PHYSICAL REVIEW A VOLUME 17, NUMBER 1 JANUARY 1978

Bound states of the potential $V(r) = -Z/(r+\beta)$

C. H. Mehta
Computer Division, Mathematics Group, Institute of Petroleum Exploration, ONGC, Dehradum 248195, India

S. H. Patil
Department of Physics, Indian Institute of Technology, Bombay-400 076, India
(Received 19 April 1977)

We have studied the bound states of the potential $V(r) = -Z/(r+\beta)$. The S-wave energy level $E(\beta)$, as a function of the parameter β, has a logarithmic singularity at the origin. The imaginary part of $E(\beta)$ for $\beta \to 0$ goes as $Z^{4+2l}\beta^{2+2l}$, where l is the angular momentum.

I. INTRODUCTION

The Coulomb potential $1/r$ has been studied extensively in the literature on quantum mechanics. It is one of the few potentials which allows exact closed solutions for the energy eigenvalues and their wave functions. Many of its interesting properties, such as the well-known "accidental" degeneracy of states and their O(4) group structure, are associated with this potential only and are intimately connected with the singularity of the potential at the origin.

In the context of quantum field theory, however, it is precisely the singularity at $r=0$ (or more generally on the light-cone) which is the crux of divergence difficulties. Indeed, it has been a suggestion of long standing[1] that if gravitational interactions of elementary particles are taken into account the singularity would shift away from the light-cone. In other words, there would be a gravitational cutoff of Coulomb interactions resulting in a finite theory of quantum fields. A nonrelativistic expression of this idea is provided by the potential

$$V(r) = -Z/(r+\beta)\,, \quad \beta > 0\,. \tag{1}$$

This potential may also serve as an approximation to the potential due to a smeared charge rather than a point charge, and may be a pertinent potential for the description of mesonic atoms.

The purpose of this paper is to study the behavior of the bound-state energy levels in the modified Coulomb potential (1), as a function of parameter β. In particular, we study the S-wave bound states which are the only states we have been able to analyze rigorously.

The eigenvalue relation for the S-wave bound states is obtained in Sec. II by imposing the usual boundary conditions on the wave functions at the origin and at infinity. It is found that the energy $E(\beta)$ has a logarithmic branch-cut at $\beta=0$ which precludes a strict perturbation series for $E(\beta)$ around $\beta=0$. In Sec. III the same problem is studied using dispersion theory. The analyticity properties of $E(\beta)$ admit a dispersion integral representation of $E(\beta)$. We derive a lemma which relates the imaginary part of $E(\beta)$ to the wave function at the singular point $r=-\beta$. In the limit of $\beta \to 0$, the wave function is given by the wave function for the corresponding Coulomb potential. By performing the dispersion integral in the limit of $\beta \to 0$, we retrieve the exact results of Sec. II.

The agreement (in the $\beta \to 0$ limit) between the exact solution and the result of the dispersion theory gives support to the analysis of the singularity structure near $\beta \simeq 0$ by the use of dispersion theory. This has been done by Bender and Wu[2] for the anharmonic oscillator. It can also be used for the evaluation of energy levels where the exact results are not easily available and where the perturbation series is not admissible.[3]

II. $\beta > 0$ AND EIGENVALUE EQUATION FOR S-WAVE BOUND STATES

A. Some general properties

We work in units with $2m = \hbar = 1$. For $\beta > 0$, the Hamiltonian

$$H(Z,\beta) = p^2 - \frac{Z}{r+\beta} + \frac{l(l+1)}{r^2} \tag{2}$$

satisfies

$$H(Z,\beta) > H(Z,0)$$

so that the energy levels satisfy the inequality

$$E_{nl}(Z,\beta) > E_{nl}(Z,0)\,. \tag{3}$$

Furthermore, since p^2 is a positive definite operator, the bound states also satisfy

$$E_{nl}(Z,\beta) > -Z/\beta\,. \tag{4}$$

Differentiating the Schrödinger equation

$$H(\beta)\,|\psi(\beta)\rangle = E(\beta)\,|\psi(\beta)\rangle \tag{5}$$

with respect to β, we get

$$\frac{\partial H}{\partial \beta}\left|\psi(\beta)\right\rangle + H(\beta)\frac{\partial}{\partial \beta}\left|\psi(\beta)\right\rangle = \frac{\partial E}{\partial \beta}\left|\psi(\beta)\right\rangle + E(\beta)\frac{\partial}{\partial \beta}\left|\psi(\beta)\right\rangle . \tag{6}$$

Taking the scalar product with $|\psi(\beta)\rangle$ and using the Hermiticity of $H(\beta)$ for $\beta > 0$, we get

$$\frac{\partial E}{\partial \beta} = \frac{Z\langle\psi(\beta)|1/(r+\beta)^2|\psi(\beta)\rangle}{\langle\psi(\beta)|\psi(\beta)\rangle} > 0 . \tag{7}$$

It follows from the last equation that the bound-state energy levels are continuous functions of β for $\beta > 0$. One can obtain similarly,

$$\frac{\partial E}{\partial Z} = -\frac{\langle\psi(\beta)|1/(r+\beta)|\psi(\beta)\rangle}{\langle\psi(\beta)|\psi(\beta)\rangle} < 0 . \tag{8}$$

We now subject our system to the Symanzik scale transformation[4]

$$p \to \alpha p , \quad r \to r/\alpha , \quad \alpha > 0$$

for which the corresponding unitary transformation takes the Hamiltonian (2) into

$$H \to H' = \alpha^2\left(p^2 - \frac{Z/\alpha}{r+\alpha\beta} + \frac{l(l+1)}{r^2}\right).$$

It follows therefore that the energy levels $E_{nl}(Z,\beta)$ satisfy the property

$$\alpha^2 E(Z/\alpha, \alpha\beta) = E(Z,\beta) . \tag{9}$$

Setting $\alpha = \beta$ and $\alpha = 1/\beta$ gives two relations

$$E(Z,\beta) = \beta^2 E(Z/\beta, \beta^2) \tag{10}$$

and

$$E(Z,\beta) = (1/\beta^2)E(Z\beta, 1) . \tag{11}$$

The last two relations will be useful in determining the asymptotic behavior of $E(\beta)$ as $\beta \to +\infty$.

B. The eigenvalue condition for S-wave

For $l = 0$ and $\beta > 0$, the wave function for a bound state with energy E satisfies the Schrödinger equation

$$\frac{d^2\chi}{dr^2} + \left(E + \frac{Z}{r+\beta}\right)\chi = 0 , \tag{12}$$

where $\chi(r) = rR(r)$, $R(r)$ being the radial wave function. In terms of the variable

$$y = 2\sqrt{-E}(r+\beta) ,$$

we obtain Whittaker's equation[5]

$$\frac{d^2\chi}{dy^2} + \left(-\frac{1}{4} + \frac{z}{2y\sqrt{-E}}\right)\chi = 0 \tag{13}$$

whose general solution is

$$\chi = ye^{-y/2}[AM(1 - z/2\sqrt{-E}, 2, y) + BU(1 - z/2\sqrt{-E}, 2, y)] , \tag{14}$$

where M and U are the confluent hypergeometric functions[5]:

$$M(a,b,x) = 1 + \frac{a}{b}\frac{x}{1!} + \frac{a(a+1)x^2}{b(b+1)2!} + \cdots \tag{15}$$

and

$$U(a,2,x) = \frac{1}{\Gamma(a-1)}\left(M(a,2,x)\ \ln x + \sum_{r=1}^{\infty}\frac{(a)_r x^r}{(2)_r r!}\ [\psi(a+r) - \psi(1+r) - \psi(2+r)]\right) + \frac{1}{\Gamma(a)x} \tag{16}$$

with

$$(a)_r = a(a+1)\cdots(a+r-1), \quad (a)_0 = 1 ,$$

$$\psi(a) = \frac{1}{\Gamma(a)}\frac{d\Gamma(a)}{da} .$$

For $y \to \infty$, the asymptotic behavior of the confluent hypergeometric functions gives for $\chi(y)$,

$$\chi(y) \underset{y\to\infty}{\sim} ye^{-y/2}\left(\frac{Ae^y y^{(-1+z/2\sqrt{-E})}}{\Gamma(1 - Z/2\sqrt{-E})} + By^{-1+Z/2\sqrt{-E}}\right). \tag{17}$$

For χ to describe a bound state we require $\chi(\infty) = 0$. Therefore we must have either

$$A = 0 \tag{18a}$$

or

$$1 - Z/2\sqrt{-E} = -n , \quad n = 0, 1, 2, \ldots . \tag{18b}$$

However, if the condition (18b) holds, then the two solutions in terms of M and U functions become linearly dependent and we get

$$\chi = ye^{-y/2}\left(\frac{A}{n!\,(n+1)} + B(-1)^n\right)L_n^1(y) , \tag{19}$$

where $L_n^1(y)$ is the associated Laguerre polynomial.[4] Further, since the radial wave function $R(r)$ has to be finite at the origin, we require $\chi(0) = 0$ giving

$$L_n^1(2\beta\sqrt{-E}) = 0 . \tag{20}$$

This last condition (20) is, in general, inconsistent with the condition (18b), and therefore we must have $A = 0$. The solution then becomes

$$\chi = Bye^{-y/2}U(1 - Z/2\sqrt{-E}, 2, y) \tag{21}$$

and $\chi(0) = 0$ yields the eigenvalue equation

$$U(1-Z/2\sqrt{-E},2,2\beta\sqrt{-E})=0\,,\quad \beta>0\,. \tag{22}$$

C. $\beta\to+0$ limit

We can obtain the S-wave energy levels in the limit of $\beta\to+0$ by using Eqs. (22) and (16). We expect that for β very small and positive, $E(\beta)$ would be only slightly different from Coulomb value $E(0)$, and therefore write

$$a=1-Z/2\sqrt{-E}=-(n-1)+\delta(\beta)\,,\quad n=1,2,\ldots, \tag{23}$$

where $\delta(\beta)\to0$ as $\beta\to0+$.

The logarithmic derivative of $\Gamma(a)$ appearing in Eq. (16) may be approximated by

$$\psi(a)\to-1/\delta \quad \text{for } a\to-(n-1)$$

and

$$\psi(a+1)\to-1/\delta+1/(-n+1+\delta)\,. \tag{24}$$

Therefore to order β, we obtain

$$-\frac{1}{Z}-\frac{\beta}{\delta}+\beta\ln\beta=0 \tag{25}$$

or equivalently,

$$\delta=-\beta Z-Z^2\beta^2\ln\beta+O(\beta^3)\,. \tag{26}$$

Substituting Eq. (26) into (23), and solving for E gives

$$E_n=-\frac{Z^2}{4n^2}\left(1-\frac{2\beta Z}{n}-\frac{2\beta^2Z^2\ln\beta}{n}\right)+O(\beta^3)\,. \tag{27}$$

The expression (27) for the S-wave energy levels exhibits logarithmic branch point at $\beta=0$ with a branch-cut along the negative real axis of β. Since $E(\beta)$ has a singularity at $\beta=0$, perturbation series for $E(\beta)$ around $\beta=0$ is not strictly possible. This result is similar to that of Isham *et al.*[1]; the electron's self mass as a function of the gravitational coupling k has a singularity at $k=0$.

III. $\beta<0$ AND THE DISPERSION RELATION FOR $E(\beta)$

A. Imaginary part of $E(\beta)$

In the last section we showed that for $\beta<0$, $E(\beta)$ has a logarithmic cut at the origin. This means that $E(\beta)$ has an imaginary part and the states are metastable. To calculate the imaginary part of $E(\beta)$ however, it is not always necessary to solve the full eigenvalue problem, and in fact for the singular potentials of the type of Eq. (1) it proves expedient to use the following lemma:

Lemma: Let

$$H=H_0-\frac{Z}{r+\beta\pm i\epsilon}\,,\quad \beta \text{ real.} \tag{28}$$

If H has bounded normalized eigenfunction $|\beta\pm i\epsilon\rangle$ and if H_0 is Hermitian, then

$$E(\beta+i\epsilon)-E(\beta-i\epsilon)=8i\pi^2Z\beta^2|R(\beta,r=-\beta)|^2 \tag{29}$$

provided $E(\beta)$ is a real analytic function of β.

Proof: From the definition we have

$$\langle\beta-i\epsilon|H(\beta-i\epsilon)|\beta-i\epsilon\rangle=E(\beta-i\epsilon)\langle\beta-i\epsilon|\beta-i\epsilon\rangle\,. \tag{30}$$

On subtracting from Eq. (30) its adjoint and using the relation

$$H^+(\beta-i\epsilon)=H(\beta+i\epsilon)\,, \tag{31}$$

we get

$$[E^*(\beta-i\epsilon)-E(\beta-i\epsilon)]\langle\beta-i\epsilon|\beta-i\epsilon\rangle =2i\pi Z\langle\beta-i\epsilon|\delta(r+\beta)|\beta-i\epsilon\rangle\,.$$

Real analyticity of $E(\beta)$ and unit normalization of states gives the desired equation (29).

For the potential of Eq. (1), we may obtain the imaginary part of $E(\beta)$ in the limit of $\beta\to0-$ by using result (29) and the normalized hydrogen-atom wave functions for the exact wave functions, which is justified in the limit of $\beta\to0-$. This immediately gives for the S-wave bound states,

$$\operatorname{Im}E_{n0}(\beta)\underset{\beta\to0}{\sim}\frac{\pi Z^4\beta^2}{2n^3} \tag{32}$$

in agreement with the imaginary part of Eq. (27).

A similar procedure for the higher angular momentum states gives

$$\operatorname{Im}E_{nl}\underset{\beta\to0}{\sim}Z^{4+2l}\beta^{2+2l}\,. \tag{33}$$

B. Dispersion relation for $E(\beta)$

We now write a dispersion relation for $E(\beta)$. We restrict ourselves to the negative energy bound states with $l=0$, and assume that $E(\beta)$ has no singularity in the complex β-plane cut along the negative real axis. Since we are assuming that there is no essential singularity at infinity, it is sufficient to determine the asymptotic behavior of $E(\beta)$ along the positive real axis. If the leading term in $E(\beta)$ is β^{-p} as $\beta\to\infty$ then from Eq. (4) of Sec. II, $p\geq1$. Furthermore, as $\beta\to\infty$

$$|E(Z/\beta,\beta^2)|<|E(Z,\beta^2)| \tag{34}$$

which follows from Eq. (8). Therefore from Eq. (10) one has

$$|E(Z,\beta)|<\beta^2|E(Z,\beta^2)| \tag{35}$$

which implies that $|E(Z,\beta)|\geq1/\beta^2$. Hence, we obtain for β going to infinity,

$$1/\beta\geq|E(Z,\beta)|\geq1/\beta^2\,. \tag{36}$$

The asymptotic behavior (36) admits a dispersion relation for $E(\beta)$ without any subtraction,

$$E(\beta)=\frac{1}{\pi}\int_{-\infty}^{0}\frac{\operatorname{Im} E(\beta')\,d\beta'}{\beta'-\beta}. \qquad (37)$$

If the imaginary part of $E(\beta')$ were exactly known for all β', then Eq. (37) would complete our knowledge of $E(\beta)$. Since we know the imaginary part of $E(\beta')$ only for $\beta'\to 0$, we use the dispersion relation for evaluating $E(\beta)$ only for $\beta\to 0$. For this purpose, however, it is more expedient to write a twice-subtracted relation

$$E(\beta)=-\frac{Z^2}{4n^2}+\frac{\beta Z^3}{2n^3}+\frac{\beta^2}{\pi}\int_{-\infty}^{0}\frac{\operatorname{Im} E(\beta')\,d\beta'}{(\beta'-\beta)\beta'^2}, \qquad (38)$$

where the first term is the unperturbed value $E(0)$ and the second term is the first-order perturbation correction to $E(0)$. Using Eq. (32) for the small part of the integral in (38), we get

$$E(\beta)_{\beta\to 0}=-\frac{Z^2}{4n^2}\left(1-\frac{2\beta Z}{n}-\frac{2\beta^2 Z^2\ln\beta}{n}\right)+O(\beta^3) \qquad (39)$$

in agreement with the exact result of Sec. II.

The agreement (in the $\beta\to 0$ limit) between the exact solution and the result of dispersion theory instills confidence in the use of dispersion theory for the analysis of singularities near $\beta=0$. This may be the reason why the Bender and Wu[2] analysis in terms of dispersion relations, of the singularities of the energy levels of anharmonic oscillator gives the correct singularity structure. It also provides some justification for the use of dispersion techniques for practical evaluation of energy levels where either the exact results are not easily obtainable or where the perturbation series is not strictly admissible. For instance, for screened Coulomb potential[3]

$$V(r)=-\frac{Z}{r}+\frac{Z-1}{r+\beta}$$

the energy levels again have singularity at $\beta=0$ and the perturbation series in β for $E(\beta)$ is not admissible. The dispersion theory, on the other hand, gives results which are in very good agreement with experiments.

IV. SUMMARY

We have discussed in this paper the bound states of the potential $-Z/(r+\beta)$. We find that the energy level as a function of the parameter β has a logarithmic singularity at the origin $\beta=0$. The imaginary part of $E(\beta)$ for the negative energy states for β going to zero goes as $Z^{4+2l}\beta^{2+2l}$, where l is the angular momentum. We have not discussed the positive energy states in this paper, but we expect that for β negative the positive energy states are all quasistationary. It would be interesting to construct the S matrix for this problem and trace its singularities as a function of β.

[1] L. Landau, in *Niels Bohr and the Development of Physics* (Pergamon, London); O. Klein, *ibid.* W. Pauli, Helv. Phys. Acta Suppl. 4, 69 (1956); S. Deser, Rev. Mod. Phys. 29, 417 (1957); C. J. Isham, A. Salam, and J. Strathdee, Phys. Rev. D 3, 1805 (1971); 5, 2548 (1972).
[2] C. M. Bender and T. T. Wu, Phys. Rev. 184, 1231 (1969).
[3] C. H. Mehta and S. H. Patil, preceding paper, Phys. Rev. A 17, 34 (1978).
[4] B. Simon, Ann. Phys. (N.Y.) 58, 79 (1970).
[5] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1968), p. 505.