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Connection between perturbation theory, projection-operator techniques, and statistical linearization for nonlinear systems

Aaron B. Budgor
Lawrence Livermore Laboratory, University of California, Livermore, California 94550

Bruce J. West
Physical Dynamics Inc., La Jolla, California

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We employ the equivalence between Zwanzig's projection-operator formalism and perturbation theory to demonstrate that the approximate-solution technique of statistical linearization for nonlinear stochastic differential equations corresponds to the lowest-order β truncation in both the consolidated perturbation expansions and in the "mass operator" of a renormalized Green's function equation. Other consolidated equations can be obtained by selectively modifying this mass operator. We particularize the results of this paper to the Duffing anharmonic oscillator equation.

I. INTRODUCTION

In the past few years there have appeared a number of allied articles[1-6] discussing approximate solutions to the Duffing anharmonic oscillator equation perturbed by Gaussian white noise $\xi(t)$,

$$\ddot{x}+\alpha\dot{x}+\omega_0^2 x+\beta x^3=\xi(t)\,. \tag{1}$$

The parent article by Morton and Corrsin[1(a)] examined this problem by making a perturbation expansion in the response variable $x(t)$, thereby converting the nonlinear ordinary differential equation to an infinite set of linear ordinary differential equations. Each of these latter linear equations corresponded to a certain order in the perturbation expansion parameter β. By assigning diagrams to various terms in the Fourier expansion of these perturbative equations, the ensuing infinite hierarchy of algebraic relations became equivalent to an infinite set of diagrammatic equations. Morton and Corrsin were able to consolidate these diagrams to obtain a set of three coupled nonlinear integral equations for the solution of the spectral density function $S(\omega)$ in terms of a renormalized Green's function $G(\omega)$ and a renormalized vertex function $\Gamma(\omega)$.[1(b)] In field theoretic terms $G(\omega)$ would be a dressed propagator and $\Gamma(\omega)$ a dressed mass. Morton and Corrsin truncated these consolidated equations to various orders and compared the resulting numerical calculations with the results of an analog computer experiment.

Bixon and Zwanzig[2] tackled the problem via the projection operator methods of statistical mechanics and found that their spectral-density calculation agreed favorably with the best of the consolidated equation approximations of Morton and Corrsin. The spectral-density function in their formulation is a simple algebraic function of the frequency. Both oscillator frequency and friction coefficient are renormalized and depend on the nonlinear part of the original equation.

Finally, Budgor *et al.*[3] showed that by using a variational prescription, a number of the integral equations derived by Morton and Corrsin could be obtained in an exceedingly simple fashion.

The purpose of this paper is to tie together these three seemingly different techniques and review their interrelationships. We begin in Secs. II and III by converting, using Hamilton's equations, a general nonlinear stochastic dynamical equation into a Liouville equation for the probability-distribution function $\rho(t)$. The equivalence between Zwanzig's projection formulas and an algebraic perturbation theory is discussed. Statistical linearization is shown to be related to the "lowest"-order β truncation in the consolidated perturbation equations and in the mass operator of Dyson's equation. Other consolidated equations can be found by using a consolidation operator Q which can be defined to selectively modify the mass operator M in Dyson's equation and in effect produce consolidated equations having quite general properties. We exemplify this discussion by applying the results to the Duffing oscillator in Sec. IV. Section V concludes our review of perturbation methods with a brief description of the theory of Martin, Siggia, and Rose[4] on the type and structure of the diagrams that go into the renormalized propagator, renormalized vertex operator, and renormalized spectral density function. Section VI concludes with some general considerations on the utility of the method of statistical linearization.

II. PROJECTION OPERATOR FORMALISM

It is well known that any dynamical system of the form

$$\dot{\underline{u}} = \underline{F}(\underline{u}, t), \tag{2.1}$$

where $\underline{u}$ is a $2n$-dimensional stochastic vector and $\underline{F}(\underline{u}, t)$ is a *nonlinear* vector function, can also be represented by the *linear* partial differential equation

$$(\partial\rho/\partial t)(\underline{u}, t) = L\rho(\underline{u}, t). \tag{2.2}$$

In classical statistical mechanics, L is the Liouville operator and $\rho(\underline{u}, t)$ is the n-particle distribution function for the system in the phase space $(\underline{u}_1, \ldots, \underline{u}_n) = (\underline{x}_1, \underline{p}_1; \ldots; \underline{x}_n, \underline{p}_n)$. In the theory of stochastic differential equations $\rho(\underline{u}, t)$ is the joint distribution function of the canonical random position and momentum vectors $(\underline{x}_1, \ldots, \underline{x}_n)$ and $(\underline{p}_1, \ldots, \underline{p}_n)$ respectively, and L can be, for example, the Fokker-Planck diffusion operator.[2] Equation (2.2) is solved with the initial condition

$$\rho(\underline{u}, t=0) = \prod_1^n \delta[\underline{x}_i - \underline{x}_i(0)]\delta[\underline{p}_i - \underline{p}_i(0)], \tag{2.3}$$

where $\underline{x}(0)$ and $\underline{p}(0)$ represent the initial positions, and momenta, respectively. The Liouville operator L is defined in terms of the Poisson bracket relation

$$L = \sum_1^n \left(\frac{\partial H}{\partial \underline{p}_i} \cdot \frac{\partial}{\partial \underline{x}_i} - \frac{\partial H}{\partial \underline{x}_i} \cdot \frac{\partial}{\partial \underline{p}_i} \right) \tag{2.4}$$

in which $H(\underline{x}, \underline{p}, t)$ is, in general, the time-dependent Hamiltonian for the dynamical system (2.1).

Equation (2.2), together with the initial condition given by (2.3), is the starting point for a projection-operator formalism introduced into statistical mechanics by Zwanzig.[5] There is also a completely equivalent perturbation-expansion technique for solving (2.1) which is applicable for quite general polynomial functions $\underline{F}(\underline{u}, t)$. In practical calculations the quantities of interest are the correlation functions and spectral densities obtained from a perturbation analysis of (2.1). Since these quantities are identical to those obtained from a comparable perturbation analysis of (2.2), we shall, for convenience, restrict our discussion to (2.2).

Given a Hilbert space $\mathcal{H}$ we decompose the Liouville Eq. (2.2) by defining projection operators P and $I-P$ with P acting on $P\mathcal{H}$ such that $P\rho \equiv \rho_0$ and $I-P$ acting on the orthogonal subspace $(I-P)\mathcal{H}$ such that $(I-P)\rho \equiv \rho_1$. We further stipulate that for an arbitrary phase function A,

$$PLPA = PL_0A = L_0PA. \tag{2.5}$$

Equation (2.5) requires that the Liouville operator L separate into two parts,

$$L = L_0 + L_1, \tag{2.6}$$

with L_1 satisfying the relations

$$\begin{aligned} &PL_1PA = 0, \quad PL_1A = PL(I-P)A, \\ &L_1PA = (I-P)LPA. \end{aligned} \tag{2.7}$$

In deriving Eqs. (2.7), we have utilized the fact that P is idempotent, $P^2 = P$. It is thus clear that L_0 is diagonal in the subspace $P\mathcal{H}$, i.e., that L_0 and P commute, and that L_1 connects $P\mathcal{H}$ with $(I-P)\mathcal{H}$.

The equations of evolution for ρ_0 and ρ_1 as obtained from (2.2) are

$$\frac{\partial\rho_0}{\partial t} = PL\rho_0 + PL\rho_1, \tag{2.8a}$$

$$\frac{\partial\rho_1}{\partial t} = (I-P)L\rho_0 + (I-P)L\rho_1. \tag{2.8b}$$

If for simplicity we assume L_1 to be time independent, (2.8b) integrates directly to

$$\rho_1(t) = \int_0^t d\tau \exp[\tau(I-P)L](I-P)L\rho_0(t-\tau)\, d\tau, \tag{2.9}$$

subject to the initial condition $(I-P)\rho(0) = 0$, i.e., the system is initially in $P\mathcal{H}$. For a time-dependent $L_1(t)$, such as in the interaction representation, the exponential in (2.9) is replaced by an infinite series of time-ordered products of the perturbation operator. Inserting (2.9) into (2.8a) yields the integro-differential equation

$$\begin{aligned} \frac{\partial\rho_0}{\partial t} = PL\rho_0 + \int_0^t d\tau\, PL \exp[\tau(I-P)L] \\ \times (I-P)L\rho_0(t-\tau)\, d\tau. \end{aligned} \tag{2.10}$$

Equation (2.10) is the projection-operator equation of evolution for the systematic or averaged part of the phase-space distribution function. It simplifies, using relations (2.5) and (2.7) to

$$\begin{aligned} \frac{\partial\rho_0}{\partial t} = L_0\rho_0 + \int_0^t d\tau\, PL_1 \exp[\tau(I-P)L](I-P) \\ \times L_1\rho_0(t-\tau)\, d\tau. \end{aligned} \tag{2.11}$$

Writing the exponential in the form

$$\begin{aligned} &\exp[\tau(I-P)L_0]\exp[\tau(I-P)L_1] \\ &\quad = \exp(\tau L_0)(I-P)\exp[\tau(I-P)L_1] \\ &\quad = \exp(\tau L_0)\exp[\tau(I-P)L_1](I-P), \end{aligned} \tag{2.12}$$

we obtain from (2.11)

$$\begin{aligned} \frac{\partial\rho_0}{\partial t} = L_0\rho_0 + \int_0^t d\tau\, PL_1 \exp(\tau L_0)\exp[\tau(I-P)L_1] \\ \times (I-P)L_1\rho_0(t-\tau)\, d\tau. \end{aligned} \tag{2.13}$$

III. PERTURBATION THEORY

The projection-operator formalism in the preceding section is based on a representation in which the Liouville operator L_0 is diagonal, i.e., P and L_0 commute. In this section we consider the perturbation solution of

$$\frac{\partial \rho}{\partial t} = (L_0 + L_1)\rho \tag{3.1}$$

directly. The operator L_0 is chosen to be the deterministic or unperturbed part of L, and L_1 the perturbation, or equivalently, the stochastic interaction.

In terms of Laplace transforms (indicated by a caret over a function) the solution to (3.1) is given by[5]

$$\hat{\rho}_0 = (1 - \hat{G}_0\hat{M})^{-1}\hat{G}_0\rho_0(0) = (\hat{G}_0^{-1} - \hat{M})^{-1}\rho_0(0), \tag{3.2}$$

where $\hat{G}_0$ is the free-particle propagator

$$\hat{G}_0 = (s - L_0)^{-1}, \tag{3.3}$$

and the operator $\hat{M}$ is found to have the series representation

$$\hat{M} = \sum_1^\infty PL_1[\hat{G}_0(I-P)L_1]^n P. \tag{3.4}$$

The perturbation operator $\hat{M}$ can be related to the projection operator in Sec. II by taking the Laplace transform of (2.11) to obtain

$$\hat{\rho}_0 = [s - L_0 - PL_1\hat{G}(I-P)L_1]^{-1}\rho_0(0), \tag{3.5}$$

with perturbed Green's function $\hat{G}$ given by

$$\hat{G} = [s - L_0 - (I-P)L_1]^{-1} = \hat{G}_0 + \hat{G}_0(I-P)L_1\hat{G}. \tag{3.6}$$

Comparing (3.2) and (3.3) with (3.5) we obtain

$$\hat{M} = PL_1\hat{G}(I-P)L_1, \tag{3.7}$$

so that in configuration space,

$$\frac{\partial \rho_0}{\partial t} = L_0\rho_0 + \int_0^t M(\tau)\rho_0(t-\tau)\,d\tau. \tag{3.8}$$

The projection operator kernel is therefore identical to the perturbation operator, as it should be since both are exact when all orders in the perturbation series are kept. The method which gives the more rapidly convergent approximate description has been the subject of much debate.

The formal structure of $\hat{M}$ as given by (3.4) is more informative than (3.7). The operator begins and ends with a P, but contains only $I-P$ in the intermediate-state factors. This implies that only the initial and final perturbations connect to the phase space $P\mathcal{K}$ with all intermediate state excitations being in the $(I-P)\mathcal{K}$ space. In diagrammatic language the operator $\hat{M}$ has the form of a "linked-cluster expansion", i.e., all diagrams are completely linked or connected and are irreducible.[5] A similar equivalence in the interaction representation between a *total* time-ordered cumulant expansion and the projection-operator technique was noted by Yoon, Deutsch, and Freed in the context of spin-relaxation theory.[6] The inclusion of the operator $I-P$ is just the concept of "connected cumulants."

If we identify the propagator in (3.5) with a Green's function $\hat{\mathcal{G}}$, then using (3.7) we can write

$$\hat{\mathcal{G}} = (s - L_0 - \hat{M})^{-1} = \hat{G}_0 + \hat{G}_0\hat{M}\hat{\mathcal{G}}. \tag{3.9}$$

Physically, $\hat{M}$ can be identified with the mass operator occuring in Dyson's equation for the renormalized propagator $\hat{\mathcal{G}}$ in quantum field theory.[7,8] The all-orders perturbation solution for the stochastic field therefore has an equivalence to quantum field theory and the attendent formal apparatus.[4]

An important approximation to the equation of evolution (3.8) can be found by truncating $M(\tau)$ at the first term $[M(\tau) = M_1(\tau)]$, i.e., neglecting the interaction term in the exponential in (2.13). The resulting equation is

$$\frac{\partial \rho_0}{\partial t} = L_0\rho_0(t) + PL_1\int_0^t \exp[L_0(t-t_1)] \times (I-P)L_1P\rho_0(t_1)\,dt_1, \tag{3.10}$$

an equation which has appeared in the stochastic theories of fluids and plasmas as the Boltzmann and Landau equations.[7] It has also been referred to as the *method of first-order smoothing*[7] since it is the linear counterpart of the Krylov-Bogoliubov method of averaging for nonlinear differential equations.[9] Thus, one can expect that when this approximation is generalized to nonlinear systems it should yield results identical to those obtained by the method of statistical linearization.[3,10]

In the next section, (3.10) will be shown to yield the same results as the first Kraichnan-Wyld approximation of Morton and Corrsin and is therefore one example of a consolidated equation. The nomenclature "consolidated equation" refers to a specific partial summation and reexpression of the equation of evolution in terms of renormalized quantities. Other consolidated equation approximations can be obtained by introducing a consolidation operator Q. We add and subtract Q from the Liouville operator $L_0 + L_1$ to obtain

$$\frac{\partial \rho_1}{\partial t} = (I-P)(L_0+Q)\rho + (I-P)(L_1-Q)\rho. \tag{3.11}$$

If we assume that P and Q commute,[11] then by Laplace transforming (3.11) we obtain

$$\hat{\rho}_1 = \hat{G}_Q(I-P)(L_1-Q)(\hat{\rho}_0 + \hat{\rho}_1) \tag{3.12}$$

with the renormalized propagator

$$\hat{G}_Q = 1/(s - L_0 - Q)\,. \tag{3.13}$$

Physically speaking, since the Liouville operator and Hamiltonian are related by (2.4), the associated effect of shifting by Q results in a modification of the underlying system Hamiltonian. The Hamiltonian is a sum of a free-particle part (H_0) and interactive part (H_1), which generate the corresponding Liouville operators via (2.4). The consolidation operator Q corresponds to adding an average potential $\overline{V}$ to H_0 and subtracting the same potential from H_1 such that

$$H = H_0 + H_1 \tag{3.14a}$$

$$= H_0' + H_1', \tag{3.14b}$$

with

$$H_0' = H_0 + \overline{V}; \quad H_1' = H_1 - \overline{V}\,. \tag{3.15}$$

Prior to the effective transformation in (3.14b) the initial phase space in (3.14a) is defined such that H_0 is diagonal and the projection operator P commutes with L_0. Upon transformation to the renormalized system (3.14b) the initial phase space is now defined such that H_0' is diagonal and the projection operator P_Q commutes with $L_0 + Q$.

The equation of motion for the diagonal part of the ensemble-distribution function in the subspace $P_Q\mathcal{H}$ is given by

$$\hat{\rho}_{0Q} = \hat{G}_Q \rho_{0Q}(0) + \hat{G}_Q \hat{M}_Q \hat{\rho}_{0Q}\,, \tag{3.16}$$

where the series representation of the renormalized mass operator is

$$\hat{M}_Q = P_Q(L_1 - Q)\sum_0^\infty [\hat{G}_Q(I - P_Q)(L_1 - Q)]^n P_Q\,, \tag{3.17}$$

and $\hat{G}_Q$ is defined by (3.13). Note that we have assumed in (3.16) that $\rho_{1Q}(t=0) = 0$ rather than $\rho_1(t=0) = 0$ as used earlier. Since this initial condition is arbitrary we can choose, with the same degree of confidence, the initial state to be diagonal in either $P\mathcal{H}$ or $P_Q\mathcal{H}$.

When Q is zero the mass operator (3.17) reduces to (3.4) since $\lim_{Q\to 0} P_Q = P$. If, on the other hand, Q is selected to cancel certain prescribed diagrams from the $Q=0$ perturbation series, then consolidated equations other than first-order smoothing can be derived. The classes of diagrams associated with Q are based on physical arguments closely tied to the system of interest. For example, in the theory of nuclear matter, or hard-sphere scattering in kinetic theory, the interaction potential can become singular. Brueckner[12] showed that by summing all two-particle interactions which appear as "ladder" diagrams, the matrix elements of the resulting "effective interaction" $(H_1 - \overline{V})$ are finite. Bloch,[13] in a comprehensive review of the diagrammatic approach to quantum statistical mechanics, showed how one does a similar renormalization of the hard-core potential in kinetic theory.

Note that the approximation leading to (3.10) in the renormalized representation yields

$$\frac{\partial \rho_{0Q}}{\partial t} = (L_0 + Q)\rho_{0Q} + P_Q L_1 \int_0^t \exp[(L_0 + Q)(t - t_1)] \times (I - P_Q)L_1 P_Q \rho_{0Q}(t - t_1)\, dt_1\,, \tag{3.18}$$

which although of the same form as (3.10) has a much different interpretation. The distribution function ρ_{0Q} has correlations at time $t=0$ corresponding to the difference in P_Q and P. These correlations are not present in ρ_0 and by construction result from the change in the definition of the "unperturbed" state of the system. When the method of statistical linearization is applied to (3.18) these correlations manifest themselves as an alteration in the mean-squared response of the "unperturbed" system.

In Sec. IV we illustrate the above statements about renormalization using the specific example of a Duffing oscillator for our nonlinear system. Extensions of these ideas to general nonlinear systems are made in Sec. V.

IV. THE DUFFING OSCILLATOR

In practical situations, especially for nonlinear systems, one is almost always limited to calculations of the stationary autocorrelation function $R_{xx}(t)$ and its Fourier transform $S_{xx}(\omega)$, the spectral-density function. In such cases the considerations of the preceding sections seem somewhat esoteric and the connection with applications obscure. To remedy this situation we recall the analysis of the Duffing oscillator as a prototype of a single-degree-of-freedom nonlinear system and investigate how to apply the preceding analysis.

The correlation and spectral-density functions for the Duffing oscillator were calculated using a diagrammatic method by Morton and Corrsin, a projection-operator technique by Bixon and Zwanzig, and a more straightforward algebraic technique by Budgor. Morton and Corrsin, and Budgor computed these quantities in the same manner, whereas Bixon and Zwanzig defined the autocorrelation function as

$$R_{xx}(t) = \int x(t) e^{Lt} x(0) \rho_0(0)\, dx\, dv\,, \tag{4.1}$$

where $x(t)$ is the displacement of the oscillator

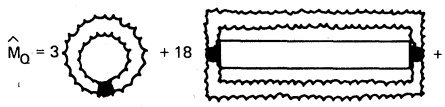
at time t, and L is the Fokker-Planck diffusion operator for the nonlinear system. Equation (4.1) was evaluated numerically after expanding $R_{xx}(t)$ in terms of its moments,

$$R_{xx}(t)=\sum_{2}^{\infty}\frac{R_m t^m}{m!}\,. \tag{4.2}$$

Truncating (4.2) at $m=2$ yields [apart from a scaling factor of 2 which results from the difference in definition of $R_{xx}(t)$] expressions identical to the first Kraichnan-Wyld approximation of Morton and Corrsin or, equivalently, the statistical-linearization result of Budgor. A detailed discussion of the consequences of this equality is made elsewhere.

We shall not go into the details of the Morton and Corrsin diagrammatic representation of the perturbation analysis for the Duffing oscillator here. It is worth pointing out a few of the salient features of their analysis, however, which may have application to other nonlinear systems. We recall their notation: an interaction at a vertex is represented by a point, a free propagator by a straight line, a renormalized propagator by a rectangle, a renormalized interaction at a vertex by a circle with a point in the center, the unperturbed spectral density by a wavy line, and finally, the renormalized spectral density by a wavy rectangle.

Dyson's equation (3.9) for the renormalized Green's function has the following terms for the mass operator $\hat{M}_Q$[14]:



(4.3)

in the perturbation analysis of the Duffing oscillator. There is a term-by-term correspondence between $\hat{M}_Q$ in (4.3) and the effective frequency Ω^2 [$\equiv(2\alpha\sigma_x^2)^{-1}$] discussed by Bixon and Zwanzig. The effective frequency is defined in terms of a truncated memory kernel, and from (3.7) and (3.8) we see that (4.3) is just the diagrammatic form of the memory kernel. The spectral density as defined by Bixon and Zwanzig is expressed in terms of $\hat{M}_Q$ as

$$S_{xx}(\omega)=\frac{\alpha(1+\hat{M}_Q)}{(1+\hat{M}_Q-\omega^2)^2+\alpha^2\omega^2}\,. \tag{4.4}$$

When $\hat{M}_Q=0$, this is the spectral density of a linear oscillator with internal frequency set equal to unity. $\hat{M}_Q$ truncated at the first term in $\hat{M}$, i.e., when $\hat{M}_Q=\hat{M}=\hat{M}_1$ [see Eq. (3.10)], becomes statistical linearization; at its second term, the Bixon-Zwanzig result for $S_{xx}(\omega)$ up to coefficient R_3, etc.

The truncation of $\hat{M}_Q$ at its first term, $\hat{M}_Q=\hat{M}_{Q1}$, becomes statistical linearization with the correlations induced by the nonlinearities included in the mean-squared response of the oscillator. That this gives a marked improvement in the calulated spectral density was observed by Budgor *et al.*[3] This approximation, referred to as the modified statistical linearization approximation, corresponds to using the steady-state solution of the nonlinear Fokker-Planck equation for the Duffing oscillator in the calculation of the mean-squared response of the oscillator.

V. COMPLETELY RENORMALIZED THEORY

Wyld's analysis[15] of the quadratic nonlinearities in turbulence and Morton and Corrsin's analysis of the cubic nonlinearity in the Duffing oscillator suggest some general features of perturbation expansions which may be used to calculate correlation functions in nonlinear systems. From these analyses one might expect that for a scalar-dynamic equation with polynomial nonlinearity (x^n; $n=2,\ 3,\ldots$) the complete characterization of the spectral density will involve three coupled nonlinear integral equations. These equations relate the renormalized (or consolidated) spectral density $S_{xx}(\omega)$ to the renormalized propagator $\hat{G}_Q$ and renormalized vertex operator $\hat{\Gamma}_Q(\omega_1,\ldots,\omega_{n-1})$. The vertex operator $\hat{\Gamma}_Q$ is related to the renormalized mass operator $\hat{M}_Q$ by means of the Ward identities.[4,15]

Martin *et al.* (MSR) point out that the renormalization technique of Wyld does not work to order higher than four and also that the renormalized expansion used by Morton and Corrsin is not systematic since the series are not expressible as coupled equations in terms of renormalized quantities *only*, i.e., their series is an expansion in terms of *both* the bare and renormalized vertex operators and not in the renormalized operators alone. MSR remedy this deficiency by introducing a variable conjugate to that in the original nonlinear equation of motion with a corresponding dynamic equation for the conjugate variable.[16] Using MSR's notation, the variables Ψ and $\hat{\Psi}$ define a two component matrix Φ, which at position $\underline{r}$ and time t obeys Hamilton's equation

$$\dot{\Phi}(1)=[\Phi(1),H], \tag{5.1}$$

with the "non-Hermitian" Hamiltonian

$$H=\int\gamma_1(1)\Phi(1)\,d1+\frac{1}{2!}\int\gamma_2(1,2)\Phi(1)\Phi(2)\,d1\,d2 +\frac{1}{3!}\int\gamma_3(1,2,3)\times\Phi(1)\Phi(2)\Phi(3)\,d1\,d2\,d3\,. \tag{5.2}$$

The notation condenses all the variables for a given particle to the single numerical index in

(5.1) and (5.2); γ_1, γ_2, and γ_3 are vertex matrices and repeated indices are summed over.

The field operator Φ satisfies the Poisson bracket relations

$$[\Phi(\alpha, l, \underline{r}, t), \Phi(\alpha', l', \underline{r}', t)] = i\sigma_2(\alpha, \alpha')\delta_{ll'}\delta(\underline{r}-\underline{r}'), \quad (5.3)$$

where σ_2 is the Pauli matrix

$$i\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (5.4)$$

and α is either +1 or −1. Using Hamiltonian (5.2) and the commutator (5.3), the equation of motion for Φ is

$$-i\sigma_2\dot{\Phi}(1) = \gamma_1(1) + \gamma_2(1,2)\Phi(2) + \tfrac{1}{2}\gamma_3(1,2,3)\Phi(2)\Phi(3), \quad (5.5)$$

which is precisely the form obtained in quantum field theory with γ_2 the bare-particle mass and γ_3 the bare-particle charge.

The equations of motion for the average value of Φ and the two-point correlation function are obtained by MSR using the generating functional

$$S = \left\{ \exp \int_{t_i}^{t_f} \Phi(3)\eta(3) \right\}_+, \quad (5.6)$$

where the brackets indicate a time ordering between t_i and t_f. The correlation functions are obtained by taking functional derivatives of (5.6) with respect to η, averaging, and then taking the limit as η goes to zero. The first- and second-moment functions are defined schematically as

$$M_1^\eta \equiv (\delta/\delta\eta)\ln\langle S\rangle = \langle S\Phi\rangle_+/\langle S\rangle,$$
$$M_2^\eta \equiv (\delta^2/\delta\eta^2)\ln\langle S\rangle = \langle S\Phi^2\rangle_+/\langle S\rangle - \langle S\Phi\rangle_+^2/\langle S\rangle^2, \quad (5.7)$$

so that

$$\langle\Phi\rangle = \lim_{\eta\to 0} M_1^\eta, \quad \text{variance } \Phi = \lim_{\eta\to 0} M_2^\eta. \quad (5.8)$$

The renormalization of the vertex functions γ_2 and γ_3 correspond to the "mass" and "charge" renormalization of quantum electrodynamics. This procedure yields two renormalized propagators and three renormalized vertex operators which appear as the nonvanishing parts of a matrix propagator and matrix vertex operator. MSR obtain a *closed* set of nonlinear integral equations for the average function M_1^η and its two-point correlation function M_2^η entirely in terms of these renormalized quantities.

The closed set of equations in the MSR formalism may be expressed in terms of projection operators by identifying the generating functional S with the projection of the matrix Φ onto the dynamic subspace $P_\eta\mathcal{K}$. The projection operator P_η can be used to rewrite (5.7) as

$$M_1^\eta = P_\eta\Phi, \quad M_2^\eta = P_\eta\Phi^2 - (P_\eta\Phi)^2. \quad (5.9)$$

Lax[17] has discussed the linked moments of a stochastic variable in a nonlinear system using the generating function for the normalized probability-density function. The generating function in that kinematic case is just the characteristic function. The MSR formalism is a generalization of that concept to dynamic nonlinear systems and the generating function is replaced by a generating functional.

VI. CONCLUSIONS

We see from the equivalence of the memory kernel and mass operator (3.7) that, in principle, there is no difference between the projection operator and perturbation techniques. Just as projection operators restrict the dynamical problem to a subspace of the Hilbert space of probability densities, the consolidated equations reflect this same restriction in the dynamical state space. Variational calculations determine the Hilbert space $P_Q\mathcal{K}$ that is stationary with respect to variations in the mass operator $\hat{M}_Q$. The generating-functional technique used in Sec. V is a procedure for making systematic the consolidations of such perturbation series. The key to using the MSR formalism is identifying the conjugate variable. MSR point out that in the case of the Duffing oscillator, the conjugate variable is the square of the oscillator displacement. In terms of this variable the coupling in the dynamic equation is cubic.

A great deal more effort is required to determine which of the above representations of the nonlinear problem will yield the more rapidly convergent solutions. It is also probably the case that the physical problem itself will dictate the method of approximation. For example, from the first-order smoothing approximation ($\hat{M}_Q = \hat{M} = \hat{M}_1$) one can infer that the average of Dyson's equation,

$$\langle\hat{\mathcal{G}}\rangle = \hat{G}_0 + \langle\hat{G}_0\hat{M}\hat{\mathcal{G}}\rangle, \quad (6.1)$$

is equivalent to

$$\langle\hat{\mathcal{G}}\rangle = \hat{G}_0 + \hat{G}_0\langle\hat{M}_1\rangle\langle\hat{\mathcal{G}}\rangle. \quad (6.2)$$

This implies statistical independence of the perturbation field and the response function whenever these occur under a configuration space or time integral. All the diagrams related to (6.2) are therefore of the same type and unconnected. The physical systems for which this should in fact be a good approximate solution is not at all clear, although in fluid mechanics it is apparently successful for Reynolds numbers $R \ll 1$.[7]

On the other hand, by reinterpreting first-order

smoothing as statistical linearization,[10] we can immediately determine the criteria which insure the success of this approximation. Similar arguments may be profferred for other consolidated equations.

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[14]Here $\hat{M}_Q$ is the Fourier transform of $M_Q(t)$, not the Laplace transform as defined in Sec. III.

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[16]This completes the analogy with quantum field theory, where in second quantized form the field variable is a creation operator and its conjugate is a destruction operator; the space-time evolution of the field requires both quantities.

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