilde{S}^{02} = O(1)$ this is no longer the case, since then

$$\frac{6L\alpha\tilde{S}^{02}}{1 + 2\alpha\tilde{S}^{02}} \frac{|\nabla\tilde{S}^0|^2}{\tilde{S}^0} \simeq 6L\frac{|\nabla\tilde{S}^0|^2}{\tilde{S}^0}, \tag{A9}$$

so that the gradient term is then comparable to $L\nabla^2\tilde{S}^0$. However, one can make a heuristic argument to suggest that nevertheless the asymptotic solution of (A7) is the same as $S^0(\vec{r}, t)$, in agreement with our previous derivation. To see this, note that in the region where the gradient term might be important we have

$$\frac{6L\alpha\tilde{S}^{02}}{1 + 2\alpha\tilde{S}^{02}}|\nabla\tilde{S}^0|^2 \sim \frac{6L\alpha\tilde{S}^{02}}{1 + 2\alpha\tilde{S}^{02}}\nabla^2\tilde{S}^0 \equiv LA(t)\nabla^2\tilde{S}^0. \tag{A10}$$

In order to simulate the effect of this gradient term we therefore consider the equation

$$\dot{\tilde{S}}^0 = L[\kappa^2 + \nabla^2 - A(t)\nabla^2]\tilde{S}^0 \tag{A11}$$

with $A(t) = \hat{g}e^{2\gamma_0 t}$ such that $\hat{g} \ll 1$ and $A(t) = O(1)$ for large t. Then the solution of (A11) is

$$\tilde{S}^0(q, t) = \exp\left(L\kappa^2 t - Lq^2 t + Lq^2 \int_0^t A(s)ds\right)\tilde{S}^0(q, 0). \tag{A12}$$

But,

$$\int_0^t A(s)ds = (\hat{g}/2\gamma_0)(e^{2\gamma_0 t} - 1) \simeq \hat{g}e^{2\gamma_0 t}/2\gamma_0 = A(t)/2\gamma_0 \ll t. \tag{A13}$$

Therefore the effect of the last term in (A11) and correspondingly the last term in (A7) has only a small effect on $S(\vec{r}, t)$.

APPENDIX B

In order to examine the validity of the approximation that led from (5.8) to (5.10) let us consider the integrand

$$\varphi(\vec{r}, t, t_1) \equiv -\frac{gL}{6}e^{\gamma_0 t_1}\sigma(e^{-\gamma_0 t_1}\sigma^0)^3 = -\frac{gL}{6}e^{\gamma_0 t_1}\frac{\exp[3\gamma_0(t - t_1)]S(\vec{r})^3}{\{1 + \alpha\exp[2\gamma_0(t - t_1)]S(\vec{r})^2\}^{3/2}}. \tag{B1}$$

Consider the ratio

$$R(\vec{r}, t, t_1) \equiv \frac{\varphi(\vec{r}, t, t_1)}{\varphi(\vec{r}, t, 0)} = e^{-2\gamma_0 t_1}\left(\frac{1 + \alpha e^{2\gamma_0 t}S(\vec{r})^2}{1 + \alpha\exp[2\gamma_0(t - t_1)]S(\vec{r})^2}\right)^{3/2}. \tag{B2}$$

In order for our approximation to be valid we must require that this quantity decreases substantially within a time which is much shorter than t. Namely, defining t_1 by

$$R(\vec{r}, t, t_1) = \eta \,, \tag{B3}$$

where η is a number less than unity, say $\frac{1}{2}$, we require

$$t_1 \ll t \,. \tag{B4}$$

We can estimate (B2) by replacing γ_0 inside the square bracket of (B2) by the differential operator $D(\vec{r})$, (S, q), and then replacing $S(\vec{r})^2$ by its average over the initial distribution. Thus (B3) becomes

$$\left(\frac{1+y(t)}{1+y(t-t_1)}\right)^{3/2} e^{-2\gamma_0 t_1} = \eta \,. \tag{B5}$$

The most dangerous case is when we are close to the complete saturation, that is, $y(t) \gg 1$. Then we have

$$y(t) \cong \eta^{2/3} e^{4\gamma_0 t_1/3}[1 + y(t - t_1)] \,. \tag{B6}$$

$y(t - t_1)$ can be consistently ignored since using $y(t) \simeq \eta^{3/2} e^{4\gamma_0 t_1/3}$ and $y(t - t_1) \simeq y(t) e^{-2\gamma_0 t_1}$ we have $y(t - t_1) \simeq \eta y(t)^{-1/2}$, which is small. Thus the condition (B4) takes the form

$$\gamma_0 t \gg \tfrac{3}{4} \ln[y(t)/\eta^{3/2}] \,. \tag{B7}$$

By choosing α to be sufficiently small, (B7) can be satisfied over a wide range of t for any fixed number η less than one even near saturation. Since in (5.8), $Lk^2 t_1 \simeq t_1/t$, our approximation of replacing D by γ_0 in (5.8) is thus justified.

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[11]Needless to say simply changing the sign of t in (2.18) and (2.19) leads to quite different results.

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$$\langle |z_{\vec{k}}|^2\rangle^{\text{th}} = (g/6\kappa^2)\langle |S_{\vec{k}}|^2\rangle^{\text{th}} \sim O(g/\kappa^4)$$

as

$$C_2^{\text{th}}(x) \simeq \int_{\vec{k}}^{k<[xl(t)]^{-1}} \langle |z_{\vec{k}}|^2\rangle^{\text{th}} \sim \frac{g}{l(t)^d \kappa^4 x^d} \,.$$

For short distances $x \ll 1$, this implies the condition $1 > g/l(t)^d \kappa^4 x^d$ If we take $g \sim \kappa^{4-d}$ (see Sec. VI), this condition reduces to $x > 1/(\kappa l(t))$ or $\kappa r > 1$. Since fluctuations with wave numbers greater than κ^{-1} have been excluded already at the beginning, this condition is automatically satisfied. For long distances the condition becomes $e^{x^2/4d}\kappa r > 1$.

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