One-exponent scaling for very high-Reynolds-number turbulence

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We show that two strong but physically plausible assumptions allow all the measurable scaling exponents for very high-Reynolds-number incompressible turbulence to be expressed simply in terms of the exponent μ characterizing the dissipation fluctuations. The first assumption, introduced by Obukhov in 1962, relates the locally fluctuating dissipation to the locally fluctuating nonlinear energy transfer. The second assumption is that the same dissipation-length scale η defines the crossover from the dissipation range to the inertial range for all of the velocity structure functions. The resulting exponent relations are the same as recently obtained from a simple geometrical model by Frisch, Sulem, and Nelkin, but the results appear less model dependent in the present context. In addition we introduce a locally defined energy transfer variable $T(\vec{x}) = \eta^2\psi^3(\vec{x})$, where $\psi(\vec{x})$ is any component of the velocity derivative tensor. We suggest that $T(\vec{x})$ has the same statistical properties as the locally defined viscous dissipation $\tilde{\epsilon}(\vec{x}) = \nu\psi^2(\vec{x})$, where ν is the kinematic viscosity. This suggestion is compatible with our other results, and is capable of experimental test.

I. INTRODUCTION

There is a great deal of experimental evidence supporting the idea of a universal small-scale regime in very high-Reynolds-number turbulence.[1] This regime is governed by an energy cascade, and is characterized by scale-similar behavior of correlation functions. In its simplest form this scaling was already given by Kolmogorov in 1941, and leads to the famous $k^{-5/3}$ behavior of the energy spectrum in the limit of zero viscosity. We now know that there are important dynamical fluctuations in the cascade process, and that these can lead to scaling exponents which are not calculable by simple dimensional analysis. In particular the $k^{-5/3}$ law is probably not exact. An important theoretical objective is the calculation of these modified exponents from the underlying Navier-Stokes equations, but this is a formidable task. A more modest goal is to develop a phenomenological scaling theory which relates measurable scaling exponents to each other.

There is a considerable history of such scaling theories. Following the 1962 suggestions of Obukhov[2] and Kolmogorov[3] attention has largely focused on how fluctuations in the cascade process are manifested in the statistical properties of the local viscous dissipation. Several authors have commented[4-6] that this attention may be misplaced since it is the nonlinear energy transfer and not the linear viscous dissipation which is the dynamically relevant quantity. This criticism is well founded, but essentially negative. In particular it does not suggest alternative experimentally measurable variables whose fluctuations would be of greater intrinsic theoretical interest.

In this paper we attempt to remedy this situation in a theoretically plausible and experimentally testable way. We begin in Sec. II with a reinterpretation of the 1962 Obukhov assumption relating velocity structure functions to averages of the local viscous dissipation over a finite region. In Sec. III we make the strong but plausible scaling assumptions that power laws occur wherever possible, and that a single length scale defines the viscous cutoff of all relevant correlation functions. This allows all of the scaling exponents to be expressed in terms of a single exponent μ which describes the dissipation fluctuations. The results agree with those of a simple geometrical model, presented recently by Frisch, Sulem, and Nelkin,[7] but in the present context they appear less model dependent.

In Sec. IV we introduce a local dynamical variable, related to nonlinear energy transfer, which we suggest has the same statistical properties as the local viscous dissipation. This is a spatially local generalization of the 1962 Obukhov assumption. We show that this suggestion is consistent with our other results, and discuss the experiments which would test it. In Sec. V we discuss our results, pointing out that they are suggestive of an assumed operator algebra for very high-Reynolds-number turbulence.

An earlier paper by one of us,[8] based on formal analogy to scaling in equilibrium critical phenomena, is in the same general spirit as the present work. In that paper, however, we did not make the 1962 Obukhov assumption, and could draw less conclusions about the relations among exponents. There is also a subtle logical error in the scaling arguments in Ref. 8. This is discussed further in Appendix A.

The 1962 suggestion of Kolmogorov[3] has led to considerable interest in the statistical properties of certain ratio variables defined in terms of the viscous dissipation averaged over an interval. In Appendix B we discuss theory and experiment relating to the properties of these variables. We suggest that recent results in this area, although internally consistent, have little relation to the scaling exponents of greatest dynamical interest.

II. THE 1962 OBUKHOV ASSUMPTION

Throughout this paper we will work with averages over a linear interval of length r, and with appropriate one-dimensional surrogates for the three-dimensional dynamical variables. This has the advantage of simplicity, and of being closer to what is actually measured. It has the disadvantage of not dealing with any of the interesting geometrical problems of turbulent flows. We expect, however, for the scaling exponents which are of primary interest here, that the results will not be affected.

As a one-dimensional surrogate for the local viscous dissipation, we introduce the variable

$$\tilde{\epsilon}(\vec{x}) = \nu\psi^2(\vec{x}), \tag{1}$$

where

$$\psi(\vec{x}) = \frac{\partial v_1}{\partial x_1} \tag{2}$$

could just as well have been any other component of the velocity derivative tensor. Assuming that the energy dissipation per unit mass ϵ is finite in the limit of zero viscosity, and that the small scales are locally isotropic, we have

$$\lim_{\nu\to 0}\langle\tilde{\epsilon}(\vec{x})\rangle = \epsilon/15 = \text{const.} \tag{3}$$

A dynamical variable whose average is constant in the limit of zero viscosity we call "regularized." This is an important property that we will seek to generalize to other variables. We make the usual assumption that the turbulence is in a statistically steady state.

Now consider the average of $\tilde{\epsilon}$ over a linear interval of length r,

$$\epsilon^*(r) = \frac{1}{r}\int_0^r \tilde{\epsilon}(x)\,dx. \tag{4}$$

For homogeneous turbulence ϵ^* will have the same average value as $\tilde{\epsilon}$. In 1962, Kolmogorov suggested that $\epsilon^*(r)$ should be a log-normal random variable. This is, at best, approximately true. In any case the viscous dissipation does not determine the essential dynamical behavior, and it is not the place where we should look first for theoretical insight.

A natural physical idea is that the fluctuations in viscous dissipation over an interval of size r should be determined by the fluctuations in energy cascade rate over the same interval. Suppose that viscosity is unimportant for intervals larger than some dissipation length scale η. For $r \gg \eta$ a natural measure of the fluctuating energy transfer is

$$\epsilon_t(r) = \Delta v^3(r)/r, \tag{5}$$

where

$$\Delta v(r) = v_1(\vec{x}+\vec{r}) - v_1(\vec{x}) \tag{6}$$

is a typical velocity difference across the interval. The average value of the nth power of $\epsilon_t(r)$ is just

$$\langle\epsilon_t^n(r)\rangle = r^{-n}\langle[\Delta v(r)]^{3n}\rangle = r^{-n}D_{3n}(r). \tag{7}$$

The quantity

$$D_n(r) = \langle[\Delta v(r)]^n\rangle$$

is called the nth-order longitudinal structure function,[1] and is a natural object for study in statistical theories of turbulence.

The assumption that the statistical properties of $\epsilon_t(r)$ and of $\epsilon^*(r)$ should be the same is just the 1962 Obukhov assumption, which is usually written in the form

$$D_n(r) = C_n r^{n/3}\langle[\epsilon^*(r)]^{n/3}\rangle. \tag{8}$$

As has been emphasized by Kraichnan,[4] there is no need for Eq. (8) to be true, but it seems to us to express a physically plausible idea: the viscosity dissipates the energy where the energy cascade deposits it. Note, however, that the physics in Eq. (8) requires reading it from right to left. There is no reason for the dissipation statistics *per se* to be simple, but they should reflect the statistics of the velocity structure functions. These latter quantities, though also not simple, are of greater theoretical interest.

We begin by looking more closely at Eq. (8) for $n=3$ and for $n=6$. For $n=3$ we have

$$D_3(r) = C_3\epsilon r, \tag{9}$$

which is known to follow from the Navier-Stokes equations.[1,9] For $n=6$ the natural assumption in light of the scale similarity of the problem is

$$\langle[\epsilon^*(r)]^2\rangle = \text{const}\times\epsilon^2(L/r)^\mu, \tag{10}$$

and

$$D_6(r) = \text{const}\times\epsilon^2 r^2(L/r)^\mu, \tag{11}$$

where μ is a universal scaling exponent, and L is an external length scale of the turbulence. Equation (10) was already proposed in 1962 by Kolmogorov.[3] For homogeneous turbulence Eq. (10) is equivalent to

$$\langle\tilde{\epsilon}(\vec{x})\tilde{\epsilon}(\vec{x}+\vec{r})\rangle = \text{const}\times\epsilon^2(L/r)^\mu. \tag{12}$$

Equation (12) is the usual experimental definition of the exponent μ. It has been measured by several workers[1,10] and has a value of about 0.5. Equation (11) suggests that the sixth-order structure function should be determined by this same exponent independent of any models for the statistics of the local dissipation field. This is consistent with the one existing experiment,[11] but is not very accurately tested.

Equations (8)–(12) apply only for r much larger than the dissipation length scale η. In this inertial range of distances, there is no natural length scale, and observable correlation functions do not depend on the value of the viscosity. It is conceptually important to recognize that they also do not depend on the detailed mechanism of viscous dissipation. A small modification of the functional form of the viscous dissipation term in the Navier-Stokes equations would affect the definition of the local dissipation in Eq. (1), but Eqs. (8)–(12) would be unchanged. In particular the exponent μ defined by Eq. (11) would be the same. The quantity to be measured in Eq. (12) would be changed, but with this change, the dissipation correlation should still be given by Eq. (12) with the same exponent μ. Such a change in the dissipation mechanism is hypothetical for the strongly turbulent fluid flows available in the laboratory, but it is useful to think about. The idea of a modified dissipation mechanism is discussed in more quantitative terms in Ref. 6.

III. SCALING RELATIONS

What are the scaling exponents defined by the structure functions $D_n(r)$, and what are the relations among them? In the inertial range $r \gg \eta$ it is reasonable to assume

$$D_n(r) \approx C_n \epsilon^{n/3} r^{n/3} (r/L)^{\zeta_n}, \tag{13}$$

where the exponents ζ_n are, in general, unknown. From our previous discussion we have $\zeta_3 = 0$, and the Obukhov assumption that $\zeta_6 = -\mu$, where μ is the exponent determined by Eq. (12). The exponent ζ_2 is of special interest since it determines the correction to the 5/3 law for the energy spectrum.

In the dissipation range $r \ll \eta$, we expect the structure functions to be analytic in r, and to be given by

$$D_n(r) \approx \langle \psi^n(\vec{x}) \rangle r^n. \tag{14}$$

The quantities $\langle \psi^n(\vec{x}) \rangle$ will diverge as the viscosity goes to zero, and it is reasonable to assume that these divergences are also described by power laws,

$$\langle \psi^n(\vec{x}) \rangle \sim \nu^{-y_n}. \tag{15}$$

The exponents y_n are also unknown except for $y_2 = 1$ which is fixed by Eqs. (1) and (3). The exponents y_3 and y_4 are perhaps more familiar in terms of the skewness and flatness of the probability distribution of $\psi(\vec{x})$. These are given by

$$S_0 = -\langle \psi^3 \rangle / \langle \psi^2 \rangle^{3/2} \sim \nu^{-\theta} \tag{16}$$

and

$$F_0 = \langle \psi^4 \rangle / \langle \psi^2 \rangle^2 \sim \nu^{-\alpha}, \tag{17}$$

where

$$\theta = y_3 - \tfrac{3}{2} \tag{18}$$

and

$$\alpha = y_4 - 2. \tag{19}$$

Since the $\langle \psi^n \rangle$ are moments of a probability distribution they satisfy the Schwarz inequalities

$$\langle \psi^{n/2} \rangle^2 \le \langle \psi^{n-k} \rangle \langle \psi^k \rangle. \tag{20}$$

When combined with Eq. (15) which is assumed valid for any sufficiently small values of ν, this gives the exponent inequalities

$$2y_{n/2} \le y_{n-k} + y_k. \tag{21}$$

The exponents y_n are a concave function of n. The special case $n=6, k=2$ gives the familiar inequality

$$\theta \le \tfrac{1}{2}\alpha, \tag{22}$$

which states that the skewness can increase no more rapidly than the square root of the flatness.

The structure functions $D_n(r)$ are also moments of a probability distribution, and satisfy the inequalities

$$[D_{n/2}(r)]^2 \le D_{n-k}(r) D_k(r). \tag{23}$$

When combined with Eq. (13), which is assumed valid for any sufficiently small value of r/L, this gives the exponent inequalities

$$2\zeta_{n/2} \ge \zeta_{n-k} + \zeta_k. \tag{24}$$

The exponents ζ_n are a convex function of n. The special cases $n=6, k=2$, and $n=8, k=2$ will be important in what follows. When combined with the result that $\zeta_3 = 0$, and the Obukhov assumption that $\zeta_6 = -\mu$, these become

$$\zeta_2 + \zeta_4 \le 0 \tag{25}$$

and

$$\zeta_2 - 2\zeta_4 \le \mu. \tag{26}$$

These can be combined to give the familiar Mandelbrot inequality[7,12]

$$\zeta_2 \le \tfrac{1}{3}\mu. \tag{27}$$

To go further we must determine the dissipation length scale η. This is most naturally done in terms of the crossover of the structure functions $D_n(r)$ from the small-r behavior of Eq. (14) to the large-r behavior of Eq. (13). In the absence of dissipation fluctuations the 1941 Kolmogorov theory applies with $\zeta_n = 0$ and $y_n = \frac{1}{2}n$. The crossover distance determined from any of the $D_n(r)$ is the same within a constant factor, and is given by the well-known expression

$$\eta_k = \epsilon^{-1/4}\nu^{3/4}. \tag{28}$$

In the presence of dissipation fluctuations we make the strong assumption that the crossover determined from any of the $D_n(r)$ has the same dependence on viscosity. From the crossover for $n = 2$ we obtain

$$\eta \sim \nu^z, \quad z = (\tfrac{4}{3} - \zeta_2)^{-1}. \tag{29}$$

Thus η depends on the unknown exponent ζ_2. From the crossover for $n = 3$ we obtain

$$\langle \psi^3(\vec{x})\rangle \sim \eta^{-2}, \tag{30}$$

a result to which we will return. Equation (30) allows the unknown exponent y_3 to be expressed in terms of the unknown exponent ζ_2. In particular, it gives the divergence of the skewness in the limit of zero viscosity in the form

$$\theta = \tfrac{3}{2}\zeta_2(\tfrac{4}{3} - \zeta_2)^{-1}. \tag{31}$$

Equation (31) has been given elsewhere.[7,8]

The crossover for arbitrary n, under the assumption of a single dissipation length η, gives the exponent relation

$$y_n = (\tfrac{2}{3}n - \zeta_n)(\tfrac{4}{3} - \zeta_2)^{-1}. \tag{32}$$

Thus the exponents y_n and ζ_n are not independent of each other, but we still have no relation among the ζ_n for different values of n. We can get such a relation by making one further scaling assumption which is very much in the spirit of what we have already assumed. As an alternative calculation of the exponent y_4 consider

$$\nu^2\langle\psi^4(\vec{x})\rangle = \lim_{r\to 0}\langle\tilde{\epsilon}(\vec{x})\tilde{\epsilon}(\vec{x}+\vec{r})\rangle, \tag{33}$$

and assume that this limit can be calculated by extrapolating the inertial range form of Eq. (12) back to $r = \eta$. This gives us

$$y_4 = 2 + \mu z, \tag{34}$$

a plausible result which we have used before.[8] If we combine Eq. (32) for $n = 4$ with Eq. (34), we obtain

$$2\zeta_2 - \zeta_4 = \mu. \tag{35}$$

The only solution consists with Eq. (35) and the exponent inequalities of Eq. (25) and (26) is

$$\zeta_2 = -\zeta_4 = \tfrac{1}{3}\mu. \tag{36}$$

This is the result obtained from a simple geometrical model by Frisch *et al.*[7] Note that the 1962 Kolmogorov result

$$\zeta_2 = -\tfrac{1}{2}\zeta_4 = \tfrac{1}{9}\mu \tag{37}$$

is not consistent with the assumption of a single dissipation length scale. This assumption leads to a very strong constraint on the relation among scaling exponents.

IV. DYNAMICALLY RELEVANT LOCAL DISSIPATION VARIABLE

Equation (8) applies to dynamical variables defined for a finite interval large compared to a dissipation length scale. We would like an extension to local dynamical variables defined at every spatial point. We want an energy transfer variable which refers only to the small scales of motion, and does not explicitly depend on viscosity. The former condition requires that we use only velocity derivatives and not velocity directly. The latter condition is more subtle. The only length at our disposal is the dissipation length scale η. This certainly depends on viscosity, but the dependence is implicit in the sense that such a length scale would still exist if the mechanism of viscous dissipation were slightly different. Using only η and velocity derivatives, we can define a local energy transfer variable by

$$T(\vec{x}) = \eta^2\psi^3(\vec{x}). \tag{38}$$

Using Eq. (30) we see that $T(\vec{x})$ is "regularized" in the sense that

$$\lim_{\nu\to 0}\eta^2\langle\psi^3(\vec{x})\rangle = \text{const}\times\epsilon. \tag{39}$$

The variable $T(\vec{x})$ is dimensionally appropriate as a dissipation rate. We now make the assumption that $T(\vec{x})$ and the local dissipation $\tilde{\epsilon}(\vec{x})$ have identical statistical properties. There is an immediate experimental consequence of this assumption. In the inertial range, $r \gg \eta$, we predict

$$\begin{aligned}\nu^2\langle\psi^2(\vec{x})\psi^2(\vec{x}+\vec{r})\rangle &\sim \eta^4\langle\psi^3(\vec{x})\psi^3(\vec{x}+\vec{r})\rangle \\ &\sim \nu\eta^2\langle\psi^2(\vec{x})\psi^3(\vec{x}+\vec{r})\rangle \\ &\sim \epsilon^2(L/r)^\mu.\end{aligned} \tag{40}$$

These three correlation functions should all be governed by the same inertial range exponent μ. There is no experimental information on this question, but it should not be too difficult to obtain.

In addition we can calculate the values of these correlation functions for zero separation by extrapolating their inertial range form back to the dissipation length scale η. Thus we expect

$$\langle\psi^4(\vec{x})\rangle \sim \nu^{-2}\eta^{-\mu}, \tag{41}$$

$$\langle\psi^5(\vec{x})\rangle \sim \nu^{-1}\eta^{-2-\mu}, \tag{42}$$

and

$$\langle \psi^6(\vec{x}) \rangle \sim \eta^{-4-\mu}. \tag{43}$$

Equation (41) has already been used to get the exponent relation of Eq. (34). Equation (43) is equivalent to Eq. (32) for $n=6$, and thus contains no new information. It does, however, demonstrate the internal consistency of the scaling arguments of the preceding section with the dynamical assumption made here. Equation (42) can be combined with Eq. (32) for $n=5$ to give us two equations in the two unknowns y_5 and ζ_5. Their solution is

$$y_5 = \tfrac{5}{2} + 9\mu/2(4-\mu), \quad \zeta_5 = -\tfrac{2}{3}\mu. \tag{44}$$

All of our results to date can be combined in the form

$$y_n = \frac{n}{2} + \frac{3\mu}{4-\mu}\left(\frac{n}{2} - 1\right) \tag{45}$$

and

$$\zeta_n = \tfrac{1}{3}(3-n)\mu. \tag{46}$$

Equations (45) and (46) apply for $2 \leqslant n \leqslant 6$, but it would be straightforward to extend to larger values of n if we choose to make the scaling assumption on $D_n(r)$ for larger n.

V. DISCUSSION

We have derived one exponent scaling from two assumptions: the Obukhov assumption of Eq. (11) relating the sixth-order structure function to the dissipation fluctuations, and the assumption that there is a single dissipation length scale describing the crossover of the structure functions from their small-r behavior to their inertial range behavior. This is sufficient to determine all of the exponents in terms of the exponent μ defined by the dissipation correlation function. In particular it is sufficient to logically exclude the 1962 Kolmogorov results.

The assumptions we have made are simple and plausible, but they give much stronger results than one would have expected. Just because these results are so strong, the underlying assumptions deserve critical analysis. There is certainly no *a priori* dynamical reason why the relations among scaling exponents should be as simple as derived here.

Although the derivation of one exponent scaling given here is new, many of the results are not. They have been obtained earlier in a variety of ways, most of which seem much more model dependent than what we have presented. Similar results were first obtained by Novikov and Stewart in 1964 (see Chap. 25 of Ref. 1 for a discussion). A description of intermittency in terms of statistical geometry has been given by Mandelbrot.[12] Those scaling relations given here which can be compared with his description correspond to a special case which he calls "fractally homogeneous turbulence." He also argues that this special case is less pathological mathematically and more plausible physically than one might expect on dynamical grounds. In a recent paper Frisch, Sulem, and Nelkin[7] have expressed the "fractally homogeneous" case of Mandelbrot in terms of a simple dynamical model, and have explicitly derived Eqs. (45) and (46).

In Sec. IV we proposed that the locally defined variable $T(\vec{x}) = \eta^2\psi^3(\vec{x})$ should have the same statistical behavior as the local dissipation $\tilde{\epsilon}(\vec{x})$. This was shown to be internally consistent with our scaling assumptions, and is more easily testable experimentally than any of our other results. It may also be a useful guide to the structure of a future dynamical theory. Consider the dynamical variable

$$A_n(\vec{x}) = \nu^{y_n}\psi^n(\vec{x}), \quad n \geqslant 2. \tag{47}$$

This variable is regularized in the sense that

$$\lim_{\nu \to 0} \langle A_n(\vec{x}) \rangle = \text{const.} \tag{48}$$

A natural generalization of our scaling results in that

$$\lim_{\nu \to 0} \langle A_p(\vec{x}) A_q(\vec{x} + \vec{r}) \rangle \sim (L/r)^{\mu}. \tag{49}$$

In other words, all of the $A_n(\vec{x})$ for $n \geqslant 2$ are regularized scaling operators with scaling dimension $\tfrac{1}{2}\mu$.

Having given a physically plausible structure for a simple scaling theory of very high-Reynolds-number turbulence, we conclude by analyzing what went wrong in two earlier attempts. In Appendix A we discuss the errors in earlier work by one of us[8] based on analogy to critical phenomena. In Appendix B we discuss how recent experiments throw doubt on the relevance of the 1962 Kolmogorov idea and its subsequent development.

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APPENDIX A: SOME ERRORS IN REFERENCE 8

In an earlier paper by one of us, some of the same relations among exponents given here were obtained by apparently quite similar reasoning. For example, Eq. (37) of Ref. 8 is the same as Eq. (31) here, Eq. (17) of Ref. 8 is the same as Eq. (34) here, and Eq. (25) of Ref. 8 is the same as replacing the inequality of Eq. (22) here by an equality. In Ref. 8, however, we made no assumption of the 1962 Obukhov type relating local dissipation to local energy transfer. In retrospect it is clear that we obtained results which could not be obtained without some physical assumption of this type. The error in Ref. 8 lies in the arguments leading to Eq. (25), particularly in Eq. (20). It is worth analyzing this error in detail since it points out some subtle features of the scaling theory. We considered the correlation function

$$C(r) = \langle \tilde{\epsilon}(\vec{x})\psi(\vec{x}+\vec{r})\rangle . \tag{A1}$$

In the limit of small r we have

$$C(0) = \nu\langle \psi^3(\vec{x})\rangle , \tag{A2}$$

so that $C(0)$ determines the skewness S_0. We made the natural assumption that $C(r)$ had the inertial range form

$$C(r) \sim (L/r)^q , \quad q = \tfrac{2}{3} - \tfrac{1}{2}\zeta_2 + \tfrac{1}{2}\mu, \tag{A3}$$

and that

$$C(0) \sim (L/\eta)^q . \tag{A4}$$

We did not recognize, however, that since $\psi(\vec{x})$ is a velocity derivative, we must have

$$\int_{-\infty}^{\infty} C(r)\, dr = \langle \tilde{\epsilon}(\vec{x})[v_1(\infty) - v_1(-\infty)]\rangle = 0 \tag{A5}$$

for homogeneous turbulence, assuming that correlation functions factor in the limit of infinite separation.

Equations (A3) and (A5) are, however, not compatible since the numerical value of the exponent q is less than 1. The most plausible resolution of this apparent paradox is that Eq. (A3) be incorrect. One would *a priori* expect the leading-order term in an asymptotic expansion of $C(r)$ to be proportional to r^{-q}, but we have a kind of orthogonality expressed by Eq. (A5) which suggests that the amplitude of this leading-order term must vanish. In Ref. 8 we suggested that Eq. (A3) be tested experimentally. We now believe that this test would be negative. Either $C(r)$ has no inertial range power law, or its large-r behavior is of the form r^{-p}, with $p > 1$. We have no theoretical idea how to calculate the exponent p if it exists. Since Eq. (A3) is not valid, the scaling argument of Eq. (A4) is equally suspect, and thus the conclusion in Ref. 8 that $\theta = \frac{1}{2}\alpha$ is not logically sound. This does not of course mean that it is incorrect. We reach the same conclusion here by a different route, but that route involves an additional physical assumption.

We might also comment on the discussion in Ref. 8 on the role of variable spatial dimensionality. The details of this discussion have no merit, but the qualitative point that a natural dimension is defined for the problem is worth repeating. The variable $\psi(\vec{x})$ has scaling dimension $\frac{2}{3}$ in the Kolmogorov mean-field theory. If as in Ref. 8 or in a similar discussion by deGennes,[13] fluctuations in ψ^2 are considered to be dynamically important, the natural critical dimension for the problem is $4(\frac{2}{3}) = \frac{8}{3}$. In light of the criticism in Ref. 6, however, and the ideas presented here, it seems that fluctuations in ψ^3 are the most natural dynamical quantity. This leads to a natural choice of critical dimension of $6(\frac{2}{3}) = 4$. What, if anything, happens in the neighborhood of either of these values for spatial dimension is a completely open question. There is no soundly based evidence that $d = 4$ or $d = \frac{8}{3}$ are special dimensions for turbulence.

APPENDIX B: STATISTICS OF RATIO VARIABLES

The idea underlying the original 1962 Kolmogorov log-normality assumption was explained by Yaglom[14] in 1966. The averaged dissipation $\epsilon^*(r)$ can be written as a product of identically distributed random variables. If these variables are approximately statistically independent, then the logarithm of their sum will have an asymptotically normal distribution. The variables which enter this product are the ratio variables

$$q(r, l) = \epsilon^*(r)/\epsilon^*(l). \tag{B1}$$

In a very interesting paper, Novikov[15] derived some remarkable properties of these ratio variables. If the strong scale-similarity condition

$$\langle [q(r, l)]^p \rangle = (l/r)^{\mu_p} \tag{B2}$$

holds for all integer values of p, then $q(r, y)$ is statistically independent of $q(y, l)$ for $r < y < l$, and $q(r, l)$ is asymptotically log-normal for $l/r \to \infty$. The scaling exponents μ_p are not, however, given by those for the asymptotic log-normal distribution,

$$\mu_p^* = \tfrac{1}{2}\mu_2 p(p-1). \tag{B3}$$

The ideas of Novikov have been tested experimentally by Van Atta and Yeh,[16] and their data agree well with scale similarity and its predicted consequences. They do not, however, pay much attention to the numerical values of their measured exponents. We will look more closely at these values, and suggest that they indicate serious problems about the dynamical relevance of the ratio variables.

Van Atta and Yeh find

$$\mu_2 = 0.22, \quad \mu_3 = 0.61, \quad \mu_4 = 1.13. \tag{B4}$$

This is approximately the quadratic dependence on p that would be obtained from a log-normal distribution. This is not surprising since approximate log-normality should lead to low-order moments which are consistent with the asymptotic distribution. It is only for high-order moments where the quadratic dependence is mathematically inadmissible and physically unexpected. More important, however, is that the value of μ_2 is much smaller than the value of $\mu = 0.5$ measured from the fluctuations in $\bar{\epsilon}(\vec{x})$. Thus taking the ratio in Eq. (B1) strongly reduces the fluctuations even when $l \gg r$ so that the denominator fluctuates much less than the numerator. This is presumably due to the long-range correlations between the random variables in numerator and denominator in Eq. (B1).

In Novikov's paper the use of ratio variables keeps the fluctuations bounded, and it is essential to his proofs. Experiments show us that this bounding of fluctuations is physically as well as mathematically important. The fact that $\mu_2 \neq \mu$ means that there is no simple relation between the statistics of the ratio variables and the statistics of dissipation fluctuations. The Kolmogorov-Yaglom-Novikov arguments seem internally consistent with the observed fluctuations of ratio variables, but they tell us very little about the physically more important dissipation fluctuations. Thus there is no imcompatibility between the approximately quadratic p dependence of the μ_p in Eq. (B4) and the predicted linear dependence of the ζ_p and y_p in Eqs. (48) and (49). It would be nice to understand better the observed exponents in Eq. (B4), but this seems peripheral to the understanding of the dynamically important scaling exponents.

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[1]A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics*, translated from the russian by J. Lumley (MIT, Cambridge, Mass., 1975), Vol. II.
[2]A. M. Obukhov, J. Fluid Mech. 13, 77 (1962).
[3]A. N. Kolmogorov, J. Fluid Mech. 13, 82 (1962).
[4]R. H. Kraichnan, J. Fluid Mech. 62, 305 (1974).
[5]M. Nelkin, Phys. Rev. A 9, 388 (1974).
[6]U. Frisch, M. Lesieur, and P. L. Sulem, Phys. Rev. Lett. 37, 895 (1976).
[7]U. Frisch, P. L. Sulem, and M. Nelkin, J. Fluid. Mech. (to be published).
[8]M. Nelkin, Phys. Rev. A 11, 1737 (1975).
[9]L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Addison-Wesley, Reading, Mass., 1959). See Eq. (33.19).
[10]C. H. Gibson, G. R. Stegen, and S. McConnell, Phys. Fluids 13, 2448 (1970).
[11]C. W. Van Atta and J. Park, in *Statistical Models and Turbulence*, edited by M. Rosenblatt and C. Van Atta (Springer, Berlin, 1972), pp. 402–426.
[12]B. Mandelbrot, *Fractals: Form, Chance and Dimension* (Freeman, San Francisco, 1977).
[13]P. G. deGennes, in *Fluctuations, Instabilities and Phase Transitions*, edited by T. Riste (Plenum, New York, 1975).
[14]A. M. Yaglom, Dokl. Akad. Nauk. SSSR 166, 49 (1966) [Sov. Phys. Dokl. 11, 26 (1966)]. See also Ref. 1, pp. 613–620.
[15]E. A. Novikov, Prikl. Mat. Mekh. 35, 266 (1971) [Appl. Math. Mech. 35, 231 (1972)]. See also Ref. 1, pp. 620–625.
[16]C. W. Van Atta and T. T. Yeh, J. Fluid Mech. 71, 417 (1975).