Stimulated emission from relativistic electrons passing through a spatially periodic transverse magnetic field

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Stimulated emission from a relativistic beam of electrons passing through a spatially periodic right-hand circularly polarized magnetic field is considered. The amplification is found to be due to a ponderomotive bunching of the electrons. The effect is completely classical and for an infinite interaction distance a dispersion relation, which takes into account space-charge effects, describing the scattered field is derived. Conditions on the pump field amplitude, beam density, and momentum spread of the beam for emission from individual electrons to occur or for emission from plasma oscillations to occur are examined. Also, emission from individual electrons over a finite interaction distance is considered and gain is determined for distances less than an e-folding length.

I. INTRODUCTION

Stimulated emission with the stimulated radiation occurring in the direction of the electron beam has recently been observed by Elias *et al.*[1] by passing a beam of relativistic electrons through a spatially periodic magnetic field polarized transversely to the beam direction. This effect has been considered theoretically by several authors. Pantell, Soncini, and Puthoff[2] have given a quantum-mechanical derivation of the gain for an infinite interaction region, while Sukhatme and Wolff[3] give a quantum derivation taking into account a finite interaction length. Madey, Schwettman, and Fairbank[4] also have given a quantum derivation and pointed out that their formula does not contain Planck's constant indicating the possibility of a classical interpretation. They compared their result with a classical traveling-wave analysis which bears little resemblance and concluded that a classical interpretation was not feasible. Hopf *et al.*[5] have subsequently shown that Sukhatme and Wolff's finite interaction length results can be derived classically. All of the above derivations neglect emission from the plasma oscillations of the electron beam (Raman scattering) which may be important for large beam densities and has been considered by Sprangle *et al.*[6] and Kwan *et al.*[7]

The motivation for the present work is the issue raised by Madey, Schwettman, and Fairbank of their quantum-mechanically derived gain formula having a limit as Planck's constant was set equal to zero which raised the question of whether their result could be obtained classically. They expressed substantial doubts that this was the case and one of the key points presented in support of these doubts was the fact that their gain expression differed in what appeared to be a fundamental way from an expression which had been derived on the basis of conventional traveling-wave amplifier theory. Although classical derivations yielding results in close agreement with the quantum-mechanically derived gain formula have appeared, the issue of disagreement with the results of traveling-wave tube analysis has not received much attention. It is our purpose to present an extended version of classical traveling-wave amplifier theory with the objective of providing a uniform framework within which one can study the various regimes of operation which exist. One result of this analysis is a resolution of the issue raised in Ref. 4. The analysis presented here takes into account the distribution in momentum along the direction of the relativistic electron beam, and the interaction between the electrons is included. Also, the effects of short amplifier lengths for the case of a sufficiently low-density beam are examined. The gain formulas for emission from individual electrons or emission from the electron's collective interactions will be derived along with conditions on the pump field strength, beam density, and fractional momentum spread of the beam for the various operating regimes.

II. DERIVATION OF DISPERSION RELATION

In the following we give a classical derivation of the emitted radiation taking into account both interaction with individual electrons and plasma oscillations by first determining in the laboratory frame of reference the induced nonlinear driving current which is then substituted in the wave equation to give a dispersion relation for the emitted radiation. As a model we consider a cold, collisionless, infinitely-long, homogeneous, neutralized, relativistic, monoenergetic electron beam traveling in the $+z$ direction through a spatially periodic, right-hand circularly polarized magnetic field

$$\vec{B}_0 = \tfrac{1}{2}[(\hat{x}+i\hat{y})B_0 e^{ik_0 z} + (\hat{x}-i\hat{y})B_0 e^{-ik_0 z}]. \quad (2.1)$$

The derivation of the growth of the emitted radiation when an electron passes through the field $\vec{B}_0$ can be calculated by treating the scattering of an electron by an electromagnetic field

$$\vec{E}_1 = \tfrac{1}{2}[(\hat{x}+i\hat{y})\exp[i(k_1 z+\omega_1 t)] + (\hat{x}-i\hat{y})\exp[-i(k_1 z+\omega_1 t)]E_1 \quad (2.2)$$

with the amplitudes B_0, E_1 and wave numbers and frequency k_0, k_1, ω_1 being related as follows. When an electron beam of velocity $v\hat{z}$ interacts with the field, its momentum is perturbed and satisfies the Lorentz force equation for the spatially periodic magnetic field

$$\frac{d}{dt}\delta\vec{p}(z) = evB_0[\hat{y}\cos k_0 z - \hat{x}\sin k_0 z], \quad (2.3)$$

and for the electromagnetic field

$$\frac{d}{dt}\delta\vec{p}(z,t) = e\left(1+\frac{k_1 v}{\omega_1}\right)\vec{E}_1(z,t). \quad (2.4)$$

Solving Eqs. (2.3) and (2.4) for the perturbed momentum yields

$$\delta\vec{p}(z) = (eB_0/k_0)(\hat{x}\cos k_0 z + \hat{y}\sin k_0 z), \quad (2.5)$$

$$\delta\vec{p}(z,t) = \frac{eE_1}{\omega_1}[\hat{x}\sin(k_1 z+\omega_1 t) + \hat{y}\cos(k_1 z+\omega_1 t)]. \quad (2.6)$$

We wish to specify the electromagnetic field in a manner which yields the same motion as occurs for the static magnetic field, therefore we equate the magnitudes of the perturbed momentum:

$$E_1/\omega_1 = B_0/k_0. \quad (2.7)$$

In addition, since the time dependence must also be the same,

$$k_0 v = \omega_1 + k_1 v \quad (2.8)$$

so that the growths calculated using an electromagnetic wave yield the growth for a spatially periodic magnetic field by making the transformation given by Eqs. (2.7) and (2.8).

The total right-hand circularly polarized electric field used in the calculation of the growth is given by

$$\vec{\mathcal{E}}(z,t) = \tfrac{1}{2}[(\hat{x}-i\hat{y})E_1\exp[-i(k_1 z+\omega_1 t)] + (\hat{x}-i\hat{y})E_2\exp[i(k_2 z-\omega_2 t)] + \text{c.c.}], \quad (2.9)$$

where the first term is the pump field which is traveling opposite to the electron beam with k_1 satisfying the linear dispersion (c = speed of light = 1),

$$k_1^2 = \omega_1^2 - \omega_p^2(m/E) \quad (2.10)$$

where $\omega_p = (4\pi n e^2/m)^{1/2}$ is the plasma frequency, n the electron number density in the laboratory frame, and $E = m\gamma$ is the electron energy. The second term in Eq. (2.9) is the backscattered radiation traveling in the same direction as the electrons with k_2 containing a small imaginary part and is Doppler shifted up in frequency by approximately

$$\omega_2 \approx 4(E/m)^2\omega_1. \quad (2.11)$$

The growth of the backscattered radiation is due to a bunching of the electrons. The electrons interacting with the electromagnetic field acquire a small oscillation velocity transverse to the beam direction. This velocity in turn interacts with the electromagnetic field through the Lorentz $\vec{V}(z,t)\times\vec{B}(z,t)$ force to provide a mechanism coupling the electrons, pump field, and scattered field allowing the electrons to give up energy to the scattered radiation.

The distribution function $f(p,z,t)$ of the electrons from which the driving current is obtained satisfies the one-dimensional relativistic collisionless Boltzmann equation[8]

$$\frac{\partial f(p,z,t)}{\partial t} + \vec{V}(z,t)\cdot\hat{z}\,\frac{\partial f(p,z,t)}{\partial z} + e[-\vec{\nabla}\phi(z,t) + \vec{\mathcal{E}}(z,t) + \vec{V}(z,t)\times\vec{B}(z,t)]\cdot\hat{z}\,\frac{\partial f(p,z,t)}{\partial p} = 0, \quad (2.12)$$

where $\phi(z,t)$ is the self-consistent space-charge potential describing the interaction between the electrons which satisfies Poisson's equation

$$\frac{\partial^2\phi(z,t)}{\partial z^2} = -4\pi n e\left(\int dp\, f(p,z,t) - 1\right), \quad (2.13)$$

and the velocity $\vec{V}$ satisfies the relativistic Lorentz force equation[9]

$$\frac{d\vec{V}}{dt} = \frac{e}{E}[\vec{\mathcal{E}} + \vec{V}\times\vec{B} - \vec{V}(\vec{V}\cdot\vec{\mathcal{E}})]. \quad (2.14)$$

The electromagnetic field is taken to be a small perturbation on the electron's motion, so the above equations are solved iteratively by expanding the velocity $\vec{V}$, distribution f, and potential ϕ in powers of the field amplitude:

$$\vec{V} = v\hat{z} + \vec{v}^{(1)}(z,t), \quad (2.15)$$

$$f = f^{(0)}(p) + f^{(1)}(p,z,t) + f^{(2)}(p,z,t), \quad (2.16)$$

$$\phi = \phi^{(0)}(z,t) + \phi^{(1)}(z,t) + \phi^{(2)}(z,t). \quad (2.17)$$

Substituting into Eqs. (2.12)–(2.14) yields

$$f^{(1)}=\phi^{(0)}=\phi^{(1)}=0\ , \tag{2.18}$$

$$E\frac{\partial\vec{v}^{(1)}}{\partial t}+vE\frac{\partial\vec{v}^{(1)}}{\partial z}=e\vec{\mathcal{E}}+ev\hat{z}\times\vec{B}\ , \tag{2.19}$$

$$\frac{\partial f^{(2)}}{\partial t}+v\frac{\partial f^{(2)}}{\partial z}+e\left(-\frac{\partial\phi^{(2)}}{\partial z}+(\vec{v}^{(1)}\times\vec{B})\cdot\hat{z}\right)\frac{\partial f^{(0)}}{\partial p}=0\ , \tag{2.20}$$

$$\frac{\partial^2\phi^{(2)}}{\partial z^2}=-4\pi ne\int dp f^{(2)}\ , \tag{2.21}$$

where v is the constant zero-order beam velocity, $f^{(0)}(p)$ is the initial distribution normalized as $\int f^{(0)}dp=1$, $\vec{v}^{(1)}$ is the transverse oscillation velocity, and $f^{(2)}$ describes the density fluctuations producing the scattering.

Next, Eqs. (2.19)–(2.21) are Fourier transformed in space and time to give

$$\vec{v}^{(1)}(k,\omega)=\frac{ie\vec{\mathcal{E}}(k,\omega)}{\omega E}\ , \tag{2.22}$$

$$f^{(2)}(p,k,\omega)=\left[\frac{iek\phi^{(2)}(k,\omega)-\int\int dq\,d\xi[(k-g)/(\omega-\zeta)]\vec{v}^{(1)}(q,\zeta)\cdot\vec{\mathcal{E}}(k-q,\omega-\zeta)}{(ikv-\omega)}\right]\frac{\partial f^{(0)}(p)}{\partial p}\ , \tag{2.23}$$

$$\phi^{(2)}(k,\omega)=4\pi en^{(2)}(k,\omega)/k^2\ . \tag{2.24}$$

For the processes of interest only the beat term between the pump field and scattered field yielding $k=k_1+k_2$, $\omega=\omega_2-\omega_1$ is relevant, so using Eqs. (2.22) and (2.24) in the expression for $f^{(2)}$ and keeping only the relevant terms gives

$$f^{(2)}(p,k,\omega)=\frac{4\pi e^2n^{(2)}(k,\omega)}{k(kv-\omega)}\frac{\partial f^{(0)}(p)}{\partial p}+\frac{e^2E_1E_2k}{2(kv-\omega)\omega_1\omega_2E}\frac{\partial f^{(0)}(p)}{\partial p}\times\delta(k-k_1-k_2)\delta(\omega+\omega_1-\omega_2)\ . \tag{2.25}$$

In order to eliminate $n^{(2)}(k,\omega)$ from the above expression each side of Eq. (2.25) is multiplied by the unperturbed electron-number density and integrated over momentum to yield

$$n^{(2)}(k,\omega)\epsilon(k,\omega)=\int dp\frac{ne^2E_1E_2k}{2(kv-\omega)\omega_1\omega_2E}\frac{\partial f^{(0)}(p)}{\partial p}\times\delta(k-k_1-k_2)\delta(\omega+\omega_1-\omega_2)\ , \tag{2.26}$$

where $\epsilon(k,\omega)$ is the dielectric function and is defined by

$$\epsilon(k,\omega)=1-\frac{m\omega_p^2}{k}\int dp\frac{\partial f^{(0)}/\partial p}{kv-\omega}\ . \tag{2.27}$$

Equation (2.26) is substituted in Eq. (2.25) to give the final expression for the transformed second-order distribution

$$f^{(2)}(p,k,\omega)=\frac{e^2E_1E_2k}{2(kv-\omega)\omega_1\omega_2}\frac{\partial f^{(0)}}{\partial p}\times\left[\frac{m\omega_p^2}{k\epsilon(k,\omega)}\int\frac{dp'}{E'}\frac{\partial f^{(0)}/\partial p'}{kv'-\omega}+\frac{1}{E}\right]\ . \tag{2.28}$$

The above expression for $f^{(2)}$ is used to calculate the nonlinear transverse driving current which is defined by

$$\vec{J}_3=en\int dp\,\vec{v}^{(1)}f^{(2)}\ . \tag{2.29}$$

The emitted field satisfies the one-dimensional wave equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial z^2}\right)\vec{E}_2=-4\pi F\frac{\partial\vec{J}_3}{\partial t}\ , \tag{2.30}$$

where F is a phenomenological filling factor which describes the coupling of the electron beam to the electromagnetic mode being amplified. For a uniform electromagnetic plane wave and homogeneous electron beam of infinite cross section, F is unity, and for the realistic case of finite beam cross section, F is unity when the electron beam completely fills the electromagnetic beam, that is to say when the electron-beam radius exceeds the electromagnetic beam radius. In the opposite case F is given to a good approximation by the ratio of the electron-beam area to the electromagnetic-beam area. Taking the Fourier transform of Eq. (2.30) with Eq. (2.29) being substituted on the right-hand side gives

$$(\omega_2^2-k_2^2)\vec{E}_2=-i\omega_2 4\pi Fne\times\int dp\left(\int\int dq\,d\zeta f^{(2)}(p,q,\zeta)\times\vec{v}^{(1)}(k_2-q,\omega_2-\zeta)\right)\ . \tag{2.31}$$

Substituting Eq. (2.22) for $\vec{v}^{(1)}$ and Eq. (2.28) for $f^{(2)}$ finally yields the dispersion relation for the emitted field

$$\omega_2^2 - k_2^2 = m\omega_p^2 kF \int \frac{dp}{E} \left[\frac{e^2E_1^2 \partial f^{(0)}/\partial p}{2\omega_1^2 (kv-\omega)} \left(\frac{m\omega_p^2}{k\epsilon(k,\omega)} \int \frac{dp'}{E'} \frac{\partial f^{(0)}/\partial p'}{kv'-\omega} + \frac{1}{E} \right) \right] . \tag{2.32}$$

Equation (2.32) determines the relation between the emitted frequency ω_2 and wave-number k_2. Since an amplifier is being studied, ω_2 is taken to be a real specifiable parameter with situations being looked for where k_2 has a negative imaginary part. The analysis of the dispersion relation is carried out in Sec. III.

III. ANALYSIS AND DISCUSSION OF GAIN COEFFICIENTS

In order to examine the different regimes, limiting cases are considered. It is first assumed the density is low in which case Compton scattering (magnetic brehsstrahlung) occurs and the interaction between the bunched electrons is not important. Next, the low-density case for a finite interaction length L is examined. Finally, the large-density limit (Raman effect), where the interaction between the bunched electrons is important, for an infinite interaction length is looked at and conditions on the various parameters for the validity of each regime of operation are obtained.

A. Compton effect (long interaction region)

In the case of low density the first term in the large parentheses of Eq. (2.32) is negligible and Eq. (2.32) becomes

$$\omega_2^2 - k_2^2 = \frac{m\omega_p^2 kFe^2E_1^2}{2\omega_1^2} \int \frac{dp}{E^2} \frac{\partial f^{(0)}/\partial p}{kv-\omega} \tag{3.1}$$

When the phase velocity of the bunching force is equal to the particle velocity

$$(k_1 + k_2)v = \omega_2 - \omega_1 \tag{3.2}$$

for Compton scattering, or

$$k_0 v = \omega_2 - k_2 v \tag{3.3}$$

for magnetic bremsstrahlung, the bunching force will appear to be stationary with respect to the electrons and for a small velocity difference the electrons will either lose or obtain energy from the wave. Equations (3.2) and (3.3) are also a consequence of energy-momentum conservation. The kinetic energy of the electrons is

$$E = m\gamma = (p^2 + |\delta\vec{p}|^2 + m^2)^{1/2} , \tag{3.4}$$

where $|\delta\vec{p}|^2 = e^2E_1^2/\omega_1^2 = e^2B_0^2/k_0^2$ is given by Eqs. (2.6) and (2.5). The momenta appearing in Eq. (3.4) are related to the zero-order velocity by $vE = p$ and the transverse oscillation velocity by $\vec{v}^{(1)} E = \delta\vec{p}$. Taking $\omega_2 \approx k_2$, $\omega_1 \approx k_1$ in Eq. (3.2) gives $\omega_1 = \omega_2(1-v^2)/(1+v)^2$. Equation (3.4) is used to determine $1 - v^2 = (1/\gamma^2)[1 + (eE_1/m\omega_1)^2]$, so for $v \approx 1$ Eq. (3.2) becomes

$$\omega_2 \approx \frac{4\gamma^2\omega_1}{1 + (eE_1/m\omega_1)^2} , \tag{3.5}$$

while in the magnetic brehsstrahlung case (3.3) gives

$$\omega_2 \approx \frac{2\gamma^2 k_0}{1 + (eB_0/mk_0)^2} . \tag{3.6}$$

Equations (3.5) and (3.6) are the Doppler-shifted frequency of the forward spontaneous synchrotron radiation as given in Ref. 1. The correction in the denominators of (3.5) and (3.6), $eE_1/m\omega_1 = eB_0/mk_0$, is typically quite large but does not threaten the approximations used to derive the dispersion relation since $|\vec{v}^{(1)}| = |\delta\vec{p}/m\gamma|$ is the quantity that has been treated as being small.

In evaluating the integral in Eq. (3.1) it is necessary to take account of the fact that k_2 may be complex and that a Landau contour[10] must be used. To this end we write $k = k' - i\Gamma$, where k' and Γ are real, so that Γ represents the amplitude gain per unit length. For $\Gamma > 0$ (and assuming $v > 0$) the integral can be evaluated along the real p axis. The value for $\Gamma < 0$ is obtained by analytically continuing the function defined by the integral from positive Γ. It is useful to write the integral in Eq. (3.1) as follows ($\Gamma > 0$):

$$\int \frac{dp}{E^2} \frac{\partial f^{(0)}}{\partial p} \frac{1}{kv-\omega} = \int \frac{dp}{E^2} \frac{\partial f^{(0)}}{\partial p} \left[\frac{k'v-\omega}{(k'v-\omega)^2 + \Gamma^2 v^2} + i \frac{\Gamma v}{(k'v-\omega)^2 + \Gamma^2 v^2} \right] \tag{3.7}$$

and to distinguish two cases depending upon which of the two factors $\partial f^{(0)}/\partial p$, $1/(kv-\omega)$ is more rapidly varying as a function of p. Evidently, for sufficiently small Γ it is $1/(kv-\omega)$ which is the more rapidly varying, while for large Γ the opposite is the case (assuming, as we do, that $k'v - \omega \approx 0$ at the peak of $f^{(0)}$). The two cases are characterized by $\Gamma/k' \ll \Delta v/v$ and $\Gamma/k' \gg \Delta v/v$, respectively, where Δv is the width of the velocity distribution.

Case 1: small Γ limit. Where useful we treat $\partial f^{(0)}/\partial p$ and smooth functions of v as slowly varying and write

$$\int \frac{dp}{E^2}\frac{\partial f^{(0)}}{\partial p}\frac{1}{kv-\omega}$$

$$=\int \frac{dp}{E^2}\frac{\partial f^{(0)}}{\partial p}\frac{k'v-\omega}{(k'v-\omega)^2+\Gamma^2v^2}$$

$$+\frac{i\Gamma\omega E}{m^2k'}\frac{\partial f^{(0)}}{\partial p}\bigg|_{v=\frac{\omega}{k'}}\int du\,\frac{1}{(k'-\omega u)^2+\Gamma^2}, \tag{3.8}$$

which yields

$$\mathrm{Im}\int \frac{dp}{E^2}\frac{\partial f^{(0)}}{\partial p}\frac{1}{kv-\omega}=\frac{\pi E}{m^2k'}\frac{\partial f^{(0)}}{\partial p}\bigg|_{v=\frac{\omega}{k'}}. \tag{3.9}$$

We note that for this limit analytic continuation to negative Γ simply amounts to using Eq. (3.8) for the real part of the integral (which depends only on Γ^2) and Eq. (3.9) for the imaginary part. Substitution into Eq. (3.1) then yields for the amplitude growth rate Γ_1

$$\Gamma_1=\frac{\pi\omega_p^2Fe^2E_1^2E}{4\omega_1^2\omega_2 m}\frac{\partial f^{(0)}}{\partial p}\bigg|_{v=\frac{\omega}{k'}}, \tag{3.10}$$

$\partial f^{(0)}/\partial p|_{v=\omega/k'}$ describes the character of the electron population levels. When it is positive the population is inverted and amplification takes place. When it is negative there is absorption.

After taking due care with notational difference, Eq. (3.10) may be compared with the quantum-mechanically derived results of Refs. 1 and 2. Equation (3.10) is a factor of 2 larger than the result of Ref. 2. The difference arises from the fact that the mode number equation [their Eq. (6)] is a factor of 2 too large due to the erroneous inclusion of both polarizations. In order to compare Eq. (3.10) with the equation given in Ref. 1 we reexpress the result in terms of the frequency distribution function $g(\omega)$ of the spontaneous radiation that would be emitted by the beam as it passes through the magnet. In this long interaction-length regime the line shape arises entirely from the distribution of Doppler shifts so that there is a direct connection between $f^{(0)}(p)$ and $g(\omega)$. Neglecting terms which vary little over the widths of the rapidly varying function we find

$$p^2\frac{\partial f^{(0)}}{\partial p}=\frac{4}{[1+(eB_0/mk_0)^2]^2}\,\omega^2\frac{\partial g}{\partial\omega}$$

$$=\frac{4}{[1+(eE_1/m\omega_1)^2]^2}\,\omega^2\frac{\partial g}{\partial\omega}, \tag{3.11}$$

where the frequency and momentum derivatives are related by Eqs. (3.3) and (3.2). Equation (3.10) then becomes ($v\approx 1$)

$$\Gamma_1=\beta\frac{E_1^2}{\omega_1^2}\left[1+\left(\frac{eE_1}{m\omega_1}\right)^2\right]^{-2}=\beta\frac{B_0^2}{k_0^2}\left[1+\left(\frac{eB_0}{mk_0}\right)^2\right]^{-2}, \tag{3.12}$$

where

$$\beta=\frac{\pi\omega_p^2Fe^2\omega^2\partial g/\partial\omega}{mvp\omega_2}.$$

Equation (3.12) is identical with the result of Ref. 1 when the order B_0^2 correction in the denominator can be neglected. It is noted that when the order B_0^2 correction in the denominator is not negligible, Eq. (3.12) and the result in Ref. 1 are in slight disagreement. The B_0^2 correction may not appear in their result due to a systematic neglect of terms of this order in their calculation. It is interesting that the agreement between the $p^2\partial f^{(0)}/\partial p$ formula is exact while only approximate in the $\omega^2\partial g/\partial\omega$ form. It will be shown later that Eq. (3.12) also applies in the case of a short interaction length as does the result in Ref. 1. Our conclusion, therefore, is that the results obtained classically and quantum mechanically are the same.

Case 2: small Δv limit. We treat $f^{(0)}$ as the only rapidly varying quantity and write

$$\int \frac{dp}{E^2}\frac{\partial f^{(0)}}{\partial p}\frac{1}{kv-\omega}\approx-\frac{\partial}{\partial p}[E^{-2}(kv-\omega)^{-1}]_{v=v_0}\int dp\,f^{(0)}$$

$$\approx\frac{km^2}{E_0^5(kv_0-\omega)^2}. \tag{3.13}$$

The dispersion relation (3.1) is then found to be

$$\omega_2^2-k_2^2=\frac{Fe^2E_1^2\omega_p^2k^2m^3}{2\omega_1^2E_0^5(kv_0-\omega)^2}. \tag{3.14}$$

Equation (3.14) is identical to the dispersion relation which is found by applying standard traveling-wave tube theory to a monoenergetic electron beam bunched by the $\vec{V}\times\vec{B}$ force. We have a quartic expression for k_2 which can, in the usual way, be replaced by a cubic by setting $\omega_2^2-k_2^2=2\omega_2(\omega_2-k_2)$ on the left-hand side. This substitution eliminates the uninteresting backward wave. Maximum gain is obtained when the velocity matching wave number k_{20}, defined by $k_{20}=(\omega_2-\omega_1)/v_0-k_1$, is also precisely equal to ω_2. This occurs when $\omega_2=\omega_{20}\equiv\omega_1(1+v_0)/(1-v_0)$. With this assumption Eq. (3.14) becomes

$$k_{20}-k_2=\frac{Fe^2E_1^2\omega_p^2k^2m^3}{4\omega_1^2\omega_2E_0^5v_0^2(k_2-k_{20})^2}\equiv\frac{\kappa^3}{(k_2-k_{20})^2}, \tag{3.15}$$

which yields the three well known roots

$$k_2-k_{20}=\begin{bmatrix}e^{-i\pi/3}\\ e^{i\pi/3}\\ -1\end{bmatrix}\kappa. \tag{3.16}$$

From this (using $k_2 \approx \omega_2$, $v_0 \approx 1$) we obtain the following expression for the maximum gain Γ_2,

$$\Gamma_2 = \frac{\sqrt{3}}{2} \frac{m}{E_0} F^{1/3} \left(\frac{eE_1}{E_0 \omega_1}\right)^{2/3} \left(\frac{\omega_p \omega_2^{1/2}}{2}\right)^{2/3} . \quad (3.17)$$

Equation (3.17) is of the same form as that quoted in Ref. 4 as Eq. (5). The contrast between Γ_1 and Γ_2 is seen not to arise as a difference between a quantum and classical description as there suggested, but rather as from the differing physical circumstances associated with case 1 and case 2. Under the condition of case 2 a correct quantum treatment must give Eq. (3.17). We note that under case 1 conditions only a portion of the electron distribution participates in the process, while under case 2 conditions all participate. It should also be mentioned that the distinction between these two cases has long been familiar in plasma physics in the context of two-stream instabilities and nonlinear Landau damping.

Equation (3.10) can be used to reexpress the condition which distinguishes case 1 from case 2 in terms of the coupling strength. Defining Δp^2 by $|\partial f^{(0)}/\partial p|_{\max} \equiv 1/\Delta p^2$, we find that the condition $\Gamma_1/k_2 \ll \Delta v/v$ implies

$$\frac{\pi \omega_p^2 F e^2 E_1^2 E}{4 \omega_1^2 \omega_2^2 m^3} \ll \left(\frac{\Delta p}{p}\right)^3 . \quad (3.18)$$

One sees that for a given $\Delta p/p$ one is always in the case 1 regime for sufficiently small $\omega_p^2 E_1^2$. However, as one increases $\omega_p^2 E_1^2$, say with the aim of increasing the gain, condition (3.18) is eventually violated and Eq. (3.17) takes over. Thus while the gain increases linearly in $\omega_p^2 E_1^2$ with small values of that parameter, it increases only as $(\omega_p^2 E_1^2)^{1/3}$ for large values.

When the frequency matching condition is not satisfied, the gain is reduced. For $|\omega_2 - \omega_{20}| \ll 2\gamma^2 \kappa$ we find

$$\Gamma_2 \approx \frac{\sqrt{3}}{2} \kappa \left(1 - \frac{(\omega_2 - \omega_{20})^2}{36 \gamma^4 \kappa^2}\right) . \quad (3.19)$$

For $\omega_2 - \omega_{20} < 0$ there is a threshold for gain, but there is no threshold for $\omega_2 - \omega_{20} > 0$. This may be expressed by the requirement for gain $\omega_2 - \omega_{20} > -2^{1/3} 3 \kappa \gamma^2$. When $\omega_2 - \omega_{20} \gg 2\gamma^2 \kappa$, the gain is given by

$$\Gamma_2 \approx \frac{\sqrt{2} \gamma \kappa^{3/2}}{(\omega_2 - \omega_{20})^{1/2}} . \quad (3.20)$$

The root of Eq. (3.14) which exhibits gain always occurs near a zero of $kv_0 - \omega$. Even when $\omega_2 - \omega_{20}$ is large and positive, the real part of $kv_0 - \omega$ is always positive. Hence, the bunching wave velocity ω/k' is always less than v_0 as expected.

B. Compton effect (short interaction region)

In order to obtain the growth for Compton scattering over a finite interaction length L Eqs. (2.19) and (2.20) with $\phi^{(2)} = 0$ are used. Since the interaction length is finite, these equations can no longer be solved using Fourier transforms but must be integrated directly over position. Keeping only the beat term, giving $k = k_1 + k_2$, $\omega = \omega_2 - \omega_1$, Eq. (2.20) becomes

$$\frac{\partial f^{(2)}}{\partial t} + v \frac{\partial f^{(2)}}{\partial z} = -\frac{ie^2 E_1 E_2 k}{2E \omega_1 \omega_2} \frac{\partial f^{(0)}}{\partial p} \exp[-i(kz - \omega t)] \quad (3.21)$$

whose solution is given when integrated from 0 to z by [neglecting z dependence of E_2 as in Eq. (3.25)]

$$f^{(2)}(p, z, t) = \frac{e^2 E_1 E_2 k}{2\omega_1 \omega_2 (kp - \omega E)} \times \frac{\partial f^{(0)}}{\partial p} [e^{-ikz} - e^{-i\omega z/v}] e^{i\omega t} . \quad (3.22)$$

Equation (2.19) gives for the transverse oscillation velocity

$$\vec{v}^{(1)}(z, t) = (eE_1/i\omega_1 E)(\hat{x} + i\hat{y}) \exp[i(k_1 z + \omega_1 t)] . \quad (3.23)$$

Substituting Eqs. (3.23) and (3.22) into the definition of the driving current (2.29) the wave equation for the emitted field Eq. (2.30) becomes

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial z^2}\right) E_2(z) \exp[-i(k_2 z - \omega_2 t)] = \frac{-2\pi e^4 n F k E_1^2 E_2(z)}{\omega_1^2} \int \frac{dp}{E} \frac{\partial f^{(0)}/\partial p}{(kp - \omega E)} \times \left\{1 - \exp\left[i\left(k - \frac{\omega}{v}\right) z\right]\right\} \exp[-i(k_2 z - \omega_2 t)] . \quad (3.24)$$

Under the assumption of small growth in the available length, $\Gamma_L L \ll 1$, integrating the wave equation (3.24) from 0 to L gives the amplitude gain per unit length

$$\Gamma_L = \frac{m \omega_p^2 F k e^2 E_1^2}{L 2 \omega_1^2 k_2} \int \frac{dp}{Ep} \frac{\partial f^{(0)}/\partial p}{(k - \omega/v)^2} \sin^2 \frac{(k - \omega/v)L}{2} . \quad (3.25)$$

This expression is a factor of 4 larger than the quantum-mechanically derived result of Sukhatme and Wolff, and it is the same as the classically derived result of Hopf *et al*. To reexpress Eq. (3.25) in terms of the sharp line approximation to the spontaneous line shape, which takes the form in this case

$$g(\omega)=\frac{2(1-v)}{\pi}\int dp f^{(0)}(p)\,\frac{\sin^2\frac{1}{2}(k-\omega/v)L}{(k-\omega/v)^2L}\,, \quad (3.26)$$

Eq. (3.25) is integrated by parts treating $E^{-1}p^{-1}$ as slowly varying and yields

$$\frac{1}{Ep}\int dp\,\frac{\partial f^{(0)}}{\partial p}\,\frac{\sin^2\frac{1}{2}(k-\omega)L}{(k-\omega/v)^2L}=\frac{\pi\omega}{2m\gamma^3(1-v)^2Ep}\,\frac{\partial g}{\partial\omega}\,, \quad (3.27)$$

where the momentum derivative has been reexpressed in terms of the frequency derivative. Using $k\approx\omega,\ k_2\approx\omega_2,\ v\approx 1$, Eq. (3.25) becomes

$$\Gamma_1=\beta\,\frac{E_1^2}{\omega_1^2}\left[1+\left(\frac{eE_1}{m\omega_1}\right)^2\right]^{-2}\doteq\beta\,\frac{B_0^2}{k_0^2}\left[1+\left(\frac{eB_0}{mk_0}\right)^2\right]^{-2}\,, \quad (3.28)$$

with

$$\beta=\frac{\pi\omega_p^2Fe^2\omega^2\partial g/\partial\omega}{mp\omega_2}\,.$$

Equations (3.28) and (3.12) are identical so that expressing the growth in terms of the spontaneous line-shape function has the virtue of giving the same result for a short and long interaction region. By taking the limit of the finite interaction length L going to infinity Eq. (3.25) yields Γ_1 and taking this limit corresponds to the case of exact momentum conservation since it gives a Dirac delta function with Eq. (3.2) as its argument. For the case of a sharp momentum distribution $f^{(0)}=\delta(p-p_0)$, Eq. (3.25) is integrated by parts to give

$$\Gamma_L=\frac{m\omega_p^2kFe^2E_1^2}{2L\omega_1^2k_2}\left(\frac{m^2\omega L^3}{8p_0^3E_0^2}\,\frac{\partial}{\partial\theta}\,\frac{\sin^2\theta}{\theta^2}-\frac{L^2\sin^2\theta}{\theta^2}\,\frac{\partial E^{-1}p^{-1}}{\partial p}\right)_{p=p_0}\,, \quad (3.29)$$

where $\theta=(\omega/v-p)\frac{1}{2}L$. Assuming that last term in Eq. (3.29) is negligible and $k\approx k_2\approx\omega_2\gg\omega_1$ gives the maximum

$$\Gamma_L\Big|_{\max}=\frac{0.54}{16}\,\frac{\omega_p^2m^3\omega_2e^2E_1^2L^2F}{\omega_1^2E_0^5}\,. \quad (3.30)$$

This expression is valid only for situations in which $\Gamma_L L$ is well below unity. As L increases one might heuristically expect the L on the right-hand side to be properly replaced by $1/\Gamma_L$, and it is interesting to observe that one then obtains an expression of the same form as Eq. (3.17).

C. Raman effect (long interaction region)

When the beam density becomes large enough, so that the interaction between the electrons becomes important, emission from the plasma oscillations of the beam (Raman effect) will take place. All of the terms in the infinite-interaction-length expression (2.32) are kept and the integrals can be performed along the real axis from $-\infty$ to ∞. Integrating by parts and using $f^{(0)}=\delta(p-p_0)$, $\epsilon=1-(\omega_p^2/\gamma_0^3)/(kv_0-\omega)^2$ gives

$$\left(\omega_2^2-k_2^2-\frac{\omega_p^2}{\gamma_0}\right)\left((kv_0-\omega)^2-\frac{\omega_p^2}{\gamma_0^3}\right)=\frac{e^2E_1^2F\omega_p^2}{2E_0^2\gamma_0^2\omega_1^2}\left[\frac{k}{\gamma_0^{1/2}}+\omega_p\right]^2\,, \quad (3.31)$$

where the linear dispersion equation (2.10) for $k_2,\ \omega_2$ has been taken into account in the first term on the left-hand side of Eq. (3.31) and $p_0\approx E_0$ was used to obtain the square-bracketed term on the right-hand side of Eq. (3.31). When $k/\gamma_0^{1/2}\gg\omega_p$ in Eq. (3.31), this expression becomes identical, for a Debye length of zero, to Eq. (25) of Ref. 7. The right-hand side of Eq. (3.31) can be thought of as a coupling between the emitted wave and the plasma wave on the electron beam. Only the Stokes case $kv_0-\omega=\omega_p/\gamma_0^{3/2}$ gives gain. Defining $\eta=(\omega_2^2-\omega_p^2/\gamma_0)^{1/2}$ and the frequency-velocity matching condition for the Stokes case $k_1v_0+\omega_1-\omega_2+\eta v_0=\omega_p/\gamma_0^{3/2}$ then give $(v_0\approx 1)$

$$(k_2-\eta)^2=\frac{-e^2E_1^2F\omega_p}{8\eta E_0^2\gamma_0^{1/2}\omega_1^2}\left[\frac{k}{\gamma_0^{1/2}}+\omega_p\right]^2\,, \quad (3.32)$$

which yields the maximum amplitude gain

$$\Gamma_3=\frac{\sqrt{2}}{4}\,\frac{eE_1}{E_0\omega_1}\left(\frac{\omega_pF}{\eta\gamma_0^{1/2}}\right)^{1/2}\left[\frac{k}{\gamma_0^{1/2}}+\omega_p\right]\,. \quad (3.33)$$

When the frequency-velocity matching condition is not satisfied, Eq. (3.33) is replaced by $(v_0\approx 1)$

$$\Gamma_3'=(\Gamma_3^2-\tfrac{1}{4}(\alpha+\eta v_0)^2)^{1/2} \quad (3.34)$$

where $\alpha=k_1v_0-\omega_2+\omega_1-\omega_p/\gamma_0^{3/2}$. In order to obtain amplification the pump field has to be large enough to overcome the frequency mismatch in Eq. (3.34) and has a threshold obtained by setting Eq (3.34) equal to zero.

By taking the limit as ω_p becomes small in Eq. (3.31) this expression is seen to become Eq. (3.14) from which Γ_2 was obtained. Thus the Compton and Raman effects are not distinct processes. There is a single peak in gain and the Raman peak at $k_1v_0+\omega_1-\omega_2+\eta v_0=\omega_p/\gamma_0^{3/2}$ moves toward the Compton position $k_1v_0+\omega_1-\omega_2(1-v_0)=0$ as ω_p goes

TABLE I. Summary of operating regimes.

Parameter ordering	Peak amplitude gain
$P \ll \Omega_p \ll \frac{\Delta p}{p}$	
or $\Omega_p \ll P \ll \frac{\Delta p}{p}$	
or $\Omega_p \ll \frac{\Delta p}{p} \ll P$ and $\Omega_p^2 P \ll \left(\frac{\Delta p}{p}\right)^3$	$\Gamma_1 = \frac{\pi}{4} P \Omega_p^2 \left(\frac{\Delta p}{p}\right)^{-2} \frac{k_2}{\gamma^2}$
$\frac{\Delta p}{p} \ll \Omega_p \ll P$	
or $\Omega_p \ll \frac{\Delta p}{p} \ll P$ and $\left(\frac{\Delta p}{p}\right)^3 \ll \Omega_p^2 P$	$\Gamma_2 = \frac{\sqrt{3}}{2} P^{1/3} \Omega_p^{2/3} \frac{k_2}{\gamma^2}$
$P \ll \frac{\Delta p}{p} \ll \Omega_p$	
or $\frac{\Delta p}{p} \ll P \ll \Omega_p$	$\Gamma_3 = \frac{\sqrt{2}}{4} P^{1/2} \Omega_p^{1/2} \frac{k_2}{\gamma^2}$

to zero. The condition for Γ_3 to occur is

$$\frac{1}{(kv_0 - \omega)^2 - \omega_p^2/\gamma_0^3} \gg \frac{-1}{(kv_0 - \omega)^2} \tag{3.35}$$

which, when frequency-velocity matching is used, becomes

$$\frac{eE_1}{E_0\omega_1} \ll \frac{(2\eta\,\omega_p/\gamma_0^{5/2}F)^{1/2}}{k/\gamma_0^{1/2} + \omega_p} \quad . \tag{3.36}$$

The Raman gain formula Γ_3 then makes a transition to the Compton effect (Γ_1 for $\omega_p\gamma_0^{1/2}/\omega_2 \sim \Delta p/p$ otherwise to Γ_2) when the condition (3.36) becomes violated.

The regimes of operation and applicable peak amplitude gain formulas are summarized in Table I. All of the gain formulas and operating regimes are characterized by a pump parameter (c = speed of light= 1) $P = e^2B_0^2F/m^2k_0^2 = e^2E_1^2F/m^2\omega_1^2$, density parameter $\Omega_p^2 = \omega_p^2\gamma/\omega_2^2$ (which is assumed to be much less than unity for the expressions given in the Table), and fractional line width $\Delta\omega/\omega$ or momentum spread $\Delta p/p = (\Delta\omega/2\omega)[1 + (P/F)^2]$. From the Table it is seen that as a rule of thumb for $\Delta p/p$ being larger than the other parameters Γ_1 is usually the applicable gain formula, Γ_2 is usually applicable for a large pump parameter P, and Γ_3 for a large density parameter Ω_p.

For concreteness we now give several numerical examples. From Ref. 1 for a magnetic field of $B_0 = 2.4 \times 10^3$ G, linewidth $\Delta\omega/2\omega = 2 \times 10^{-3}$, pump field wave number $k_0 = 1.96$ cm^{-1}, emitted wave number $k_2 = 5.93 \times 10^3$ cm^{-1}, plasma frequency $\omega_p = 2.43 \times 10^9$ sec^{-1}, filling factor $F \approx 2 \times 10^{-2}$, and $\gamma = 47.1$ the relevant parameters are then $\Omega_p = 9.4 \times 10^{-5}$, $P = 10^{-2}$, $\Delta p/p = 2.5 \times 10^{-3}$. Examining Table I indicates that Γ_1 is the relevant gain formula as was used in Ref. 1. In Ref. 7 the example of $\gamma = 2$, $k_0/\omega_p = 2.2$, $\omega_2 = 14.22\,\omega_p$, $F = 1$ gives $\Omega_p = 9.94 \times 10^{-2}$, $P = 0.207\,(\omega_c/\omega_p)^2$ ($\omega_c = eB_0/m$) so that at $\omega_c/\omega_p = 0.69$, $P = \Omega_p$ and Table I indicates that a transition between Γ_3 and Γ_2 should occur. This may be the reason for the discrepancy between the theoretical Raman growth-rate predictions and numerical simulation in Fig. 12 of Ref. 7. As a final numerical example the parameters given in Ref. 7 for an astron beam with $\omega_p = 9.77 \times 10^{10}$ sec^{-1} and $B_0 = 5 \times 10^3$ G, $k_0 = \pi$ cm^{-1}, $k_2 = 2\pi \times 10^2$ cm^{-1}, $\gamma = 10$, $\Delta p/p = 10^{-3}$, $F = 1$ give $\Omega_p = 1.64 \times 10^{-2}$, $P = 0.87$ so that the region where the gain is given by Γ_2 is being realized and not the Raman effect Γ_3.

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