Extension of the Rosen-Zener solution to the two-level problem*

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We solve the problem of two discrete quantum levels which are coupled by a time-dependent radio-frequency pulse $W(t) = V(t)\, e^{i\nu t}$, where the envelope function is of a form suggested by Rosen and Zener: $V(t) = V_0 \operatorname{sech}(\pi t/T)$. When a level damping constant γ is included, in the manner of Bethe-Lamb theory, the solutions show new features which are not expected on the basis of a sudden-approximation theory, where $V(t) = \text{const}$ over the pulse duration T. Various transient effects such as "ringing" are not present in the extended Rosen-Zener solution; these effects are related to the large impulsive forces at the step discontinuities in the sudden approximation. The final-state level amplitudes can be quite different depending on the size of the pulse rise time T as compared with the system Bohr period $1/\omega$. Our results allow a continuous and quantitatively exact comparison between the extremes of the sudden ($\omega T \ll 1$) and adiabatic ($\omega T \gg 1$) approximations. A model of a "quasisudden" step function is also constructed, and remarks are made on the validity of a certain conjecture by Rosen and Zener.

I. INTRODUCTION: TWO-LEVEL PROBLEM

Solutions to the two-level problem of quantum mechanics, where two discrete quantum levels are coupled to one another by a time-dependent interaction with an external field, have been used in the approximate analysis of many different quantum-mechanical systems. For example, two-level models have been used as a starting point to analyze problems in radiative decay and level shifts for excited states,[1] level-crossing and collision phenomena,[2] methods of magnetic resonance,[3] radio-frequency spectroscopy,[4] laser physics and quantum optics,[5] and general principles of quantum mechanics.[6] This list is by no means complete, and undoubtedly the reader can cite additional examples from his own specialty.

In its simplest form, a solution to the two-level problem requires a solution to a system of two coupled differential equations of the type

$$i\hbar \dot{s} = V^*(t) p e^{i\omega t},\quad i\hbar \dot{p} = V(t) s e^{-i\omega t}. \tag{1}$$

Here the two quantum levels are represented by time-dependent amplitudes $s(t)$ and $p(t)$, with the s level separated in energy from the p level by $\hbar\omega$. The s level is coupled to the p level by an off-diagonal matrix element of interaction with an external field—we have denoted this by $V(t)$, stressing that the coupling interaction is time dependent in general. If one or both of the levels exhibits spontaneous radiative decay, this may be accounted for (at least approximately) by adding appropriate damping terms to the right-hand side of the equations[7]; a simple transformation of $s(t)$ and $p(t)$ then allows the damping to be incorporated as an imaginary term added to ω, and the transformed equations are of the same form as Eqs. (1), with ω a complex constant.

A solution to Eqs. (1) normally proceeds by first decoupling the equations, which leads to second-order differential equations for $s(t)$ and $p(t)$ alone. The coefficients of the various derivatives in these second-order equations are functions of the coupling $V(t)$ and its first derivative, so that exact solutions to Eqs. (1) are possible only for rather specific choices of coupling. One frequently used choice,[3,4] which leads to exact solutions for the amplitudes $s(t)$ and $p(t)$, is to represent the coupling as a monochromatic, rotating field of constant amplitude over a finite time interval, that is, $V(t) = \text{const} \times e^{i\nu t}$, where the constant is V_0 over, say, $-\frac{1}{2}T < t < \frac{1}{2}T$, and is zero otherwise. For such a "rectangular pulse," the frequency ν of the applied field combines simply with ω (as $\omega - \nu$) in Eqs. (1), and the decoupled second-order equations for $s(t)$ and $p(t)$ have constant coefficients. Such equations can be solved exactly and in all generality, with arbitrary initial conditions and arbitrary variation of all relevant parameters. It is for these reasons that the "rectangular pulse" solutions are widely applied.

Although the rectangular pulse solutions are mathematically exact, they can be at best only approximations to the effects of a physically realizable coupling $V(t)$. The reason is simple: a physical $V(t)$ cannot be turned on and off instantaneously. Even if "instantaneously" is taken to mean "in a time interval which is small compared to the Bohr period $1/\omega$," it is rarely possible to arrange this experimentally, except perhaps in collisions at sufficiently high energy. In this sense, the rectangular pulse solutions are a kind of sudden approximation[8] for many situations in which they are applied. We can expect that at least the transient behavior of the two-level system of interest will be quite different for a rectangular pulse than for a more physical $V(t)$ which varies smoothly in time and which has a sort of "fringe field" characteristic, in that it is turned on and off over time intervals which are *not* small compared to $1/\omega$.

A coupling pulse shape which varies smoothly in time, and for which the two-level problem is exactly solvable, was suggested some time ago by Rosen and Zener.[9] The pulse shape is

$$V(t)=V_0 \operatorname{sech}\frac{\pi t}{T}, \quad \int_{-\infty}^{+\infty} V(t)\,dt = V_0 T. \tag{2}$$

This pulse rises exponentially from zero at time $t\to -\infty$ to a maximum of V_0 at $t=0$, and then falls off symmetrically to zero as $t\to+\infty$, exhibiting a full width at half maximum of $0.8384T$, and a total pulse "area" of V_0T as shown. Rosen and Zener used this pulse shape to analyze the results of a Stern-Gerlach type experiment which measured "spin flips" induced when a beam of initially oriented spin-$\frac{1}{2}$ particles passed through a region of magnetic field where the field was constant in magnitude but rotated in direction. They showed that the spin-flip probability (i.e., spin-up to spin-down transition probability) corresponding to this smoothly varying pulse shape was quite different than that which had been calculated for a rectangular pulse model of the field rotation, and they explained an experimental result which had been considered anomalous in terms of rectangular pulse theory.

The Rosen-Zener solution—that is, the solution for the level amplitudes $s(t)$ and $p(t)$ of Eqs. (1) for the coupling pulse $V(t)$ of Eq. (2)—may be of considerable interest in more recent work,[3,4] to the extent that a smoothly varying pulse represents (more realistically than a rectangular pulse) the interaction of a two-level system with an actual laboratory field having a "fringing" characteristic. Our purpose in this paper is (i) to extend the Rosen-Zener solution to include a simple rotating field time dependence for the coupling, and to include decaying levels (in the manner of Bethe-Lamb theory[7]); and then (ii) to compare the extended Rosen-Zener solution to the well-known results of the rectangular pulse approximation. We anticipate rather different results for the level amplitudes.

In Sec. II, we solve the two-level problem, including damping terms, for a coupling pulse which has $V(t)$ of Eq. (2) as an envelope, and in addition has a simple rotating field time dependence. We find exact solutions in terms of hypergeometric functions for initial conditions corresponding to full occupation of the s level at early times. The extended Rosen-Zener solution for the final-state p-level population reduces to their previous result in the limit of zero damping. In Sec. III, we compare the extended Rosen-Zener solution to the results for an equivalent rectangular pulse coupling. This is done for pulses of duration T both short and long compared to the system damping time scale, and for cases where the rotating field is both on and off resonance ($\omega=0, \omega\neq 0$). Next, in Sec. IV, we employ the extended Rosen-Zener solution to construct a model where an initially pure level is subjected to a coupling $U(t)$ which is turned on smoothly, that is, $U(t)$ rises smoothly from zero to a constant value V_0 in a time which is arbitrary compared to any natural time scale for the two-level system. In all cases of interest, exact or nearly exact results for the final-state level amplitudes can be obtained. These results for the extended Rosen-Zener solution quantitatively span the full range from the limit of the adiabatic approximation ($\omega T \gg 1$) to the sudden approximation ($\omega T \ll 1$). Finally, in Sec. V, we close with some remarks concerning a certain conjecture made by Rosen and Zener which allows the final-state induced level population to be calculated for a quite general class of coupling pulses, namely those for which a Fourier-transform integral exists.

II. ROSEN-ZENER SOLUTION

We represent the coupled two-level system by equations similar to Eqs. (1), namely

$$\begin{aligned} i\hbar\dot{s} &= W^*(t)p e^{i\omega_{sp}t} - \tfrac{1}{2}i\hbar\gamma_s s, \\ i\hbar\dot{p} &= W(t)s e^{-i\omega_{sp}t} - \tfrac{1}{2}i\hbar\gamma_p p. \end{aligned} \tag{3}$$

Here the two levels have time-dependent amplitudes $s(t)$ and $p(t)$, are separated in energy by $\hbar\omega_{sp}$, and are coupled by a time-dependent off-diagonal matrix element $W(t)$ of interaction with an external field. The terms in $W(t)$ are present as an exact equivalent of the time-dependent Schrödinger equation.[8] In addition, in the manner of Bethe-Lamb theory,[7] we have introduced damping terms which represent the spontaneous decay of the s and p levels at rates γ_s and γ_p, respectively.

We assume that the coupling $W(t)$ depends on time in two distinct ways: (i) it has a simple rotating field time factor $e^{i\nu t}$, where ν is the angular frequency of the applied field; (ii) it has an "envelope" factor $V(t)$, which represents the shape of the applied ac pulse. Thus we write

$$W(t)=V(t)e^{i\nu t}. \tag{4}$$

Evidently the rotating field time factor here will combine simply with the $e^{\mp i\omega_{sp}t}$ factors in Eqs. (3), producing a frequency difference $\omega=\omega_{sp}-\nu$, which may be interpreted as the distance "off resonance" in frequency units. If we now transform the physical amplitudes $s(t)$ and $p(t)$ to new amplitudes $S(t)$ and $P(t)$ defined by

$$s(t)=S(t)e^{-\gamma_s t/2}, \quad p(t)=P(t)e^{-\gamma_p t/2}, \tag{5}$$

then, with $W(t)$ of Eq. (4), Eqs. (3) become

$$i\hbar\dot{S}=V^*(t)Pe^{i\Omega t}, \quad i\hbar\dot{P}=V(t)Se^{-i\Omega t}, \tag{6}$$

where

$$\Omega=\omega+\tfrac{1}{2}i\gamma, \quad \omega=\omega_{sp}-\nu, \quad \gamma=\gamma_p-\gamma_s. \tag{7}$$

Equations (6) are identical in form to those solved by Rosen and Zener for the pulse $V(t)$ of Eq. (2), except that here the frequency Ω is complex rather than real, and the amplitudes $S(t)$ and $P(t)$ measure the amount of level mixing due to $V(t)$ in the otherwise freely decaying physical amplitudes of Eq. (5). We shall solve Eqs. (6) for the initial conditions

$$|S(t\to-\infty)|=1, \quad P(t\to-\infty)=0, \tag{8}$$

which represent full occupation of the (freely decaying) s level at early times.

It is worth noting that because of the damping terms, probability is not conserved for either Eqs. (3) or Eqs. (6). Starting from Eqs. (3), it is easy to show that for the total level population $(|s|^2+|p|^2)$

$$\frac{d(|s|^2+|p|^2)}{dt}=-(\gamma_s|s|^2+\gamma_p|p|^2). \tag{9}$$

This means that it is *not* correct to solve Eqs. (6) for, say $|P|^2$ and then set $|S|^2=1-|P|^2$. When the decay rates are nonzero, each amplitude must be solved for by itself. Of course we can still use the interrelations of Eqs. (6)—for example, it is sufficient to solve for $P(t)$, and then calculate $S(t)\propto\dot{P}(t)$ from the second of Eqs. (6).

Decoupling Eqs. (6) straightforwardly, we find

$$\begin{aligned}&\ddot{S}-[(\dot{V}/V)^*+i\Omega]\dot{S}+|V/\hbar|^2S=0,\\&\ddot{P}-[(\dot{V}/V)-i\Omega]\dot{P}+|V/\hbar|^2P=0.\end{aligned} \tag{10}$$

We now make the specific choice of the Rosen-Zener pulse of Eq. (2) for the envelope function $V(t)$. Following Rosen and Zener, we also make a change of variables from time t to

$$z(t)=\tfrac{1}{2}[\tanh(\pi t/T)+1]. \tag{11}$$

Equations (10) are then transformed to the hypergeometric equation, which takes the form [where $f(z)=S(t)$ or $P(t)$]

$$z(1-z)f''+[c-(1+a+b)z]f'-abf=0, \tag{12}$$

where

$$c=\tfrac{1}{2}\mp i(\Omega T/2\pi), \quad a=-b=V_0T/\pi\hbar. \tag{13}$$

The minus sign in c corresponds to $f(z)=S(t)$, while the plus sign corresponds to $f(z)=P(t)$. The range of variation of the variable z in Eq. (12) is $0\le z\le 1$, which corresponds to the time variation $-\infty\le t\le+\infty$. The general solution to Eq. (12) defined in this range is

$$\begin{aligned}f(z)=\;&\alpha F(a,b;c;z)\\&+\beta z^{1-c}F(a-c+1,b-c+1;2-c;z),\end{aligned} \tag{14}$$

where F is the hypergeometric function,[10] and α and β are integration constants.

We now want to impose the initial conditions of Eqs. (8) on the solutions of Eq. (14). Working on $P(t)$, for which

$$1-c=\tfrac{1}{2}+(\gamma T/4\pi)-i(\omega T/2\pi)=\phi, \tag{15}$$

we note that as $t\to-\infty, z\simeq\exp(2\pi t/T)\to 0$, so that

$$\begin{aligned}P(t\to-\infty)&\simeq\alpha+\beta z^{\phi}\\&\simeq\alpha+\beta\exp[(\pi/T+\tfrac{1}{2}\gamma-i\omega)t].\end{aligned} \tag{16}$$

We assume that the s level is the longer lived level, so that $\gamma=\gamma_p-\gamma_s$ is positive. Then the second term here goes to zero as $t\to-\infty$, and $P(t\to-\infty)=0$ only if we choose the constant $\alpha=0$. The solution for P then involves only the second of the two hypergeometric functions in Eq. (14). To fix the constant β, we first calculate $S(t)$ from the second of Eqs. (6), and then impose $|S(t\to-\infty)|=1$. In this way, we find $|\beta|=|a/\phi|$. Thus the desired solutions to Eqs. (6) for the coupling pulse of Eq. (2), which obey the initial conditions of Eq. (8), are

$$\begin{aligned}P(t)&=(a/|\phi|)z^{\phi}F(\phi+a,\phi-a;\phi+1;z),\\S(t)&=(i\phi/|\phi|)z^{\phi}\exp[-(\pi/T+\tfrac{1}{2}\gamma-i\omega)t]\\&\quad\times F(\phi+a,\phi-a;\phi;z).\end{aligned} \tag{17}$$

Here the connection between time t and the variable z is given by Eq. (11), $a=V_0T/\pi\hbar$ is proportional to the integrated pulse "area" V_0T, and ϕ is defined by Eq. (15).

The solutions of Eq. (17) appear to be identical in all important respects to the previous Rosen-Zener solution,[9] except that now the effects of damping are accounted for by the terms in γ, and the frequency ω may be interpreted as the frequency deviation from the expected transition resonance at an applied frequency $\nu=\omega_{sp}$. The change in the interpretation of ω is of no great consequence, except that we will be mainly interested in the near-resonant condition $\omega\le\tfrac{1}{2}\gamma$. The inclusion of the damping terms in γ have a substantially greater effect in altering the nature of the previous Rosen-Zener solution, as we shall see.

The final-state amplitudes are found in the behavior of the functions $P(t)$ and $S(t)$ of Eqs. (17)

as $t\to+\infty$, i.e., at times $t\gg T$, where T is the approximate duration of the coupling pulse. This corresponds to $z\to 1$, and here we must be careful in handling the hypergeometric functions. It is known[10] that $F(A,B;C;z)$ is absolutely convergent as $|z|\to 1$ only if $\mathrm{Re}(C-A-B)>0$. For the functions of Eqs. (17), this condition translates to

$$\text{for } P(t):\ \mathrm{Re}(1-\phi)=\tfrac{1}{2}-(\gamma T/4\pi)>0,\quad \text{i.e., } \gamma T<2\pi; \qquad (18)$$
$$\text{for } S(t):\ \mathrm{Re}(-\phi)=-\tfrac{1}{2}-(\gamma T/4\pi)>0.$$

The first condition can be met for short pulses, $T<2\pi/\gamma$, but the second condition cannot be satisfied as long as $\gamma=\gamma_p-\gamma_s>0$, as we have assumed. The lack of convergence in this case is more apparent than real, however, as the *physical* amplitudes [Eqs. (5)] are both well behaved for $t\gg T$.

To better see what happens for $t\gg T$, or $z\to 1$, we use the transformation formula for hypergeometric functions[10]

$$F(A,B;C;z)=\frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}\times F(A,B;A+B-C+1;1-z)+\frac{\Gamma(C)\Gamma(A+B-C)}{\Gamma(A)\Gamma(B)}(1-z)^{C-A-B}\times F(C-A,C-B;C-A-B+1;1-z), \qquad (19)$$

where the Γ's denote the γ function.[11] The F functions on the right-hand side of Eq. (19) are well behaved as $z\to 1$, and we need only worry about the factor in $(1-z)$. Applying this transformation to $P(t)$ of Eqs. (17), we find

$$P(t)=\frac{a}{|\phi|}z^{\phi}\left(\frac{\Gamma(1+\phi)\Gamma(1-\phi)}{\Gamma(1+a)\Gamma(1-a)}F(\phi+a,\phi-a;\phi;1-z)+\frac{\Gamma(\phi+1)\Gamma(\phi-1)}{\Gamma(\phi+a)\Gamma(\phi-a)}(1-z)^{1-\phi}\times F(1-a,1+a;2-\phi;1-z)\right). \qquad (20)$$

For $t\gg T$, z^{ϕ} goes uniformly to one, but the $(1-z)$ factor goes as

$$(1-z)^{1-\phi}\simeq\exp[(\tfrac{1}{2}\gamma-\pi/T-i\omega)t],\quad t\gg T. \qquad (21)$$

This diverges exponentially for $\gamma T>2\pi$. But the physical amplitude is $p(t)=P(t)e^{-\gamma_p t/2}$, which is well behaved as $t\to+\infty$. Forming $p(t)$ from Eq. (20), letting $z\to 1$ (or $t\gg T$), and doing some γ-function algebra, we find

$$p(t\gg T)=\frac{\phi}{|\phi|}\left\{\exp(-\tfrac{1}{2}\gamma_p t)\sin\pi a\csc\pi\phi+\left(\frac{a}{\phi-1}\right)\left(\frac{\Gamma^2(\phi)}{\Gamma(\phi+a)\Gamma(\phi-a)}\right)\times\exp\left[-\left(\frac{\pi}{T}+\tfrac{1}{2}\gamma_s+i\omega\right)t\right]\right\}. \qquad (22)$$

Now both terms in $p(t\gg T)$ vanish exponentially as $t\to+\infty$, so $p(t)$ is well behaved, and this expression for the induced-state amplitude should be correct for all values of γ_p, γ_s, and T. It vanishes as it must when the pulse "area" parameter $a\to 0$ (zero coupling). The first term on the right-hand side of Eq. (22) is dominant for $(\gamma_p-\gamma_s)T<2\pi$, while the second term is dominant for $(\gamma_p-\gamma_s)T>2\pi$. For $(\gamma_p-\gamma_s)T\ll 2\pi$, Eq. (22) gives the induced-level population

$$|p(t\gg T)|^2\simeq e^{-\gamma_p t}\sin^2\pi a\,|\mathrm{sech}\tfrac{1}{2}(\omega T+\tfrac{1}{2}i\gamma T)|^2. \qquad (23)$$

This is identical to the previous Rosen-Zener solution[9] in the limit of nondecaying levels, γ_p and $\gamma_s\to 0$.

If we treat $S(t)$ of Eq. (17) similarly, applying the transformation of Eq. (19), forming the physical amplitude $s(t)=S(t)e^{-\gamma_s t/2}$, and letting t become large compared to the pulse duration T, we find

$$s(t\gg T)=\frac{i\phi}{|\phi|}\left\{\exp(-\tfrac{1}{2}\gamma_s t)\left(\frac{\Gamma^2(\phi)}{\Gamma(\phi+a)\Gamma(\phi-a)}\right)+\frac{a}{\phi}(\sin\pi a\csc\pi\phi)\times\exp\left[-\left(\frac{\pi}{T}+\tfrac{1}{2}\gamma_p-i\omega\right)t\right]\right\}. \qquad (24)$$

Again, both terms in $s(t\gg T)$ vanish exponentially as $t\to+\infty$, so $s(t)$ is well-behaved in this limit, and this expression for $s(t\gg T)$ should be correct for all values of γ_p, γ_s, and T. It shows the correct behavior that $|s(t\gg T)|=e^{-\gamma_s t/2}$ in the limit of zero coupling, $a\to 0$. The first term on the right-hand side of Eq. (24) is always dominant if $\gamma_p>\gamma_s$. The limit of Eq. (24) as γ_p and $\gamma_s\to 0$ is not easy to get, nor particularly enlightening here; we shall deal with it in Sec. III.

To summarize this section, we have solved the two-level problem including damping terms, Eqs. (3), for a coupling pulse which has a simple rotating field time dependence $e^{i\nu t}$ times a bell-shaped envelope function $V(t)$ of the form of the Rosen-Zener pulse of Eq. (2). Exact solutions for the level-mixing amplitudes $P(t)$ and $S(t)$ are given in terms of hypergeometric functions in Eqs. (17), for a choice of initial conditions corresponding to full occupation of the s level at early times. The physical (Schrödinger) amplitudes, given by Eqs.

(5), are expressed at times well after the coupling pulse has died away by Eqs. (22) and (24). These amplitudes are well behaved as $t \to +\infty$, and appear to be correct for all values of the pulse duration T and choice of damping constants γ_p and γ_s. In the limit that the damping constants go to zero, the induced level population of Eq. (23) reduces to the previous Rosen-Zener solution. As we shall see, however, the inclusion of nonnegligible damping considerably changes the previous Rosen-Zener solution.

III. COMPARISON WITH THE RECTANGULAR PULSE SOLUTION

In this section we wish to compare the extended Rosen-Zener solutions of Eqs. (22) and (24) with the solutions for an equivalent rectangular pulse. By the rectangular pulse, we mean a coupling with the same rotating field dependence $e^{i\nu t}$ as in Eq. (4), but with a rectangular envelope function $V(t)$ of the form

$$V(t)=\begin{cases} V_0 & \text{for } -\tfrac{1}{2}T<t<+\tfrac{1}{2}T, \\ 0 & \text{otherwise.}\end{cases} \tag{25}$$

This pulse has the same time-integrated "area" V_0T as the Rosen-Zener pulse of Eq. (2), but differs from the Rosen-Zener pulse in that it is turned on and off effectively instantaneously.

Thus we are interested in the solutions to Eqs. (6) for the pulse shape of Eq. (25). The solution is straightforward and we merely quote the results. We choose the initial conditions: $S(t)=1, P(t)=0$ for $t<-\frac{1}{2}T$, which represents full occupation of the s level before the pulse is applied; this is equivalent to the choice of initial conditions in Eq. (8) for the Rosen-Zener pulse. Then, for $-\frac{1}{2}T<t<+\frac{1}{2}T$, we find

$$P_R(t)=-ie^{-i\omega t}[q/(1-i\kappa)Q]e^{\gamma t/2} \times\{\exp[-\mu_1(t+\tfrac{1}{2}T)]-\exp[-\mu_2(t+\tfrac{1}{2}T)]\}, \tag{26}$$

$$S_R(t)=[(Q+1)/2Q]\exp[-\mu_1(t+\tfrac{1}{2}T)] +[(Q-1)/2Q]\exp[-\mu_2(t+\tfrac{1}{2}T)]. \tag{27}$$

The parameters $\omega=\omega_{sp}-\nu$, and $\gamma=\gamma_p-\gamma_s$, have the same meaning as before [Eqs. (7)]. The new parameters are[12]

$$q=2V_0/\hbar\gamma, \quad \kappa=2\omega/\gamma; \quad Q=\{1-[4q^2/(1-i\kappa)^2]\}^{1/2}, \quad \mu_{1,2}=\tfrac{1}{4}\gamma(1-i\kappa)(1\mp Q). \tag{28}$$

We have added a subscript R to these expressions for the level-mixing amplitudes to distinguish them as rectangular pulse amplitudes; hereafter, we shall add a subscript Z to the amplitudes corresponding to the Rosen-Zener bell-shaped pulse. The final-state physical amplitudes, after the R pulse is over, will be

$$p_R(t\gg T)=P_R(\tfrac{1}{2}T)e^{-\gamma_p t/2}, \tag{29}$$

$$s_R(t\gg T)=S_R(\tfrac{1}{2}T)e^{-\gamma_s t/2}. \tag{30}$$

These are to be compared with the Z-pulse amplitudes of Eqs. (22) and (24).

A. Short pulses

Perhaps the simplest comparison between the Z (Rosen-Zener) pulse solutions and the R (rectangular) pulse solutions is in the case where the detailed pulse shape should be of no consequence, that is, the δ function limit, where the pulse amplitude $V_0\to\infty$ while the pulse duration $T\to 0$ in such a way that the pulse area V_0T is constant. In this limit, the Z-pulse amplitude of Eqs. (22) and (24) reduce to

$$p_Z(t\gg T)=e^{-\gamma_p t/2}\sin(V_0T/\hbar), \quad s_Z(t\gg T)=ie^{-\gamma_s t/2}\cos(V_0T/\hbar). \tag{31}$$

In the same limit, the R-pulse amplitudes of Eqs. (29) and (30) reduce to

$$p_R(t\gg T)=-ie^{-\gamma_p t/2}\sin(V_0T/\hbar), \quad s_R(t\gg T)=e^{-\gamma_s t/2}\cos(V_0T/\hbar). \tag{32}$$

As expected, the Z-pulse and R-pulse results are the same, except for the physically unimportant relative phase factor i.

The next simplest comparison between the Z-pulse and R-pulse results appears to be in the case where the pulse duration T is short compared to the decay rate time scale $1/\gamma$, but not short compared to the Bohr period $1/\omega$. For the Z-pulse results, this may be interpreted as $\gamma T\ll 2\pi$, but ωT not necessarily small. Then Eqs. (22) and (24) reduce to

$$p_Z(t\gg T)\simeq(\phi/|\phi|)e^{-\gamma_p t/2} \times\sin(V_0T/\hbar)\,\mathrm{sech}(\tfrac{1}{2}\omega T), \quad s_Z(t\gg T)\simeq(i\phi/|\phi|)e^{-\gamma_s t/2} \times[\Gamma^2(\phi)/\Gamma(\phi+a)\Gamma(\phi-a)], \tag{33}$$

where

$$a=V_0T/\pi\hbar; \quad \phi\simeq\tfrac{1}{2}-i(\omega T/2\pi) \text{ for } \gamma T\ll 2\pi. \tag{34}$$

For comparison with the equivalent R-pulse results, it is easiest to look at the level-mixing populations, that is, the above $|p_Z|^2$ and $|s_Z|^2$ with the free decay factors $e^{-\gamma_{p,s}t}$ factored out. These are

$$|P_Z(t\gg T)|^2 \simeq \sin^2(V_0T/\hbar)\,\mathrm{sech}^2(\tfrac{1}{2}\omega T),$$
$$|S_Z(t\gg T)|^2 \simeq |\cos[(V_0T/\hbar)+i(\tfrac{1}{2}\omega T)]|^2 \times \mathrm{sech}^2(\tfrac{1}{2}\omega T). \quad (35)$$

We note that $|P_Z|^2+|S_Z|^2=1$ here, as we would expect when the decay rate time scale $1/\gamma$ is large compared to the pulse duration T. This will be true for sufficiently short pulses ($\gamma T \ll 2\pi$), or when the decay rates are equal ($\gamma=\gamma_p-\gamma_s=0$), or when neither state decays ($\gamma_p=\gamma_s=0$). In the last case here, Eqs. (35) are identical to the previous Rosen-Zener results[9] for the final level populations, and probability is conserved as it must be.

The R-pulse results which are comparable to Eqs. (35) can be found from Eqs. (26) and (27) evaluated at $t=\frac{1}{2}T$ and approximated in the limit that $\gamma\to 0$. We find the usual quantum beat results,[13] namely

$$|P_R(t=\tfrac{1}{2}T)|^2 \simeq (2V_0/\hbar Q\omega)^2 e^{\gamma T/2}\sin^2(\tfrac{1}{2}Q\omega T),$$
$$|S_R(t=\tfrac{1}{2}T)|^2 \simeq \left|\left(\frac{Q+1}{2Q}\right)e^{-iQ\omega T/2}+\left(\frac{Q-1}{2Q}\right)e^{iQ\omega T/2}\right|^2, \quad (36)$$

where

$$Q=[1+(2V_0/\hbar\omega)^2]^{1/2}.$$

Again, when $\gamma=0$, $|P_R|^2+|S_R|^2=1$, so that the level mixing probability is conserved, as it must be.

It is not easy to directly compare the Z-pulse results of Eq. (35) with the R-pulse results of Eqs. (36), because the functions involved have rather different arguments. In any event, it is sufficient to look at the induced level populations $|P|^2$ alone, since we have $|S|^2=1-|P|^2$, when $\gamma\to 0$ in both cases. The most immediate difference between $|P_Z|^2$ and $|P_R|^2$ is that, considered as functions of the coupling amplitude V_0, they are modulated by sine factors with quite different arguments: $|P_Z|^2$ is always modulated by $\sin^2(V_0T/\hbar)$, no matter what the value of the off-resonance frequency ω, while $|P_R|^2$ is modulated by $\sin^2(\frac{1}{2}Q\omega T)$, which depends on the value of ω. The two expressions are equivalent only when $\omega\to 0$, i.e.,

$$|P_Z(t\gg T)|^2 \simeq \sin^2(V_0T/\hbar) \simeq |P_R(t=\tfrac{1}{2}T)|^2 \text{ for } \omega\to 0. \quad (37)$$

This result is consistent with the results of Eqs. (31) and (32) in that here we have let both $\gamma\to 0$ and $\omega\to 0$, which is equivalent to choosing a pulse (any pulse) whose duration T is short compared to both system time scales $1/\gamma$ and $1/\omega$.

When $\omega\neq 0$, the situation is quite different. If we assume we are sufficiently far off resonance so that $\omega T\gg 1$, then the Z-pulse result of Eqs. (35) becomes

$$|P_Z(t\gg T)|^2 \simeq 4e^{-\omega T}\sin^2 A, \quad (38)$$

where A is the pulse "area" as defined by Rosen and Zener

$$A=\frac{1}{\hbar}\int_{-\infty}^{+\infty} V(t)\,dt=\frac{V_0T}{\hbar}. \quad (39)$$

In the same limit, the R-pulse result of Eqs. (36) may be approximated by replacing the sine squared factor with its average value of $\frac{1}{2}$, so that

$$|P_R(t=\tfrac{1}{2}T)|^2 \simeq \frac{2A^2e^{\gamma T/2}}{(\omega T)^2+4A^2}. \quad (40)$$

This result is rather dramatically different than that of Eq. (38) for the Z pulse, as was noted by Rosen and Zener. Here the modulation factor has disappeared, and for $\omega T\gg 1$, $|P_R|^2$ is substantially larger than $|P_Z|^2$. We can conclude, therefore, that for short pulses ($\gamma T\ll 2\pi$) the Z-pulse and R-pulse results are similar only near resonance ($\omega T\ll 1$), and they differ substantially off resonance ($\omega T\gg 1$). Experimentally, if we were measuring an s-level to p-level transition resonance, i.e., $|P|^2$ as a function of ω, we would see different line shapes depending on whether the applied coupling pulse was more nearly rectangular or bell shaped. The differences could become substantial at off-resonance frequencies $\omega\sim\frac{1}{2}\gamma$.[14]

B. Longer pulses

When the reduced decay rate $\gamma=\gamma_p-\gamma_s$ and pulse duration T are such that γT is not small, comparison between the Z-pulse and R-pulse results becomes more difficult. One choice of pulse duration which leads to a reasonably simple analytic result for the Z pulse is

$$\gamma T=2\pi, \quad \phi=1-i\omega T/2\pi. \quad (41)$$

In this case, the induced level amplitude of Eq. (22) is

$$p_Z(t\gg T)=(i\phi/|\phi|)e^{-\gamma_p t/2} \times[(2A/\omega T)G-\sin A\,\mathrm{csch}(\tfrac{1}{2}\omega T)], \quad (42)$$

where $A=\pi a=V_0T/\hbar$ is the pulse area of Eq. (39), and

$$G=\Gamma^2(\phi)e^{-i\omega t}/\Gamma(\phi+a)\Gamma(\phi-a). \quad (43)$$

Forming the induced level-mixed population $|P_Z|^2$ by factoring the free decay multiplier $e^{-\gamma_p t}$ out of $|p_Z|^2$, we find

$$|P_Z(t\gg T)|^2=A^2\operatorname{csch}^2(\tfrac{1}{2}\omega T)\times\left[\left|\frac{\sin B}{B}\right|^2+\left(\frac{\sin A}{A}\right)^2-2\left|\frac{\sin B}{B}\right|\left(\frac{\sin A}{A}\right)\cos\theta\right], \quad (44)$$

where

$$B=A+\tfrac{1}{2}i\omega T, \quad \theta=\arg G. \quad (45)$$

We shall be interested in the behavior of $|P_Z|^2$ of Eq. (44) for the case of near resonance ($\omega T=2\pi\omega/\gamma\ll 1$), and far off resonance ($2\pi\omega/\gamma\gg 1$).

For the near resonance case, $\omega\ll\gamma/2\pi$, but $\omega\neq 0$, it is necessary to expand $|P_Z|^2$ and retain terms of order $(\frac{1}{2}\omega T)^4$ as well as order $(\frac{1}{2}\omega T)^2$. This leads to extremely messy algebra and is not particularly enlightening, so we shall do only the order $(\frac{1}{2}\omega T)^2$ expansion, which is sufficient to display the behavior of $|P_Z|^2$ on resonance, $\omega=0$. First, we note that θ of Eq. (45) is of order ωt at least, because of the $e^{-i\omega t}$ factor in G of Eq. (43). Then, to order ω^2, it is sufficient to take

$$|P_Z(t\gg T)|^2\simeq 4(2/\omega T)^2\sin^2 A\,\sin^2(\tfrac{1}{2}\theta). \quad (46)$$

We now expand θ in powers of $(\frac{1}{2}\omega T)$, discarding terms of order $(\frac{1}{2}\omega T)^3$ and higher. We find

$$\theta\simeq-\omega T[\Psi(a)+(t/T)],$$
$$\Psi(a)=\psi(1)-\tfrac{1}{2}[\psi(1+a)+\psi(1-a)], \quad (47)$$

where $\pi a=A$ is the pulse area and the ψ's are digamma functions.[11] Thus, on resonance, we get

$$|P_Z(t\gg T)|^2=4[\Psi(a)+(t/T)]^2\sin^2 A. \quad (48)$$

This result diverges for $t\to+\infty$, but the physical induced amplitude is $|p_Z|^2=|P_Z|^2e^{-\gamma_p t}$, so that

$$|p_Z(t\gg T)|^2\simeq 4e^{-\gamma_p t}[\Psi(a)+(t/T)]^2\sin^2 A \quad (49)$$

remains finite. This result is similar to that for the square of the amplitude of a critically damped oscillator.[15]

The R-pulse result comparable to Eq. (49) is found from Eq. (26) by forming the physical amplitude of Eq. (29) and then setting $\omega=0$ (or $\kappa=0$). The induced level population for an R pulse with $\gamma T=2\pi$ and on-resonance is then

$$|p_R(t\gg T)|^2=[4a^2e^{-\gamma_p t}/(1-4a^2)]\times\sinh^2[\tfrac{1}{2}\pi(1-4a^2)^{1/2}], \quad (50)$$

which is exact. Unlike the result for short pulses ($\gamma T\ll 2\pi$) where the Z-pulse and R-pulse results were identical on resonance, Eq. (37), it is clear by comparison of Eqs. (49) and (50) that the on-resonance results for a longer pulse ($\gamma T=2\pi$) are very much different. For weak coupling (i.e., $a\ll 1$), $|p_R|^2$ shows no modulation at all as a function of the coupling, while $|p_Z|^2$ has the characteristic Rosen-Zener $\sin^2 A$ modulation factor. Again, for weak coupling, Eqs. (49) and (50) can be written

$$|p_Z(t\gg T)|^2\simeq 4a^2(\pi t/T)^2e^{-\gamma_p t},$$
$$|p_R(t\gg T)|^2\simeq 4a^2e^{-\gamma_p t}\sinh^2[\tfrac{1}{2}\pi(1-2a^2)], \quad (51)$$

to order a^2. For an observation time $t\gg T$, $|p_Z|^2$ will be much larger than $|p_R|^2$, which is somewhat surprising. The appearance of the multiplicative factor $(\pi t/T)^2$ in $|p_Z|^2$ indicates that $p_Z(t)$ carries with it a physically important relative phase factor between the s and p levels which is not present as such in $p_R(t)$. In fact, a relative phase factor such as $e^{-i\omega t}$ cannot be factored out of the exact Z-pulse result of Eq. (22), while it does factor out of Eq. (26).

Another point of comparison between the Z- and R-pulse results of Eqs. (49) and (50) is that although the Rosen-Zener modulation factor $\sin^2 A$ is not present in $|p_R|^2$ for weak coupling, it does appear in the strong-coupling limit. From Eq. (50), we have (for $\gamma T=2\pi$, $\omega=0$)

$$a=\tfrac{1}{2}:\quad |p_R(t\gg T)|^2=(\tfrac{1}{2}\pi)^2e^{-\gamma_p t},$$
$$a>\tfrac{1}{2}:\quad |p_R(t\gg T)|^2=[4a^2e^{-\gamma_p t}/(4a^2-1)]\times\sin^2[\tfrac{1}{2}\pi(4a^2-1)^{1/2}], \quad (52)$$
$$a\gg\tfrac{1}{2}:\quad |p_R(t\gg T)|^2\simeq e^{-\gamma_p t}\sin^2 A.$$

The parameter $a=\frac{1}{2}$ translates to a coupling strength $V_0=\frac{1}{4}\hbar\gamma$ for the present pulse. As has been discussed by Lamb,[16] this is a critical coupling strength at resonance for the rotating field R-pulse theory with decaying states: for $V_0<\frac{1}{4}\hbar\gamma$, such a system is overdamped; for $V_0=\frac{1}{4}\hbar\gamma$, it is critically damped; for $V_0>\frac{1}{4}\hbar\gamma$, it is underdamped. In the last case only, we can expect to see a modulation of the induced level population at $\sin^2 A$, where $A=V_0T/\hbar$ is the usual result of perturbation theory for degenerate levels. The appearance of the modulation factor $\sin^2 A$ in $|p_Z|^2$ of Eq. (49) for *all* relative values of the coupling strength V_0 and damping constant γ is thus somewhat surprising, as is the fact that $|p_Z|^2$ shows the character of a critically damped oscillation with time t for all relative values of V_0 and γ. The Z-pulse theory is thus sort of a hybrid version of R-pulse theory with respect to oscillatory modulation at resonance.

Far off resonance ($\omega T\gg 1$), the comparison between the Rosen-Zener solution and rectangular pulse theory again shows quite different results for the induced level population (for a $\gamma T=2\pi$ pulse). If we neglect terms of order $(1/\omega T)^2$, Eq. (44) becomes

$$|P_Z(t \gg T)|^2 \simeq 4e^{-\omega T}\sin^2 A. \qquad (53)$$

This is the same as the off-resonance result of Eq. (38) for short pulses ($\gamma T \ll 2\pi$), which indicates that in Z-pulse theory it is the system time scale $1/\omega$ rather than $1/\gamma$ which controls the off-resonance behavior. The corresponding result from R-pulse theory is found from Eq. (26) by evaluating $P_R(t)$ at time $t=\frac{1}{2}T$, and taking the limit $\kappa = 2\omega/\gamma \gg 1$ (i.e., $\omega T \gg \pi$ for a $\gamma T = 2\pi$ pulse). Neglecting terms of order $1/\kappa^2$, we find[17]

$$|P_R(t=\tfrac{1}{2}T)|^2 \simeq [2\epsilon^2/(1+4\epsilon^2)] \times\{\cosh[\tfrac{1}{2}\gamma T(1+4\epsilon^2)^{-1/2}] - \cos[\omega T(1+4\epsilon^2)^{1/2}]\}, \qquad (54)$$

where

$$\epsilon = q/\kappa = V_0/\hbar\omega = A/\omega T. \qquad (55)$$

This result is good for any value of γT and any coupling strength ϵ. It is clear here that, unlike Z-pulse theory, the off-resonance behavior in R-pulse theory is controlled by both the $1/\omega$ and $1/\gamma$ time scales relative to T. To put $|P_R|^2$ of Eq. (54) at the same order of approximation as $|P_Z|^2$ of Eq. (53), we include only terms up to order ϵ^2, and so obtain

$$|P_R(t=\tfrac{1}{2}T)|^2 \simeq 2\epsilon^2\{\cosh(\tfrac{1}{2}\gamma T) - \cos[\omega T(1+2\epsilon^2)]\}, \qquad (56)$$

where $\gamma T = 2\pi$ for the particular pulse in question. As in the case of short pulses [$\gamma T \ll 2\pi$, Eqs. (38) and (40)], $|P_R|^2$ is substantially larger than $|P_Z|^2$ off resonance. As well, $|P_R|^2$ shows an oscillatory behavior as a function of the off-resonance frequency ω, while $|P_Z|^2$ apparently does not. This was also true for short pulses [compare Eqs. (56) with (36), and Eqs. (53) with (40)], and it indicates a kind of "ringing" behavior present in R-pulse theory which does not appear in Z-pulse theory. Such "ringing" can be expected in R-pulse theory as a consequence of the large impulsive forces generated when a sudden perturbation is applied to the system.

We can multiply examples of the differences between the Z- (Rosen-Zener) pulse solutions and R- (rectangular) pulse solutions for the final-state level amplitudes for long pulses. However, it seems sufficient to consider briefly just one more example, namely the case of a very long pulse ($\gamma T \gg 1$), at resonance ($\omega = 0$), for weak coupling ($A \ll \gamma T$). We find the induced level populations given, to terms of order $(A/\gamma T)^2$, by

$$|p_Z(t \gg T)|^2 \simeq (4A/\gamma T)^2 e^{-\gamma_s t} e^{-2\pi t/T}, \qquad |p_R(t \gg T)|^2 \simeq (4A/\gamma T)^2 e^{-\gamma_p t}\sinh^2(\tfrac{1}{4}\gamma T). \qquad (57)$$

Although these results depend upon the relative coupling strength $A/\gamma T = V_0/\hbar\gamma$ in the same manner, they may be of quite different sizes depending on the relative values of the decay rates γ_p and γ_s.

We can summarize this section very briefly by concluding that for coupling pulse durations T both short and long compared to the system decay rate time scale $1/\gamma$, there are substantial quantitative and qualitative differences between the final-state level amplitudes predicted by the theory for the Rosen-Zener pulse [Eq. (2)] and the rectangular pulse [Eq. (25)]. The results for the two pulses are similar only for very short pulses, $\gamma T \ll 1$; for longer pulses, the final-state level amplitudes are quite different (particularly off resonance) in both magnitude and phase, indicating substantial differences in transient behavior of the coupled two-level system. Although some of the differences (e.g., "ringing") can be expected qualitatively on general principles, in that the solutions for the Rosen-Zener pulse bear the same general relationship to the adiabatic approximation as do the solutions for the rectangular pulse to the sudden approximation, the present comparison allows exact quantitative differences to be calculated. This is potentially of value in those types of precision experiments[3,4] where a quantitative theory of resonance line shapes is required, and in which a coupling pulse with a "fringing" characteristic is present.

IV. "QUASISUDDEN" SOLUTION

In some applications, the two-level system of Eqs. (3) may be coupled by a pulse $W(t)$ which rises from zero at early times to a maximum magnitude of V_0, say at time $t=0$, and then remains at V_0 indefinitely for $t>0$. If we allow a rotating field time dependence for $W(t)$ during this time, i.e., a multiplicative factor of $e^{i\nu t}$ as in Eq. (4), we may model such a coupling pulse by the envelope function

$$U(t) = \begin{cases} V_0\,\mathrm{sech}(\pi t/T), & -\infty < t \le 0, \\ V_0, & t \ge 0. \end{cases} \qquad (58)$$

The overall coupling is $W(t) = U(t)e^{i\nu t}$, where ν may be zero for application of a "static" field. The envelope function $U(t)$ rises smoothly (quasi-exponentially) from zero for $t \to -\infty$ to the constant value V_0 at $t=0$, having achieved its half-maximum value at time $t=-\Delta t$, where

$$\Delta t = (T/\pi)\ln(2+\sqrt{3}) = 0.4192T \qquad (59)$$

may be taken as a sort of pulse rise time. For $T \to 0$, this rise time vanishes, and $U(t)$ then represents a step function of magnitude V_0 at time $t=0$.

For $T \to 0$, or the application of a step function coupling, we expect that the sudden approximation will work well; this is equivalent for a two-level system to expressing the level-mixing amplitudes by solutions for the leading edge of a rectangular pulse as in Eqs. (26) and (27). But when T is comparable to the system's natural time scales, either the effective Bohr period $1/\omega$ or the characteristic damping time $1/\gamma$, the sudden approximation breaks down, becoming successively worse as ωT or γT increases. The adiabatic approximation should be used for long rise times, $\omega T \gg 1$, but even this will not work near resonance, $\omega \to 0$. And for $\omega T \sim 1$, neither the sudden nor the adiabatic approximation to the actual coupling pulse shape can be expected to give reliable results for the level amplitudes at times $t > T$. Among other things, this means that for rise times T such that $\gamma T \sim 1$ one can expect to see a breakdown in the step-function theory of a resonance transition line shape (i.e., a calculation of the final-state level populations as a function of ω) at off-resonance frequencies $\omega \sim \gamma$, which is near the line half maximum. Effects of this sort are of potential importance in certain types of atomic beam experiments where the beam enters an interaction region which may have a "fringe field" characteristic, i.e., a region of gradually increasing coupling strength with a rise time T of the order of $1/\omega$ and/or $1/\gamma$.[18]

An evident improvement over a step-function (or sudden approximation) theory of the final-state level amplitudes for the coupled two-level system would be to solve the amplitude Eqs. (3) for the coupling $U(t)$ of Eq. (58), where the pulse rise time T may be varied arbitrarily from $T \to 0$ to $T \to$ "large" (as compared with $1/\omega$ and/or $1/\gamma$). That is what we will do in this section, simply by evaluating the Rosen-Zener solutions of Sec. II at time $t = 0$, which accounts for the system coupling over $-\infty < t \leq 0$, and then using the $t = 0$ Rosen-Zener solutions for the amplitudes as the "initial conditions" for the well-known step-function solutions at $t \geq 0$. This works precisely because we have *exact* solutions for the level amplitudes, namely Eqs. (17), in a region such as $-\infty < t \leq 0$ for $U(t)$, where the system coupling is represented by the envelope function $V_0 \operatorname{sech}(\pi t/T)$. One can expect, as shown in Sec. III, that there will be significant new features (as compared with step-function theory) in both the magnitude and relative phase of the final-state level amplitudes for pulse rise times T comparable to the system's damping time $1/\gamma$ and for off-resonance frequencies $\omega \lesssim \gamma$. We shall call these modified step-function type solutions the "quasisudden" solutions, since for arbitrary T neither the sudden nor the adiabatic approximation is involved as such; we have a sort of intermediate case.

The solutions to the amplitude Eqs. (3) for the coupling pulse envelope of Eq. (58) are given by the level-mixing amplitudes of Eqs. (17) for $-\infty < t \leq 0$. The physical amplitudes are those of Eqs. (5). Since the variable z of Eq. (11) is equal to one-half when $t = 0$, the physical amplitudes are then (exactly)

$$p(0) = (a/|\phi|)2^{-\phi}F(\phi + a, \phi - a; \phi + 1; \tfrac{1}{2}),$$
$$s(0) = (i\phi/|\phi|)2^{-\phi}F(\phi + a, \phi - a; \phi; \tfrac{1}{2}), \qquad (60)$$

where F is the hypergeometric function,[10] and

$$\pi a = V_0 T/\hbar = A, \quad \phi = \tfrac{1}{2} + (\gamma T/4\pi) - i(\omega T/2\pi), \qquad (61)$$

with $\gamma = \gamma_p - \gamma_s$ the system damping constant and $\omega = \omega_{sp} - \nu$ the off-resonance frequency as defined in Eqs. (7). The parameter A is now twice the pulse "area" over $-\infty < t \leq 0$ as it was defined in Eq. (39). Let us assume that the pulse maximum magnitude V_0 is a real parameter; a more general treatment would allow V_0 to be complex, say $|V_0| e^{i\delta}$ for a pulse whose rotating field factor had an initial phase δ.[12] Finally, the amplitudes of Eqs. (60) satisfy the initial conditions of Eqs. (8), namely the (freely decaying) s level is fully occupied and the p level is unoccupied as $t \to -\infty$.

The F functions of Eqs. (60) can be expressed in terms of γ functions: for example,[19]

$$p(0) = \frac{\phi}{|\phi|} \frac{\sqrt{\pi}\,\Gamma(\phi)}{2^{\phi}} \left\{ \left[\Gamma\left(\frac{\phi + a}{2}\right) \Gamma\left(\tfrac{1}{2} + \frac{\phi - a}{2}\right) \right]^{-1} - \left[\Gamma\left(\frac{\phi - a}{2}\right) \Gamma\left(\tfrac{1}{2} + \frac{\phi + a}{2}\right) \right]^{-1} \right\}. \qquad (62)$$

The expression for $s(0)$ is more complicated. After some algebra,[20] we find

$$s(0) = \frac{i\phi}{|\phi|} \frac{\sqrt{\pi}\,\Gamma(\phi)}{2^{\phi-1}} \left[\Gamma\left(\frac{\phi + a}{2}\right) \Gamma\left(\tfrac{1}{2} + \frac{\phi - a}{2}\right) \right]^{-1} - ip(0). \qquad (63)$$

Equations (62) and (63) are exact. They behave correctly in the limit of zero coupling, $V_0 \to 0$ or $a \to 0$ (but $T \neq 0$) as

$$|p(0)| \to 0, \quad |s(0)| \to 1 \text{ for } a \to 0. \qquad (64)$$

These are the usual initial conditions used for the solution to the step-function pulse. An interesting change occurs in the limit that $T \to 0$ but $\pi a = V_0 T/\hbar = A \neq 0$, i.e., a sort of "rounded off" step function. We find (for $\phi = \tfrac{1}{2}$)

$$p(0) = \sin(\tfrac{1}{2}A), \quad s(0) = i\cos(\tfrac{1}{2}A), \quad \text{for } T \to 0. \qquad (65)$$

These results are similar to those for the δ function limit of the Rosen-Zener pulse calculated in

Eqs. (31) of Sec. III A; in fact we may quote those results as

$$p_Z(0+)=\sin A,\quad s_Z(0+)=i\cos A, \tag{66}$$

where 0+ means that p_Z and s_Z are evaluated at a time t infinitesimally greater than zero. The behavior of the initial amplitudes $p(0)$ and $s(0)$ of Eq. (65) for our smoothly varying step function ($T\to 0$, but $A\neq 0$), which is—in a manner of speaking—intermediate between that for an instantaneous step (V_0 finite and $dV/dt\to\infty$) and a true δ function, indicates that any degree of rounding off at the leading edge of a step function has a critical effect on what initial conditions should be employed. The critical parameter is the effective area of the leading edge of the step

$$\tfrac{1}{2}A=\int_{-\infty}^{0}U(t)\,dt=\frac{V_0T}{2\hbar}. \tag{67}$$

It is only when $\frac{1}{2}A$ is truly zero that the usual step-function initial conditions of Eq. (64) are valid.

When the rise-time of the leading edge of the pulse is not negligible compared to the natural time scales of the coupled two-level system (i.e., T not "small" compared to the effective Bohr period $1/\omega$ or damping time $1/\gamma$), then the behavior of the initial amplitudes $p(0)$ and $s(0)$ is given by the much more complicated expressions in Eqs. (62) and (63). We shall deal with the complexities of $p(0)$ and $s(0)$ later (see Appendix A); here we wish to write down the final-state level amplitudes for the pulse $U(t)$ of Eq. (58). We have the results for the coupling from $t>-\infty$ up to $t=0$ as expressed by $p(0)$ and $s(0)$ of Eqs. (62) and (63); we will now use these as the input amplitudes for the solution to the $U(t)=V_0$ problem for $t\geq 0$.

The general solutions to Eqs. (3) for the coupling pulse magnitude V_0= const have been calculated by Lamb,[21] Ramsey,[22] and most recently by Fabjan and Pipkin.[18] If the amplitudes of the p and s levels are $p(0)$ and $s(0)$ upon application of a constant pulse of amplitude $V_0e^{i\delta}$, then at time $t\geq 0$ the level amplitudes are

$$\begin{aligned} p(t)&=\exp[-\tfrac{1}{2}i\omega t-\tfrac{1}{4}(\gamma_p+\gamma_s)t]\\ &\quad\times\{[\cosh\lambda t-(1/Q)\sinh\lambda t]p(0)\\ &\qquad-e^{i\delta}[(1-Q^{-2})^{1/2}\sinh\lambda t]s(0)\},\\ s(t)&=\exp[\tfrac{1}{2}i\omega t-\tfrac{1}{4}(\gamma_p+\gamma_s)t]\\ &\quad\times\{[\cosh\lambda t+(1/Q)\sinh\lambda t]s(0)\\ &\qquad-e^{-i\delta}[(1-Q^{-2})^{1/2}\sinh\lambda t]p(0)\}, \end{aligned} \tag{68}$$

where

$$\begin{aligned} &\lambda=\tfrac{1}{4}\gamma(1-i\kappa)Q,\quad Q=\{1-[4|q|^2/(1-i\kappa)^2]\}^{1/2};\\ &\gamma=\gamma_p-\gamma_s,\quad \omega=\omega_{sp}-\nu;\\ &q=2V_0e^{i\delta}/\hbar\gamma,\quad \kappa=2\omega/\gamma. \end{aligned} \tag{69}$$

These results are the same as Eqs. (9b) and (9a) of the Fabjan-Pipkin paper[18]; they have been transformed to our notation [see Eqs. (28)] for convenience. We note that γ is the same system damping constant and ω the same off-resonance frequency as we have previously used [see Eq. (7)]. We have previously chosen the initial rotating field phase[12] $\delta=0$; to be consistent with the $p(0)$ and $s(0)$ results of Eqs. (62) and (63) we must choose $\delta=0$ here also. Equations (68) yield results identical to the rectangular pulse results of Eqs. (26) and (27) when $p(0)=0$ and $s(0)=1$, and when the free decay terms $e^{-\gamma_p t/2}$ and $e^{-\gamma_s t/2}$ are factored out.

Equations (68) are the desired "quasisudden" solutions to the problem discussed at the beginning of this section, i.e., they are general and exact solutions to the coupled two-level problem of Eqs. (3) for a coupling pulse $W(t)=U(t)e^{i\nu t}$, with $U(t)$ being the quasi-step-function of Eq. (58). We need only substitute the exact values of $p(0)$ and $s(0)$ from Eqs. (62) and (63). The critical parameters γ, ω, T, and $A=V_0T/\hbar$ may be varied arbitrarily since the solutions are exact. Evidently the complete expressions for $p(t)$ and $s(t)$ are quite complicated. However, we can expect quite different results for the quasisudden solutions with $p(0)$ and $s(0)$ given by Eqs. (62) and (63), as compared with the step-function solutions with much simpler initial conditions such as $p(0)=0$, and $s(0)=1$, in Eq. (64). Evidence for this is already apparent in the discussion above on "rounding" in Eqs. (65)–(67).

We shall not discuss in detail the differences between the quasisudden solution and step-function solution which arise because of the different boundary conditions chosen. We shall make a few general remarks, however. If we look at the weak-coupling limit, where we expect the differences between the two solutions should be minimal, even then these differences may not be negligible. For if we expand $p(0)$ and $s(0)$ of Eqs. (62) and (63) in a series of powers of the coupling parameter $a=V_0T/\pi\hbar$, we find

$$\begin{aligned} p(0)&\simeq(\phi/|\phi|)[a\beta(\phi)+\mathcal{O}(a^3)],\\ s(0)&\simeq(i\phi/|\phi|)[1+\mathcal{O}(a^2)], \end{aligned} \tag{70}$$

where

$$\beta(\phi)=\tfrac{1}{2}\left[\psi\left(\frac{\phi+1}{2}\right)-\psi(\tfrac{1}{2}\phi)\right], \tag{71}$$

with $\psi(\phi)$ the digamma function.[11] Details are given in Appendix A, where the higher-order terms are also calculated. Now if we use these as input amplitudes in the final-state level populations corresponding to the amplitudes of Eqs. (68), namely

$$|p(t)|^2 = \exp[-\tfrac{1}{2}(\gamma_p+\gamma_s)t] \times\{|X_- p(0)|^2 + |\Sigma s(0)|^2 - 2\,\mathrm{Re}[X_-^*\Sigma p^*(0)s(0)]\},$$

$$|s(t)|^2 = \exp[-\tfrac{1}{2}(\gamma_p+\gamma_s)t] \times\{|X_+ s(0)|^2 + |\Sigma p(0)|^2 - 2\,\mathrm{Re}[X_+^*\Sigma s^*(0)p(0)]\}, \qquad (72)$$

where

$$X_\pm = \cosh\lambda t \pm (1/Q)\sinh\lambda t, \qquad \Sigma = (1-Q^{-2})^{1/2}\sinh\lambda t, \qquad (73)$$

we can see that the cross terms will be quite different, even in lowest order in a, depending on the choice of $p(0)$ and $s(0)$. If we take $p(0)$ of order a and $s(0)$ of order unity as in the quasisudden input amplitudes of Eqs. (70), the cross terms will be of order a^2 (since Σ is of order a). Such cross terms vanish in the step-function solution when we choose $p(0)=0, s(0)=1$. Since $|p(t)|^2$ itself is of order a^2, then clearly the inclusion or exclusion of the cross term is critical.

In this section, we have solved the two-level problem of Eqs. (3) for an applied coupling $W(t) = U(t)e^{i\nu t}$, where $U(t)$ is the envelope function of Eq. (58). This $W(t)$ represents a modified radiofrequency step function where the pulse rise time T is arbitrary. The solutions, which we have termed the "quasisudden" solutions, are the final-state amplitudes of Eqs. (68), where $p(0)$ and $s(0)$ are the input amplitudes of Eqs. (62) and (63). We have shown that the choice of input amplitudes depends critically on the structure of the leading edge of the pulse, depending in this specific example on the integrated "area" from $t > -\infty$ up to the point in time ($t=0$) where the pulse actually achieves constant amplitude; see Eqs. (65)–(67). Finally, we have remarked that generally the induced level population $|p(t)|^2$ will depend rather critically on the choice of input amplitudes $p(0)$ and $s(0)$, so that substantial differences in $|p(t)|^2$ may be seen depending on whether the "quasisudden" or sudden approximation solutions are used.

V. DISCUSSION: ROSEN-ZENER CONJECTURE

The detailed conclusions regarding the present calculation are set forth in the final paragraphs of Secs. II-IV. They are therefore not repeated here. However, we can remark that the general program set forth in the last paragraph of Sec. I has been accomplished. Namely, the quantum-mechanical two-level problem has been solved for a new type of coupling, by extension of the previous solution of Rosen and Zener[9] to include a rotating field dependence for the coupling pulse and to include spontaneous decay for the coupled levels. This new type of coupling pulse leads to exact solutions for the final-state level amplitudes in which all relevant parameters can be varied arbitrarily. In particular, the duration or rise time T of the applied pulse can be continuously varied from the regime of the sudden approximation (where $T \ll$ either the system Bohr period $1/\omega$ or characteristic damping time $1/\gamma$) to the regime of the adiabatic solution (where $T \gg 1/\omega$ or $1/\gamma$).[23] Over this range of variation in T, rather substantial differences occur between our "quasisudden" solutions for the final-state level amplitudes and the well-known solutions for a rectangular or step-function coupling pulse. At least some of the differences are nonphysical in that they are traceable to the nonphysical step discontinuity or infinite force impulse connected with the rectangular pulse. Although this is expected from the general principles characterizing sudden versus adiabatic approximations, our quasisudden solutions allow exact or nearly exact comparisons between these extremes, and they also quantitatively cover the intermediate case (ωT or $\gamma T \sim 1$). The quasisudden solutions are thus of potential value for a more complete and accurate version of the theory of resonance transition line shapes in those cases where the coupling pulse has a fringe-field characteristic.

We close with some remarks on an interesting conjecture made by Rosen and Zener.[9] They observed that the solution for the final-state induced level population for the coupled two-level system of Eqs. (6), namely

$$\dot{S} = -i[V^*(t)/\hbar]Pe^{i\Omega t}, \qquad \dot{P} = -i[V(t)/\hbar]Se^{-i\Omega t}, \qquad (74)$$

could be written in the form

$$|P(\text{final})|^2 = |(\sin A/A)\mathfrak{F}(T;\Omega)|^2, \qquad (75)$$

where A is the pulse area as defined in Eq. (39), and $\mathfrak{F}(T;\Omega)$ is the Fourier transform of the pulse

$$\mathfrak{F}(T;\Omega) = \frac{1}{\hbar}\int_{-\infty}^{+\infty} V(t)e^{i\Omega t}\,dt. \qquad (76)$$

Rosen and Zener showed that this rather general formulation was correct for their choice of coupling pulse, i.e., $V(t) = V_0\,\mathrm{sech}(\pi t/T)$, and for Ω a real variable. They speculated that the result of Eq. (75) held "for all nonsingular (coupling pulses), i.e., all functions which are continuous and whose first derivatives are continuous." To the extent this conjecture is valid,[24] we can use Eq. (75) to handle a quite general class of couplings, namely those for which a Fourier-transform exists.

First, we note that Eq. (75) as stated cannot be generally valid for coupling pulses $V(t)$ which are odd functions of time, for then the Fourier transform vanishes when $\Omega=0$, so that on resonance we would have $|P(\text{final})|^2\equiv 0$, which does not make sense. Perhaps one could patch up Eq. (75) for such pulses by replacing the Fourier integral over $-\infty\le t\le\infty$ by twice the integral over $0\le t\le\infty$.

Second, one can ask whether Eq. (75) is valid when Ω is a complex variable as in Eq. (7), i.e., when a nonzero damping constant $\gamma=\gamma_p-\gamma_s$ is included. The first response to this question is to see whether Eq. (75) holds for the Rosen-Zener pulse $V(t)=V_0\,\text{sech}(\pi t/T)$, when $\gamma\neq 0$. In this case, the Fourier integral becomes

$$\mathfrak{F}(T;\omega,\gamma)=\frac{V_0}{\hbar}\int_{-\infty}^{+\infty}\left[e^{-\gamma t/2}\,\text{sech}\left(\frac{\pi t}{T}\right)\right]e^{i\omega t}\,dt, \quad (77)$$

where $\Omega=\omega+\frac{1}{2}i\gamma$ as in Eqs. (7). As is shown in Appendix B, this integral exists and is tabulated for $|\gamma T|<2\pi$, and moreover

$$|(\sin A/A)\mathfrak{F}(T;\omega,\gamma)|^2=\sin^2 A\,|\text{sech}(\tfrac{1}{2}\Omega T)|^2. \quad (78)$$

This is identical with the approximate result of Eq. (23), Sec. II, which is (in the same terms)

$$|P(t\gg T)|^2\simeq\sin^2 A\,|\text{sech}\tfrac{1}{2}(\omega+\tfrac{1}{2}i\gamma)T|^2, \quad (79)$$

for $\gamma T\ll 2\pi$. If $|P(t\gg T)|$ is identified with $|P(\text{final})|$ of Eq. (75), we see that the Rosen-Zener conjecture works even when nonzero damping is included. More precisely, Eq. (75) is approximately correct for the coupling pulse $V(t)=V_0\,\text{sech}(\pi t/T)$, and for damping such that $|\gamma T|<2\pi$. Evidently the validity of Eq. (75) in general will depend on the way in which $V(t)$ goes to zero as $|t|\to\infty$, as compared with the behavior of $e^{-\gamma t/2}$: the product $V(t)e^{-\gamma t/2}$ must vanish as $|t|\to\infty$ in order for the Fourier transform to exist. We conclude, with these reservations, that the Rosen-Zener conjecture is at least approximately correct for well-behaved coupling pulses even with a (small) amount of damping present in the two-level system.

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APPENDIX A: INPUT AMPLITUDES FOR THE "QUASISUDDEN" SOLUTION

In Sec. IV, we used the amplitudes $p(0)$ and $s(0)$ of Eqs. (62) and (63), which result after application of the leading edge of a $V_0\,\text{sech}(\pi t/T)$ coupling, as the input amplitudes for our modified step-function or "quasisudden" solution of Eqs. (68) for the two-level problem. Evidently $p(0)$ and $s(0)$ are quite complicated functions of the coupling parameter a and time scale parameter ϕ, which are

$$a=V_0T/\pi\hbar,\quad \phi=\tfrac{1}{2}+(\gamma T/4\pi)-i(\omega T/2\pi), \quad (\text{A}1)$$

[from Eqs. (61)]. Here we wish to expand $p(0)$ and $s(0)$ for the case of weak coupling, $\frac{1}{2}a\ll 1$.

First note that if one defines a function f by

$$f(\phi,a)=\frac{\phi}{|\phi|}\,\frac{\sqrt{\pi}\,\Gamma(\phi)}{2^{\phi}}\left[\Gamma\left(\frac{\phi+a}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\phi-a}{2}\right)\right]^{-1}, \quad (\text{A}2)$$

then $p(0)$ and $s(0)$ of Eqs. (62) and (63) can be written

$$\begin{aligned}p(0)&=f(\phi,a)-f(\phi,-a),\\ s(0)&=i[f(\phi,a)+f(\phi,-a)].\end{aligned} \quad (\text{A}3)$$

Evidently, if we expand $f(\phi,a)$ as a Taylor series in powers of the coupling parameter a, $p(0)$ will have a leading term of order a and will contain only odd powers of a, while $s(0)$ will have a leading term of order unity and will contain only even powers of a. We shall carry out such an expansion to terms of order a^3.

To expand $f(\phi,a)$, one needs a Taylor series for the γ function.[25] Straightforwardly, to third order

$$\Gamma(x+\epsilon)\simeq\Gamma(x)[1+\epsilon\psi(x)+\tfrac{1}{2}\epsilon^2\alpha(x)+\tfrac{1}{6}\epsilon^3\xi(x)], \quad (\text{A}4)$$

where $\psi(x)$ is the digamma function,[11] and the coefficients $\alpha(x)$ and $\xi(x)$ are

$$\begin{aligned}\alpha(x)&=\Gamma''(x)/\Gamma(x)=\psi'(x)+\psi^2(x),\\ \xi(x)&=\Gamma'''(x)/\Gamma(x)=\psi''(x)+3\psi'(x)\psi(x)+\psi^3(x).\end{aligned} \quad (\text{A}5)$$

The derivatives of $\psi(x)$ are the polygamma functions.[11]

With the help of Eq. (A4), we can expand $f(\phi,a)$ of Eq. (A2) in a series of powers of $\epsilon=\frac{1}{2}a$. It is convenient to define two additional functions, namely

$$\begin{aligned}\beta(\phi)&=\tfrac{1}{2}\left[\psi\left(\frac{\phi+1}{2}\right)-\psi(\tfrac{1}{2}\phi)\right],\\ B(\phi)&=\tfrac{1}{2}\left[\psi'\left(\frac{\phi+1}{2}\right)+\psi'(\tfrac{1}{2}\phi)\right].\end{aligned} \quad (\text{A}6)$$

The properties of $\beta(\phi)$ are discussed in Gradshteyn and Ryzhik[26]; we have not been able to find such a tabulation for $B(\phi)$. After some algebra, we find that $f(\phi, a)$ may be written

$$f(\phi,a) \simeq (\phi/2|\phi|)[1+2\epsilon\beta(\phi) + \epsilon^2 C_2(\phi) + \epsilon^3 C_3(\phi)], \tag{A7}$$

where $\epsilon = \frac{1}{2}a$, and

$$C_2(\phi) = 2\beta^2(\phi) - B(\phi),$$
$$C_3(\phi) = \tfrac{4}{3}\beta^3(\phi) + \tfrac{1}{3}\beta''(\phi) - 2\beta(\phi)B(\phi). \tag{A8}$$

The amplitudes of Eq. (A3) are then

$$p(0) \simeq (\phi/|\phi|)[2\epsilon\beta(\phi) + \epsilon^3 C_3(\phi)],$$
$$s(0) \simeq (i\phi/|\phi|)[1+\epsilon^2 C_2(\phi)]. \tag{A9}$$

In the absence of coupling, $\epsilon \to 0$, we have $p(0) = 0$ and $|s(0)| = 1$, as must be.

APPENDIX B: FOURIER TRANSFORM OF THE ROSEN-ZENER PULSE

We wish to evaluate the Fourier transform of the Rosen-Zener pulse of Eq. (77), namely

$$\mathfrak{F}(T;\omega,\gamma) = \frac{V_0}{\hbar}\int_{-\infty}^{+\infty}\left[e^{-\gamma t/2}\,\mathrm{sech}\!\left(\frac{\pi t}{T}\right)\right]e^{i\omega t}\,dt. \tag{B1}$$

Here ω is the system effective Bohr frequency and is real, and γ is the system damping constant and is also real. We find a general class of similar tabulated integrals in Gradshteyn and Ryzhik,[27] namely

$$\int_0^\infty \frac{e^{-\mu x}}{\cosh x - \cos\tau}\,dx = 2\csc\tau \sum_{k=1}^{\infty}\frac{\sin k\tau}{\mu + k}, \tag{B2}$$

good for $\mathrm{Re}\,\mu > -1$, and $\tau \neq 2n\pi$. If we take $\tau = \frac{1}{2}\pi$, and make appropriate changes of variables, we find

$$\int_0^\infty e^{i\Omega t}\,\mathrm{sech}\!\left(\frac{\pi t}{T}\right)dt = \frac{2T}{\pi}\sum_{k=1}^{\infty}\frac{\sin(\frac{1}{2}k\pi)}{k - i(\Omega T/\pi)}, \tag{B3}$$

where $\Omega = \omega + \frac{1}{2}i\gamma$ is the exponential coefficient in Eq. (B1). The integral exists for $\gamma T > -2\pi$. Using this result in Eq. (B1), we find

$$\mathfrak{F}(T;\omega,\gamma) = \frac{4A}{\pi}\sum_{k=1}^{\infty}\frac{k\sin(\frac{1}{2}k\pi)}{k^2 + (\Omega T/\pi)^2}, \tag{B4}$$

where $A = V_0 T/\hbar$ is the pulse area as defined in Eq. (39), and the integral exists if $|\gamma T| < 2\pi$. The sum is proportional to the Fourier series for[28] $\mathrm{sech}(\frac{1}{2}\Omega T)$; we get

$$(\sin A/A)\mathfrak{F}(T;\omega,\gamma) = \sin A\,\mathrm{sech}(\tfrac{1}{2}\Omega T) \tag{B5}$$

for $\Omega = \omega + \frac{1}{2}i\gamma$, and $|\gamma T| < 2\pi$. This result approximately agrees with the Rosen-Zener conjecture of Eq. (75) for such short pulses, for then $|P(\mathrm{final})|^2$ should be given by the absolute square of Eq. (B5)

$$|P(\mathrm{final})|^2 = \sin^2\pi a\,|\mathrm{sech}(\tfrac{1}{2}\Omega T)|^2, \tag{B6}$$

which is approximately the same as Eq. (23) of Sec. II.

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[12]In what follows, it is assumed that the amplitude V_0 of the rectangular pulse is real, as it was for the Rosen-Zener pulse in Sec. II. A more general treatment would allow V_0 to be complex in both cases. If the amplitude were written $V_0 e^{i\delta}$, this would be equivalent (in both cases) to choosing a phase δ for the rotating field, i.e., the rotating field part of the total pulse would be $\exp[i(\nu t + \delta)]$. Different choices of δ can lead to some interesting effects, as has been shown by Fabjan and Pipkin (as cited in Ref. 4). The present work is for $\delta = 0$, however.

[13]Keller and Robiscoe, Ref. 1. Many different authors have previously obtained these results. See the papers cited in Refs. 3 and 4.

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