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Exact and approximate differential renormalization-group generators. II. The equation of state

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We apply an approximate form of the exact one-particle-irreducible renormalization-group generator to the calculation of the equation of state. Several approaches are explored. (i) Global nonlinear trajectories for the Hamiltonian parameters are exploited to give nonlinear crossover equations of state. Logarithms of the reduced temperature $t \equiv (T-T_c)/T_c$ are automatically exponentiated to give power-law behavior. (ii) The form and asymptotic properties of the equation of state are described for those systems whose complete nonlinear trajectories cannot be explicitly obtained. (iii) Nonlinear trajectories for the irreducible Green's functions are solved to give a fully exponentiated equation of state containing no logarithmic terms. (iv) Simple critical (noncrossover) equations of state are obtained by iteration of the generator with linear and quasilinear trajectories. (v) Operator nonlinear equations are solved to give the crossover equation of state for an arbitrary order $\mathcal{O}$ critical system, both in an ϵ expansion and at the borderline dimension. (vi) Combination of the Green's functions are formed into Green's eigenfunctions or operators and their nonlinear trajectories used to calculate fully exponentiated equations of state for order $\mathcal{O}$ Ising systems. Systems considered include the usual Wilson-Fisher Hamiltonian [methods (i)–(iv)], the Sak-model compressible ferromagnet [(i)–(ii)], the nm-hypercubical model (of which the dilute quenched random ferromagnet forms the $n \to 0$ limit) [(ii)], and the isotropic n-component order $\mathcal{O}$ ferromagnet [(iv)–(vi)]. The tricritical ($\mathcal{O} = 3$) case is given explicitly as an example of the general formalism of methods (iv)–(vi). Throughout, we use a bare critical propagator which is an arbitrary generalized homogeneous function of the components of the wave vector. This allows us to simultaneously describe ordinary critical systems and anisotropic Lifschitz points as well as certain structural and spin-reorientation phase transitions.

I. INTRODUCTION

In an earlier paper[1] we used a variety of exact and approximate differential renormalization-group generators to describe critical systems, calculate the critical-point exponents, and solve nonlinear renormalization-group equations. Here, we wish to discuss various methods which can be used to apply differential generators to the problem of the calculation of the free energy and magnetization equation of state. Therefore, our emphasis is not on the fixed point and scaling features of the renormalization group, but rather on its usefulness in the direct calculation of the partition function. Our methods differ from other techniques in being entirely differential in nature rather than employing field-theoretic diagram expansions alone[2,3] or in combination with some recursive or differential renormalization of the Hamiltonian parameters.[4–7] Of course, certain aspects of our solutions have diagrammatic analogs, but the generator itself is self-contained. For the calculations detailed here, we will use an approximate form[8] of an exact one-particle-irreducible generator[9] to demonstrate the solution techniques for several model systems.

The exact generator is derived by performing an infinitesimal saddle-point expansion (in the spirit of the Wegner-Houghton generator[10]) about a nonzero magnetization. This magnetization is adjusted at each stage to conform with the physical magnetization and the loop expansion is employed to remove reducible diagrams. In fact, it can be shown that it is an infinitesimalization of the loop expansion described in Ref. 3. The exact generator is a transformation on the Hamiltonian, but it can be interpreted as an evolution equation for the Helmholtz potential $A(\vec{M}_{\vec{k}})$ for an arbitrary magnetization $\vec{M}_{\vec{k}}$.

Thus, the generator is capable of calculating all the one-particle-irreducible Green's functions. To describe the exact generator we first cover the full range of wave vectors with a one-parameter family of surfaces we call shells. Each wave vector lies on exactly one shell. For systems isotropic in $\vec{k}$, we can conveniently choose the lth shell to be specified by $|\vec{k}| = \exp[-l]$. For more complex systems, other shell systems are appropriate. At the lth stage of renormalization, the generator involves derivatives taken with respect to spin fluctuations on the lth shell and on the previously treated shells (loosely, larger wave vectors or greater momenta). We have the result

$$\frac{\partial A}{\partial l} = \operatorname*{tr}_{\text{shell}} \ln\{A_{ss'} - A_{sg}[A^{-1}]_{gg'}A_{g's'}\}, \tag{1.1}$$

where $A_{pq} = \delta^2 A/\delta M_p \delta M_{-q}$ (we suppress component indices), $\{s, s'\}$ denotes on-shell momenta and $\{g, g'\}$ denotes off- and above- (greater-) shell momenta. The inverse matrix $[A^{-1}]_{gg'}$ is computed within the above (gg') shell subspace only. The physical A is the limit of $A(l)$ as l tends to infinity; that is, as the family of shells completes its sweep through $\vec{k}$ space.

Although (1.1) is exact, it is useful for the purpose of practical calculations to approximate it. The solution techniques we will use have analogs for the exact generator. The basic approximation is made to guarantee the renormalization invariance of the critical propagator [which is the k-dependent portion of the quadratic term in A, $\frac{1}{2}\sum\Gamma(\vec{k})\vec{M}_k\cdot\vec{M}_{-k}$]. Ignoring corrections to this term amounts to neglecting those terms which are the source of critical exponents such as η. The remaining dependence on $\vec{k}$ is assumed to be trivial (see below). To make this self-consistent, we take the limit of zero external momenta, that is, setting to zero all momenta not summed or traced over explicitly in (1.1). We use a propagator Γ which is a generalized homogeneous function

$$\Gamma(\{\lambda^{1/\sigma_i}k_i\})=\lambda\Gamma(\{k_i\}), \tag{1.2a}$$

where each $\vec{k}_i$ is a d_i-dimensional vector; the lattice dimension $d=\sum_i d_i$. An important example is the anisotropic Lifshitz point, $\Gamma=\sum|k_i|^{\sigma_i}$, which arises in the modeling of the onset of helical order.[11,12] Other forms are obtained for structural and spin reorientation phase transitions.[11(b)] We have termed the $\{\sigma_i\}$, propagator exponents, as suggested by the Lifshitz Γ. We define our shells by

$$\Gamma(\{k_i\})=e^{-\lambda_2 l} \tag{1.2b}$$

for some λ_2 (λ_2 will be the Gaussian eigenvalue of the $\vec{s}^2$ operator). One choice is $\lambda_2=\sigma_>$, where $\sigma_>=\max\{\sigma_i\}$, however other choices are sometimes more convenient; we will leave the normalization of l (and hence, λ_2) free. The zero-momentum limit is taken in such a way that $\Gamma(k_i)\to 0$.

We wish to preserve the possibility of considering Helmholtz potentials of the form

$$A=A(\{\langle M_{i_1}\cdots M_{i_j}\rangle\})+\frac{1}{2}\sum\Gamma M_k M_{-k}, \tag{1.3a}$$

where we use the notation

$$\langle F\rangle=\int d^dx\,F(x). \tag{1.3b}$$

This amounts to permitting the expansion coefficients to contain extra δ functions in the wave vectors which group the spins.[1] Of course we are only interested in the thermodynamic limit $\Omega(=\langle 1\rangle)\to\infty$. This allows some freedom in the approximate expression since we can include terms which vanish in the thermodynamic limit in order to give the generator a simple form.

When we take the zero-momentum limit, the matrix A_{pq} depends on p and q only in the propagator term and through δ functions. Moreover the sum over greater than shell momenta inside the log is compressed as the momenta labeled g are forced onto the shell; these terms do, however, make (1.1) continuous for *small* external momenta which enables us to sort out the spin-grouping δ functions (which are all identically satisfied in the zero-momentum limit). We obtain

$$\frac{\partial A}{\partial l}=e^{-\lambda_0 l}\left\langle\operatorname{tr}\ln\left(e^{-\lambda_2 l}\delta_{ij}+\frac{\partial^2 A}{\partial M_i\partial M_j}\right)\right\rangle, \tag{1.4}$$

where $\lambda_0\equiv\lambda_2\sum d_i/\sigma_i$, and we define

$$\frac{\partial}{\partial M_i}\langle F\rangle\langle G\rangle=\langle F\rangle\frac{\partial G}{\partial M_i}+\langle G\rangle\frac{\partial F}{\partial M_i}. \tag{1.5}$$

Component indices for M have been restored. The factor $\exp[-\lambda_0 l]$ simply is the density of states on the shell, and $\exp[-\lambda_2 l]$ is exactly the value of the propagator $\Gamma(k)$ on the shell. To use (1.4) for systems without spin groupings, $A=\langle A(M)\rangle$, we can simply drop the brackets. On the other hand, we have shown[1] that (1.4) becomes exact for totally paired systems $A=A(\langle M^2\rangle)$. We can convert (1.4) into the usual sort of fixed-point generator by the scale charges $\vec{s}=\vec{M}\exp[\frac{1}{2}(\lambda_0-\lambda_2)l]$ and $H(\vec{s},l)=\exp[\lambda_0 l]A(\vec{M},l)$. Throughout the remainder of this paper H and $\vec{s}$ will refer to the scaled Hamiltonian and spin while A and $\vec{M}$ refer to the free energy and physical magnetization.

Since (1.4) is a closed-form expression, it is possible to show, for example, that for any non-Ising isotropical interaction spin systems, the longitudinal susceptibility, Γ_2^{-1}, diverges on the coexistence surface

$$\Gamma_2\propto h^{\lambda_4/\lambda_2}, \tag{1.6}$$

where $\lambda_4=2\lambda_2-\lambda_0$ and h is the magnetic field. This divergence has been discussed for $\sigma_i=2$ by various authors[13,14] and will be explicitly demonstrated for the usual critical point in Sec. III. Similarly, we can show that the critical-point exponent δ is always given by

$$\delta=\frac{\lambda_0+\lambda_2}{\lambda_0-\lambda_2}=\left(\sum\frac{d_i}{\sigma_i}+1\right)\Big/\left(\sum\frac{d_i}{\sigma_i}-1\right). \tag{1.7}$$

Section II is devoted to the calculation of one-loop crossover equations of state for several spin systems with quartic interactions. Nonlinear solutions[15] of the Hamiltonian parameters are used to calculate the completely renormalized coupling constants[16] which are nonlinear scaling fields.[17] Iteration of (1.4) with the nonlinear trajectories gives a $M\neq 0$ generalization of the trajectory integral method.[18] The full crossover solution is given for the isotropic case and for a model compressible ferromagnet,[1,19,20] while the solution in terms of the (unknown) nonlinear scaling fields is given for the mn hypercubical model (which becomes the quenched random model for $n\to 0$). These calculations are rigorously justified only for

$T > T_c$; for $T < T_c$, a continuation of the trajectories is required. This is remedied in Sec. III where trajectories for the Green's functions $\Gamma_n = \partial^n A/\partial M^n$ are used. This method increases the number of nonlinear equations, but has the advantage of applying equally well above and below the critical temperature and justifies *a posteriori* the continuations used in Sec. II. We apply this method to the Wilson-Fisher Hamiltonian and obtain a completely exponentiated equation of state (no log terms). This solution has several interesting properties. It is exact for $n = \infty$, and, if exponents correct to second order in $\epsilon \equiv \lambda_4 \equiv 4 - d$ are used, it is exact to $O(\epsilon^2)$ for $n = 1$ when expanded to that order. For general $n \neq 1$, it correctly gives the singularities on the coexistence surface, and close to the coexistence surface, it correctly gives the $\epsilon^2 \log^2$ terms of the exact result.[3]

In Sec. IV we consider higher-order critical points (nonquartic interactions). To begin with, we use a linear approach which cannot include crossover effects, but which is straightforward to apply both above and below T_c. This method (and small quasilinear modifications of it) has a close relationship to the multiple loop-expansion. The order Θ, isotropic case is described with the $\Theta = 3$ tricritical case worked out explicitly.

[For a system described by an initial Hamiltonian of degree 2Θ in $\vec{s}$, the "ϵ expansion" is in terms of the Gaussian eigenvalue $\lambda_{2\Theta}$. Thus, for quartic systems we could define $\epsilon \equiv \lambda_4$. However, since the usual $\epsilon = 4 - d$ only corresponds to quartic systems with k^2 propagators, we will reserve ϵ for that case alone, using λ_4 (or $\lambda_{2\Theta}$) for the general case.]

In Sec. V nonlinear operator trajectories[21] are calculated for the order Θ isotropic case. In this case, an equation of state which incorporates the mean field and true critical-fixed-point crossover can be obtained within a quadrature. The method also applies at the borderline dimensions for such systems where logarithmic corrections are obtained. The $\Theta = 3$ tricritical case is done explicitly to show that the approximate generators do correctly contain such logarithmic terms.[22] We also calculate the constant term in A for general Θ. The projection coefficients used in the trajectory equations can also be calculated with the Wilson-Kogut generator.[23,24] In Sec. VI, the approaches of Secs. III and V are combined to give nonlinear trajectories for Green's eigenfunctions. For the Ising case, the results of Sec. V can be applied directly to give fully exponentiated equations of state. In Appendix A, the properties of the nonlinear trajectory solutions used in Sec. II are derived.

II. ONE-LOOP EQUATIONS OF STATE: HAMILTONIAN PARAMETER TRAJECTORIES

Elsewhere we have developed techniques to solve the renormalization-group equations for Hamiltonian parameters in a nonlinear, global fashion.[15] The asymptotic behavior of these solutions give the completely renormalized coupling constants.[16] For the ordinary Wilson-Fisher (WF) model $H = \frac{1}{2} r s^2 + \frac{1}{4} u s^4$, we define

$$r_\infty = \lim_{l \to \infty} r(l) e^{-\lambda_2 l} , \tag{2.1a}$$

$$u_\infty = \lim_{l \to \infty} u(l) e^{-\lambda_4 l} . \tag{2.1b}$$

The functions r_∞ and u_∞ are nonlinear scaling fields[17] and they define a renormalized zero-loop approximation[7] to A, $A_0 = \frac{1}{2} r_\infty M^2 + \frac{1}{4} u_\infty M^4$.

Using the parameter trajectories on the right-hand side of (1.4) defines the one-loop correction[8] A_1,

$$A_1 = \int_0^\infty \left(\frac{\partial A}{\partial l} (A_0) - \frac{\partial A_0}{\partial l} \right) dl . \tag{2.2}$$

Thus, for the WF case, the quadratic and quartic terms are subtracted. These correspond to the mass and vertex renormalizations in field theory, but here they change the bare parameters to nonlinear scaling fields, e.g., $r \to r_\infty \propto t^\gamma$ (t is the reduced temperature and γ is the susceptibility exponent).

In this section, we emphasize the calculation of the magnetization equation of state, $h = \partial A/\partial M$, since it can be expressed entirely in terms of the nonlinear scaling fields. Therefore, we will give either the equation for h or will ignore the M-independent terms in A.[8,18] The constant or M-independent term will be calculated for various systems in Secs. III-V.

By applying the method described above, we obtain the equation of state for a model system in terms of the completely renormalized coupling constants. This is true regardless of whether we are able to *calculate* the $l \to \infty$ limits. For example, for the m-component WF model we obtain

$$h = r_\infty M + u_\infty M^3 + \frac{2}{\lambda_2} M |r_\infty|^{\lambda_0/\lambda_2} \left\{ (m-1) \frac{u_\infty}{r_\infty} \left[\left(1 + \frac{u_\infty M^2}{r_\infty}\right) \ln \left| 1 + \frac{u_\infty M^2}{r_\infty} \right| - \frac{u_\infty M^2}{r_\infty} \right] + \frac{3u_\infty}{r_\infty} \left[\left(1 + \frac{3u_\infty M^2}{r_\infty}\right) \ln \left| 1 + \frac{3u_\infty M^2}{r_\infty} \right| - \frac{3u_\infty M^2}{r_\infty} \right] \right\} . \tag{2.3}$$

If we make the estimates (cf. Ref. 7 and below) $r_\infty \sim \operatorname{sgn} t|t^\gamma|$, $u_\infty \sim u^*|t|^{\gamma-2\beta}$, where $u^* \equiv \lambda_4/2(m+8)$ is the fixed-point value of u and β is the magnetization exponent, we have

$$h = |t|^\gamma (\operatorname{sgn} t)M + |t|^{\gamma-2\beta}M^3 + \frac{\lambda_4}{\lambda_2}\frac{|t|^\gamma}{m+8} M\{(m-1)(\operatorname{sgn} t + M^2|t|^{-2\beta})\ln|1+(\operatorname{sgn} t)M^2|t|^{-2\beta}| + 3(\operatorname{sgn} t + 3M^2|t|^{-2\beta})\ln|1+3(\operatorname{sgn} t)M^2|t|^{-2\beta}|\}. \quad (2.4)$$

The scale of M and h has been adjusted to keep $M, h \sim O(\lambda_4^0)$. If we ϵ-expand (2.4), we obtain the usual result; as written, (2.4) is automatically expressed in Griffiths asymptotic form.[25]

If we wish to examine crossover competition between the Wilson-Fisher fixed point[26] and the Gaussian fixed point, we must use a more precise form for r_∞ and u_∞. Defining the (positive) crossover exponent ϕ by the ratio of the magnitude of the first irrelevant eigenvalue to the temperature eigenvalue, we have $\phi = \nu[\lambda_4 + O(\lambda_4^2)] \simeq \lambda_4/\lambda_2$. The leading crossover power-law behavior is given by

$$r_\infty = t\{|t|^\varphi/[\bar{u} + (1-\bar{u})|t|^\varphi]\}^{\Delta_m}, \quad (2.5a)$$

$$u_\infty = \bar{u}|t|^\varphi/[\bar{u} + (1-\bar{u})|t|^\varphi], \quad (2.5b)$$

where $\Delta_m \equiv (m+2)/(m+8)$ and $\bar{u} = u/u^*$. The competition and double-power-law scaling behavior[27] are clearly displayed. It is interesting that the nonlinear scaling fields in (2.5) are very nearly the simplest imaginable scaling fields with the required properties.[15] This is characteristic of all the crossover equations studied in this paper.

We also have the complete nonlinear solution[1] for the Sak[20] model compressible magnet. The Hamiltonian is given by

$$\mathcal{H} = \tfrac{1}{2}r\langle \vec{S}^2\rangle + \tfrac{1}{4}u\langle \vec{S}^2\vec{S}^2\rangle + \frac{v}{4\Omega}\langle \vec{S}^2\rangle^2 + (\text{gradient terms}). \quad (2.6)$$

We let $\Omega \to \infty$ in the resulting renormalization equations to obtain the thermodynamic limit. The term proportional to v reflects the coupling between the spin and elastic degree of freedom. In this case, we will give the free energy:

$$A = \tfrac{1}{2}r_\infty M^2 + \tfrac{1}{4}(u_\infty + v_\infty)M^4 + (1/\lambda_2)|r_\infty|^{\lambda_0/\lambda_2} \times \left[(m-1)L_1\left(\frac{(u_\infty+v_\infty)M^2}{r_\infty}\right) + L_1\left(\frac{(3u_\infty+v_\infty)M^2}{r_\infty}\right)\right] + (\text{terms in } r_\infty \text{ and } t). \quad (2.7a)$$

Here $v_\infty \equiv \lim v(l)\exp(-\lambda_4 l)$ and

$$L_1(x) = \tfrac{1}{2}(1+x)^2\ln|1+x| - \tfrac{1}{2}x - \tfrac{3}{4}x^2. \quad (2.7b)$$

For $v_\infty = 0$, we return to the Wilson Hamiltonian result. The nonlinear scaling fields are

$$(r_\infty, u_\infty, v_\infty) = [tq^{\Delta_m}f(q), uq, vq^{2\Delta_m}f(q)], \quad (2.8a)$$

where

$$f(q) = \frac{q^{(1-2\Delta_m)}}{\bar{v}(1-q^{(1-2\Delta_m)})/\bar{u}(1-2\Delta_m) + q^{(1-2\Delta_m)}}, \quad (2.8b)$$

and $\bar{v} \equiv 2mv/\lambda_4$. Equations (2.8) are valid for all values of u, v, and m. In the region $\bar{u} \geq \bar{v}/(1-2\Delta_m) \geq 0$, it is sufficient for illustrative purposes to pick $q = q_0$, where q_0 is the crossover function which appears in (2.5):

$$q_0 = |t|^\varphi/[\bar{u} + (1-\bar{u})|t|^\varphi]. \quad (2.9)$$

Just as a double-power-law scaling function arises in the description of a twofold competition between fixed points, the full description of the crossover phenomena of the three competiting fixed points of this problem (a critical, tricritical, and fourth-order fixed point) is characterized by triple-power-law scaling functions.[27] These incorporate the three different critical behaviors appropriate to the Gaussian ($\bar{u} = \bar{v} = 0$), Wilson-Fisher ($\bar{u} = 1$, $\bar{v} = 0$), and Fisher renormalized[28] ($\bar{u} = 1$, $\bar{v} = 1 - 2\Delta_m$) fixed points. This is most easily seen in the zero-loop approximation to the magnetization equation of state,

$$h = Mt\left(\frac{|t|^\varphi}{\bar{u}+(1-\bar{u})|t|^\varphi}\right)^{1-\Delta_m} \frac{\bar{u}(1-2\Delta_m)}{\bar{v} + [\bar{u}(1-2\Delta_m) - \bar{v}]\{|t|^\varphi/[\bar{u}+(1-\bar{u})|t|^\varphi]\}^{1-2\Delta_m}} + \frac{M^3\bar{u}|t|^\varphi}{\bar{u}+(1-\bar{u})|t|^\varphi}\left(1 + \frac{\bar{v}(4-m)/m}{\bar{v} + [\bar{u}(1-2\Delta_m) - \bar{v}]\{|t|^\varphi/[\bar{u}+(1-\bar{u})|t|^\varphi]\}^{1-2\Delta_m}}\right). \quad (2.10)$$

The asymptotic behavior of this expression depends on the sign of the critical-point exponent α at the Wilson-Fisher fixed point: $\alpha_{WF} \equiv \lambda_4(1-2\Delta_m)/\lambda_2$. For $\alpha_{WF} > 0$, the Wilson-Fisher point is unstable with respect to the Fisher-renormalized point. The asymptotic values of the exponents are changed in the usual way, i.e., $\gamma \to \gamma/(1-\alpha_{WF})$, $\beta \to \beta(1-\alpha_{WF})$, and $\alpha \to -\alpha_{WF}/(1-\alpha_{WF})$. In this case, $\bar{v} \neq 0$ is a surface of ordinary critical points with Fisher-renormalized exponents. There is a line of tricritical points with Wilson-Fisher exponents along $\bar{v} = 0$. Finally, the Gaussian fixed point appears as a point of order 4.

On the other hand, for $\alpha_{WF} < 0$, the Wilson-Fisher fixed point is the more stable. The second-, third-, and fourth-order critical hypersurfaces are given by $\bar{v}\neq\bar{u}(1-2\Delta_m)$, $\bar{v}=\bar{u}(1-2\Delta_m)$, and $\bar{u}=\bar{v}=0$, respectively. The relative simplicity of the full nonlinear solutions of the renormalization-group equations (cf. Appendix A) is reflected in the fact that the equation of state (2.10) involves the product of two double-power-law crossover terms rather than the most general triple-power-law.

Our third example is the nm-hypercubical model.[29] Although we have not derived the complete nonlinear solution, we can give the form of the free energy in terms of the scaling fields. The Hamiltonian is

$$H=\frac{1}{2}r\sum_{J=1}^{n} s_J^2+\frac{1}{4}u\Big(\sum_{J=1}^{n} s_J^2\Big)^2+\frac{1}{4}c\Big(\sum_{J=1}^{n}(s_J^2)^2\Big). \qquad (2.11)$$

Each $\vec{s}_J$ is an m-component spin vector. The free energy is given by

$$A=\frac{r_\infty}{2}\sum_I M_I^2+\frac{u_\infty}{4}\Big(\sum_I M_I^2\Big)^2+\frac{c_\infty}{4}\sum_I M_I^4+\frac{|r_\infty|^{\lambda_0/\lambda_2}}{\lambda_2}\sum_I\Big[(m-1)L_1\Big(\frac{u_\infty}{r_\infty}\sum_J M_J^2+\frac{c_\infty}{r_\infty}M_I^2\Big)+L_1(z_I)\Big]. \qquad (2.12)$$

The z_I are defined by

$$\ln\left|1+\sum_J\frac{2u_\infty r_\infty^{-1}M_J^2}{1+z+u_\infty r_\infty^{-1}\sum_I M_I^2+3c_\infty r_\infty^{-1}M_J^2}\right|=\sum_J\Big[\ln(1+z+z_J)-\ln\Big(1+z+u_\infty r_\infty^{-1}\sum_I M_I^2+3c_\infty r_\infty^{-1}M_J^2\Big)\Big]. \qquad (2.13)$$

For the special case of $n=2$ [cf. Ref. 15(b) and Appendix A], the roots are given by

$$z_{1,2}=\frac{1}{2r_\infty}\{(M_1^2+M_2^2)(4u_\infty+3c_\infty)\pm[4u_\infty^2(M_1^2+M_2^2)^2+3c_\infty(4u_\infty+3c_\infty)\times(M_1^2-M_2^2)^2]^{1/2}\}. \qquad (2.14)$$

The detailed behavior is quite complicated!

We can simplify the result in special cases. First, consider $M_J=M\delta_{IJ}$. Then we have

$$A=\tfrac{1}{2}r_\infty M^2+\tfrac{1}{4}(u_\infty+c_\infty)M^4+\frac{|r_\infty|^{\lambda_0/\lambda_2}}{\lambda_2}\Big[(m-1)L_1\Big(\frac{(u_\infty+c_\infty)}{r_\infty}M^2\Big)+L_1\Big(\frac{3(u_\infty+c_\infty)}{r_\infty}M^2\Big)+m(n-1)L_1\Big(\frac{u_\infty}{r_\infty}M^2\Big)\Big]. \qquad (2.15)$$

Note that for $n=1$, this reduces to the previous case (2.3) with $u_\infty+c_\infty$ replacing u_∞.

Second, consider $M_J=M$ for all J. We can now take the limit of A/n as $n\to 0$ to obtain the free energy of the random model.[30] The equation of state becomes

$$h=r_\infty M+c_\infty M^3+\frac{2}{\lambda_2}M|r_\infty|^{\lambda_0/\lambda_2}\times\Big[(m-1)\frac{c_\infty}{r_\infty}L_1'\Big(\frac{c_\infty M^2}{r_\infty}\Big)+\frac{3c_\infty+2u_\infty}{r_\infty}L_1'\Big(\frac{3c_\infty}{r_\infty}M^2\Big)+6\frac{u_\infty c_\infty}{r_\infty^2}M^2\ln\Big|1+\frac{3c_\infty}{r_\infty}M^2\Big|\Big], \qquad (2.16)$$

where $L_1'(x)=dL_1/dx=(1+x)\ln|1+x|-x$. For $u_\infty=0$, we return to (2.3).

III. GREEN'S-FUNCTION TRAJECTORIES: FULLY EXPONENTIATED EQUATIONS OF STATE

The Hamiltonian parameter trajectories discussed in Sec. II give partially exponential equations of state in that t is replaced by t^γ, etc.; however, the M dependence is unexponentiated, being expressed by logarithmic terms. In this section, we will use trajectory equations for the Green's functions to give fully exponentiated results. As a by-product, we will show that the results of Sec. II were correctly continued to $T<T_c$ [through the singularity at $r(l)=-1$].

We consider the n-component Wilson-Fisher model. All the information needed for the equation of state is provided by (1.4). We define

$$\Gamma_p=\partial^p A/\partial M^p. \qquad (3.1)$$

The function Γ_p is the p-point irreducible Green's function evaluated at zero momenta; Γ_1 is simply the magnetic field h, Γ_2 is the inverse longitudinal susceptibility and so forth. For $n\neq 1$ we expect $\Gamma_2\propto\Gamma_1^{\lambda_4/\lambda_2}$ near the coexistence surface; this will be demonstrated explicitly below.

By differentiating (1.4) with respect to M, we obtain renormalization-group equations for Γ_p. For example,

$$\dot{\Gamma}_1=e^{(\lambda_4-\lambda_2)l}[\Gamma_3 g_2+(n-1)(\Gamma_1/M)'g_1], \qquad (3.2a)$$

$$\dot{\Gamma}_2=e^{(\lambda_4-\lambda_2)l}[\Gamma_4 g_2+(n-1)(\Gamma_1/M)''g_1]-e^{\lambda_4 l}\{\Gamma_3^2 g_2^2+(n-1)[(\Gamma_1/M)']^2 g_1^2\}, \qquad (3.2b)$$

where a prime denotes differentiation with respect to M and $g_1\equiv(1+(\Gamma_1/M)e^{\lambda_2 l})^{-1}$, $g_2\equiv(1+\Gamma_2 e^{\lambda_2 l})^{-1}$. The equations for Γ_3 and Γ_4 are more complicated. Making the approximation that $\Gamma_5=\Gamma_6=0$ and keeping only quadratic terms, we have

$$\dot{\Gamma}_3=-3e^{\lambda_4 l}\Big[\Gamma_3\Gamma_4 g_2^2+(n-1)\Big(\frac{\Gamma_1}{M}\Big)'\Big(\frac{\Gamma_1}{M}\Big)''g_1^2\Big], \qquad (3.2c)$$

$$\dot{\Gamma}_4 = -3e^{\lambda_4 l}\left\{\Gamma_4^2 g_2^2 + (n-1)\left[\left(\frac{\Gamma_1}{M}\right)''\right]^2 g_1^2\right\}. \tag{3.2d}$$

Equations (3.2) preserve the relations

$$\Gamma_2 = \Gamma_1/M + \tfrac{1}{3}\Gamma_4 M^2 , \tag{3.3a}$$

$$\Gamma_3 = \Gamma_4 M , \tag{3.3b}$$

which are strictly true only as initial conditions to the exact trajectories.

We solve (3.2) in an analogous fashion to the systems in Sec. II. We first define linearized Green's functions

$$\tilde{\Gamma}_1 \equiv \Gamma_1 + \frac{e^{(\lambda_4-\lambda_2)l}}{\lambda_2-\lambda_4}\left[\Gamma_3 g_2 + (n-1)\left(\frac{\Gamma_1}{M}\right)' g_1\right], \tag{3.4a}$$

$$\tilde{\Gamma}_2 \equiv \Gamma_2 + \frac{e^{(\lambda_4-\lambda_2)l}}{\lambda_2-\lambda_4}\left[\Gamma_4 g_2 + (n-1)\left(\frac{\Gamma_1}{M}\right)'' g_1\right]. \tag{3.4b}$$

As $l\to\infty$, $\tilde{\Gamma}_1 \to \Gamma_1$, $\tilde{\Gamma}_2\to\Gamma_2$. The flow equations for $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are (to quadratic terms)

$$\frac{\dot{\tilde{\Gamma}}_1}{M} = -e^{\lambda_4 l}\,\frac{\tilde{\Gamma}_1}{M}\,\Gamma_4\left(\frac{n-1}{3}\,g_1^2 + g_2^2\right) - e^{\lambda_4 l}\,\tfrac{1}{3}\Gamma_4^2 M^2 g_2^2 , \tag{3.5a}$$

$$\dot{\tilde{\Gamma}}_2 = -e^{\lambda_4 l}\tilde{\Gamma}_2\Gamma_4\left(\frac{n-1}{3}\,g_1^2+g_2^2\right) - e^{\lambda_4 l}\Gamma_4^2 M^2 g_2^2 . \tag{3.5b}$$

The system of equations (3.2)–(3.5) can be solved by quadrature functions. Define Y_1, Y_2, and Y by

$$\dot{Y}_1/Y_1 = -\tfrac{1}{3}(n+8)\Gamma_4 e^{\lambda_4 l} g_1^2 , \tag{3.6a}$$

$$\dot{Y}_2/Y_2 = -\tfrac{1}{3}(n+8)\Gamma_4 e^{\lambda_4 l} g_2^2 , \tag{3.6b}$$

$$\dot{Y}/Y = -\Gamma_4 e^{\lambda_4 l}\left[3g_2^2 + \tfrac{1}{3}(n-1)g_1^2\right], \tag{3.6c}$$

with the boundary condition $Y_1(0)=Y_2(0)=Y(0)=1$. Of course $Y=Y_1^{(n-1)/(n+8)}Y_2^{9/(n+8)}$. Examining (3.2) and (3.5), we have immediately

$$\tilde{\Gamma}_1/M = Y(tY_2^{-6/(n+8)} + \tfrac{1}{6}u_4 M^2) , \tag{3.7a}$$

$$\tilde{\Gamma}_2 = Y(tY_2^{-6/(n+8)} + \tfrac{1}{2}u_4 M^2) , \tag{3.7b}$$

$$\Gamma_3 = u_4 Y M , \tag{3.7c}$$

$$\Gamma_4 = u_4 Y . \tag{3.7d}$$

Explicit dependence on λ_4 has been dropped; t and u_4 define the initial conditions for $\tilde{\Gamma}_1$ and Γ_4.

We cannot expect these expressions to be equally valid since successively more severe truncation was applied to the $\tilde{\Gamma}_2$, Γ_3, and Γ_4 equations. For example $\partial\Gamma_3/\partial M \neq \Gamma_4$. The expressions in (3.7b)–(3.7d) should be considered as auxillary functions to be used in the primary result (3.7a) for $\tilde{\Gamma}_1$.

Rather than attempting to directly solve the nonlinear problem to compute Y and Y_2, we will estimate them.[6,7] Combining (3.6c) with (3.7d), we have (exactly)

$$Y^{-1} = 1 + 3u_4\int_0^l e^{\lambda_4 l} g_2^2\,dl + \frac{n-1}{3}\,u_4\int_0^l e^{\lambda_4 l} g_1^2\,dl . \tag{3.8}$$

In the critical region, $\Gamma_2 \ll 1$, $\Gamma_1/M \ll 1$, so that $g_2 \approx 1$ up to a value of $l=l_1$ (and $g_2\approx 1$ up to $l=l_2$), where $(\Gamma_1/M)e^{\lambda_2 l_1} = 1(\Gamma_2 e^{\lambda_2 l_2}=1)$. Because $\Gamma_2 > \Gamma_1/M$, l_2 is less than l_1. For points near the coexistence surface, $l_2 \ll l_1$, but in any case, we have $1\ll l_2 \leq l_1$.

At a renormalization time l_2 the longitudinal fluctuations have been incorporated (or integrated out). However, this is not all the renormalization. Until $l\sim l_1$, the transverse fluctuations are still important. At $l\sim l_1$, we may assume that Γ_1/M has reached its asymptotic value, but at $l\sim l_2$, Γ_2 reflects only the longitudinal effects. Making an asymptotic expansion of the integrals in (3.8), therefore, gives as $l\to\infty$

$$Y^{-1} \cong 1 + \frac{3u_4}{\lambda_4}\left(\bar{\Gamma}_2^{-\lambda_4/\lambda_2} - 1\right) + \frac{n-1}{3}\,\frac{u_4}{\lambda_4}\left[\left(\frac{\Gamma_1}{M}\right)^{-\lambda_4/\lambda_2} - 1\right], \tag{3.9}$$

where $\bar{\Gamma}_2$ represents the value of Γ_2 at the point $l\sim l_2$.

If we now turn to Y_2, we have no simple expression like (3.8) to approximate. However, from its definition (3.6b), we see that $\dot{Y}_2 \to 0$ exponentially for $l>l_2$. Therefore, the integral for Y_2 is cut off by the longitudinal propagator at $l\cong l_2$. In the region $l<l_2$, we cannot distinguish between g_1 and g_2. Thus, the asymptotic expression for Y_2 is given by

$$Y_2^{-1} \cong 1 + \tfrac{1}{3}(n+8)(u_4/\lambda_4)(\bar{\Gamma}_2^{-\lambda_4/\lambda_2} - 1) . \tag{3.10}$$

To close our system, we need $\bar{\Gamma}_2$. Since $l_2 \gg 1$, we may assume that $\Gamma_2 \sim \tilde{\Gamma}_2$ so that (3.7b) can be used. At the renormalization time $l=l_2$, $Y\approx Y_2$ so that

$$\bar{\Gamma}_2 = Y_2\left[tY_2^{-6/(n+8)} + \tfrac{1}{2}u_4 M^2\right]. \tag{3.11}$$

Equations (3.9)–(3.11) combined with (3.7a) give the complete solution for Γ_1. The estimates used are supported by the exact nonlinear solution trajectories studied in Sec. II and Appendix A.[31]

These expressions give the full crossover equation of state containing the competition between the critical fixed point at $u_4 = 3\lambda_4/(n+8)$ and the Gaussian fixed point at $u_4=0$. We note that we can recover the Hamiltonian parameter expression of Sec. II by setting $\bar{\Gamma}_2 = t_\infty + 3u_\infty M^2$, $\Gamma_1/M = t_\infty + u_\infty M^2$ in the expressions for Y and Y_2 and expanding the M dependence ($u_4=6u$). We obtain exactly the logarithmic terms of (2.3), thus confirming that the continuations employed are justified.

To explore the result further we set $u_4 = 3\lambda_4/(n+8)$. Then, we have

$$\frac{\Gamma_1}{M} = \frac{t\bar{\Gamma}_2^{-[6/(n+8)](\lambda_4/\lambda_2)} + \frac{1}{6}M^2}{[(n-1)/(n+8)](\Gamma_1/M)^{-\lambda_4/\lambda_2} + [9/(n+8)]\bar{\Gamma}_2^{-\lambda_4/\lambda_2}} , \tag{3.12a}$$

$$\overline{\Gamma}_2=\overline{\Gamma}_2^{\lambda_4/\lambda_2}(t\overline{\Gamma}_2^{-6\lambda_4/(n+8)\lambda_2}+\tfrac{1}{2}M^2)\,. \tag{3.12b}$$

We have absorbed the fixed-point value of u_4 into a scale change of M. There are several points to note about (3.12).

First, as $n\to\infty$, the $\overline{\Gamma}_2$ dependence in (3.12a) disappears giving a simple equation for Γ_1/M

$$\Gamma_1/M=(\Gamma_1/M)^{\lambda_4/\lambda_2}(t+\tfrac{1}{6}M^2)\,. \tag{3.13a}$$

This can be solved to give the exact spherical result

$$\Gamma_1/M=(t+\tfrac{1}{6}M^2)^{1/(1-\lambda_4/\lambda_2)}\,. \tag{3.13b}$$

This reduction to an exact result does not apply to Γ_2, Γ_3, and Γ_4 as given in (3.7b)–(3.7d) as previously indicated and confirms that the expression for Γ_1 should be considered central. It also suggests that the implicit expressions such as (3.12) should be taken as written and not promiscuously ϵ-expanded.

Second, the coexistence curve singularity is correctly given. Equation (3.12b) guarantees that $\overline{\Gamma}_2$ is finite on the coexistence surface and $\Gamma_1\sim[\text{const}\,t+M^2]^{1/(1-\lambda_4/\lambda_2)}$ as required to give $\Gamma_2\sim\Gamma_1^{\lambda_4/\lambda_2}$. The amplitude for the crossover between analytic and nonanalytic behavior is given correctly to lowest order; i.e., $(n-1)\frac{1}{9}$ as can be read off directly from (3.12).[32]

Third, in the Ising case we can write (3.12a) as

$$\frac{\Gamma_1}{M}=\Gamma_2^{(\delta-3)/(\delta-1)}\left[t\,\Gamma_2^{(2\beta-1)/\gamma}+\tfrac{1}{6}M^2\right], \tag{3.14}$$

where the critical point exponents have been inserted in the manner required by scaling. The distinction between Γ_2 and $\overline{\Gamma}_2$ is of course absent. Equation (3.14) is interesting since we can use the critical-point exponents as measured or calculated separately. If we use the second-order exponents for the k^2 propagator case ($\lambda_4=\epsilon=4-d$) and expand (3.14) in an ϵ expansion, we recover the exact ϵ^2 equation of state. This fails at $O(\epsilon^3)$, however, due to the introduction of new classes of diagrams (and neglected terms of the trajectory equations) at that order.[33] In expanding to $O(\epsilon^2)$, it does not matter whether Γ_2 is obtained from (3.12) or by iteration from $\Gamma_2=\partial\Gamma_1/\partial M$.

Fourth, we may put (3.12) into normalized form. Define

$$\Gamma_1/M=M^{\delta-1}f(x),\quad \overline{\Gamma}_2=M^{\delta-1}\overline{g}(x),\quad x=t/M^{1/\beta}\,. \tag{3.15}$$

We wish to normalize such that $f(0)=1$ and $f(-1)=0$. We have

$$f(x)=\frac{n-1+9\overline{g}(0)^{-\lambda_4/\lambda_2}}{(n-1)f(x)^{-\lambda_4/\lambda_2}+9\overline{g}(x)^{-\lambda_4/\lambda_2}}\times\left[x\left(\frac{\overline{g}(x)}{\overline{g}(-1)}\right)^{-6\lambda_4/(n+8)\lambda_2}+1\right], \tag{3.16a}$$

$$\overline{g}(x)=\left(\frac{\overline{g}(x)}{\overline{g}(0)}\right)^{\lambda_4/\lambda_2}\left[x\left(\frac{\overline{g}(x)}{\overline{g}(-1)}\right)^{-6\lambda_4/(n+8)\lambda_2}+3\right]. \tag{3.16b}$$

We cannot "improve" this result by the insertion of critical-point exponents as easily as in the Ising case since there are several places where ambiguities arise. However, we can compare (3.16) as it is with the exact ϵ^2 result of Ref. 3 for the k^2 propagator case. We cannot expect to get all the $\epsilon^2\ln$ terms without correcting our exponents. However, $\epsilon^2\ln^2$ terms can be compared to check the exponentiation. A partial check has been provided by the exact spherical limit and the exact to $O(\epsilon^2)$ Ising result. We can use $f=x+1$ and $\overline{g}=x+3$ on the right-hand side of (3.16) since $O(\epsilon)$ corrections to these expressions do not affect the $\epsilon^2\ln^2$ terms. We obtain the $\epsilon^2\ln^2$ terms of Ref. 3 plus the following term:

$$\Delta=-\frac{\epsilon^2(n-1)}{(n+8)^2}(x+1)\ln^2\left(\frac{x+1}{x+3}\right). \tag{3.17}$$

Note that this vanishes for $n=1$, $n=\infty$ and on the coexistence surface $x=-1$. However, the result given in Ref. 3 also contains a quadrature integral Q, given by

$$Q=-\frac{\epsilon^2(n-1)}{2(n+8)^2}\left[(x+4)I_1(\rho)-I_2(\rho)\right], \tag{3.18}$$

where $\rho=(x+3)/4(x+1)$ and $I_1(\rho)$ and $I_2(\rho)$ are given in Ref. 3.

As $x\to-1$, $\rho\to\infty$, and we can make an asymptotic expansion of Q which gives

$$Q=\Delta+\frac{n-1}{(n+8)^2}\epsilon^2O\left(\frac{\ln^2\rho}{\rho^2},\frac{\ln\rho}{\rho}\right). \tag{3.19}$$

The term Δ thus represents the leading behavior of the integral near the coexistence surface. The remaining terms of the asymptotic expansion of Y_2 [neglected in (3.10) and subsequently] and $O(\epsilon^2)$ terms dropped from (3.2) are of the general form of the integrals of (3.18).

Fifth, Ref. 14 has also considered the exponentiated equation of state from a different approach. Equations equivalent to (3.2) are obtained by a spin shift. Approximate nonlinear trajectories are followed up to an l corresponding to our l_2; the transverse fluctuations are then considered in a diagram expansion which is approximately summed. A result different from (3.12) is obtained which agrees with it for $n=1$ but does not pass to the spherical limit for $n=\infty$. If we ϵ-expand both results for the k^2 case considered by Ref. 14, we find that Ref. 14 contains all of the $\epsilon^2\ln^2$ terms of (3.12) plus an additional term

$$\Delta'=\frac{\epsilon^2}{4}\,\frac{n^2-1}{(n+8)^2}\,\frac{(x+1)^2}{x+3}\ln^2\left(\frac{x+1}{x+3}\right). \tag{3.20}$$

This term spoils the spherical limit and is of a form which cannot arise from the integral Q (being too large by a factor of $n+1$). However, the term is small near the coexistence surface due to the factor $(x+1)^2$. The result of Ref. 14 also has change in the character of the temperature singularity at $n=5$ about which we cannot comment. The difference between the two results may be attributable to the use of the matching procedure used in Ref. 14 which can be shown to be good to $O(\epsilon)$, but which is perhaps not as uniform to $O(\epsilon^2)$ as a pure renormalization-group approach.

Finally, the free energy A can be calculated from (3.7). We cannot avoid the case of the asymptotic expansion about the break points $l=l_1$, and $l=l_2$, however, even if we express things in terms of Y and Y_2. We write

$$\dot{A}=e^{-\lambda_0 l}\{\ln(1+\Gamma_2 e^{\lambda_2 l}) + (n-1)\ln[1+(\Gamma_1/M)e^{\lambda_2 l}]\}. \tag{3.21}$$

Integrating by parts, we drop terms which are analytic at the critical point and terms with explicit λ_4 prefactors. Defining

$$\tilde{A}=A+\frac{\lambda_2}{\lambda_0(\lambda_2-\lambda_4)}e^{(\lambda_4-\lambda_2)l}\left(\tilde{\Gamma}_2 g_2+(n-1)\frac{\tilde{\Gamma}_1}{M}g_1\right), \tag{3.22}$$

we have (keeping only quadratic terms)

$$\dot{\tilde{A}}=-\frac{\lambda_2}{\lambda_0}e^{\lambda_4 l}\left[\tilde{\Gamma}_2^2 g_2^2+(n-1)\left(\frac{\tilde{\Gamma}_1}{M}\right)^2 g_1^2\right]. \tag{3.23}$$

We break the integration of (3.23) into $l<l_2$ and $l_2<l<l_1$. For $l<l_2$, we take $Y_1\approx Y_2$ and $g_1\approx g_2\approx 1$. For $l>l_2$, we set $g_2=0$, fix Y_2, and integrate Y_1 from Y_2 to its asymptotic value. To these results, we add the initial condition, $\tilde{A}=tM^2/2+u_4M^4/4!$, to obtain[34]

$$A=Y\left(t\frac{M^2}{2}Y_2^{-6/(n+8)}+\frac{u_4M^4}{4!}\right) + \frac{3n}{2(n-4)u_4}t^2(Y_2^{(n-4)/(n+8)}-1) + \frac{3}{2u_4}(Y-Y_2)t^2Y_2^{-12/(n+8)}. \tag{3.24}$$

Since $\alpha=(\lambda_4/\lambda_2)(4-n)/(n+8)$, we see that the specific-heat term is inversely proportional to α as expected. Differentiation of the explicitly shown M dependence gives (3.7).

In the spherical limit, (3.24) becomes

$$A=\tfrac{3}{2}(Y/u_4)(t+\tfrac{1}{6}u_4M^2)^2+(\text{analytic terms}). \tag{3.25}$$

For $u=u^*$, we rescale A and M ($A\to A/u$, $uM^2\to M^2$), to give

$$A=\tfrac{3}{2}(t+\tfrac{1}{6}M^2)^{2+\lambda_4/(\lambda_2-\lambda_4)}, \tag{3.26}$$

which is the exact spherical result.

We conclude this section by remarking that our results (3.7)–(3.12) and (3.24) satisfy both the Griffiths condition for large x and analytically for small x if they are treated as implicit equations. If a simple analytic guess is made as in the discussion following (3.16) then only the leading behavior will be in the Griffiths form. An iterated solution will improve this order by order (in $x^{-2\beta}$). In the spherical limit (3.13), for which the complete solutions of the implicit equations are easily obtained, both the small-x and large-x regions are represented by a single simple expression.

IV. HIGHER-ORDER CRITICAL POINTS: LINEAR AND QUASILINEAR APPROACH

The techniques of Secs. II and III give the nonlinear crossover trajectories and equation of state. At higher-order critical points (by which we mean systems not described by s^4 Landau theories), the nonlinear equations are more complex and there are different approaches which emphasize different aspects of the critical behavior. In this section, we use a simple iterative solution which gives a critical noncrossover equation of state in an ϵ expansion. The method closely parallels the field-theoretic approach as will be seen below. However, some information about the nature of the nonlinear trajectories can be employed to advantage in considering the result. We recalculate the $\Theta=2$ (s^4) equation of state as an example, and then give the result for the $\Theta=3$ (s^6) case. This method can be applied generally to any critical system.

We will consider the isotropic Θth-order equation of state for which $\lambda_{2\Theta}\ll 1$. The zero-loop approximation for A is a polynomial of degree Θ in M^2:

$$A_0=\frac{tM^2}{2}+\frac{uM^4}{4}+\cdots+\frac{v_\Theta M^{2\Theta}}{2\Theta}. \tag{4.1}$$

At the fixed point, v is $O(\lambda_{2\Theta})$; if t is considered to be $O(1)$, we have $M^2=O(\lambda_{2\Theta}^{-1/(\Theta-1)})$. We wish to have equations in which M is also $O(1)$ and we expect to be able to extract scaling behavior (in this linear approach) only if we are near the fixed point. Therefore, we set $v_\Theta=v^*\lambda_{2\Theta}$, where v^* is an $O(1)$ number which can be calculated as in Ref. 1 or which can be determined by enforcing scaling on the resulting equation of state. We rescale A and M accordingly, and define a variable z by

$$z^{\Theta-1}\equiv e^{-\lambda_2 l}. \tag{4.2}$$

Then to order $\lambda_{2\Theta}$, the generator for the free energy becomes[35]

$$\frac{\partial A}{\partial z}=-\frac{(v^*\lambda_{2\Theta})^{1/(\Theta-1)}}{4}z^{\Theta-1}\operatorname{tr}\ln\left(\delta_{ij}+z^{1-\Theta}\frac{\partial^2 A}{\partial M_i\partial M_j}\right). \tag{4.3}$$

For $l \in [0,\infty]$, we have $z \in [1,0]$. For convenience and to facilitate comparisons with other methods, we replace the cutoff 1 with a general cutoff Λ. This allows us to explore the limit $\Lambda \to \infty$.

From the form of (4.3), we see that to obtain all the $O(\lambda_{2\Theta})$ parts of A we must compute $\Theta - 1$ loops (iterations) of the equivalent integral equation. This is to be expected since only diagrams with Θ internal lines have logarithmic divergences in a field-theoretic approach.

We begin with the $\Theta = 2$ case. First, we will choose a "bare power" expression for A_0, $A_0 = \frac{1}{2}tM^2 + \frac{1}{4}M^4$, with t fixed. This corresponds to purely Gaussian trajectories of the scaled variables. We get from (4.3) a one-loop correction

$$A_1 = \tfrac{1}{4}v^*\lambda_4[\tfrac{1}{2}(t+3M^2)^2\ln(t+3M^2) + \tfrac{1}{2}(n-1)(t+M^2)^2\ln(t+M^2) + F(\Lambda)]. \quad (4.4)$$

In Eq. (4.4), $F(\Lambda)$ contains analytic corrections to A as well as terms proportional to Λ and $\ln\Lambda$ as $\Lambda \to \infty$. If we ignore $F(\Lambda)$, we reproduce the usual result.

The M^2 and M^4 portions of $F(\Lambda)$ could be incorporated into a redefinition of t and the scale of M. A more enlightening approach is to choose an initial A_0 which has some z dependence. A "bare eigenfunction" expansion in which s^2 and s^4 are replaced by the corresponding eigenfunctions Q_2 and Q_4 gives

$$A_0(z) = \tfrac{1}{2}t[M^2 - \tfrac{1}{2}n\lambda_4 v^* z] + \tfrac{1}{4}[M^4 - (n+2)\lambda_4 v^* z M^2] + O(\lambda_4^2). \quad (4.5)$$

This reduces the Λ divergence of F from linear to logarithmic. The $M^2\ln\Lambda$ and $M^4\ln\Lambda$ terms are handled by the use of pieces of the nonlinear trajectories; the full trajectories are not needed. We incorporate propagator factors into the eigenfunctions, and define[36]

$$Q_{2p} = (1/2p) f^p(z,t) L_p^{n/2-1}[-M^2/f(z,t)], \quad (4.6)$$

with $f(z,t) = \lambda_4 v^*[z - t\ln(t+z)]$. This cancels the $M^2\ln\Lambda$ term (and modifies the constant term). The $M^4\ln\Lambda$ term arises from the u^2 contribution to the u (four-point) equation; it indicates the presence of the fixed point whose effect on the trajectories was inadvisibly ignored. Recalling that the coefficient of the M^4 term (or Q_4 eigenfunction) is actually $u(l)\exp[-\lambda_4 l]$, we realize that as $l \to \infty$ on the critical separatrix, we have $u(l) \to 1$, its fixed-point value; thus, we are misrepresenting the form of the trajectories as $z \to 0$. Picking $A_0(z) = tQ_2 + z^{\lambda_4/\lambda_2}Q_4$ properly gives the asymptotic behavior of A_0 (many other choices would do as well) and cancels the $M^4\ln\Lambda$ term.

The remaining terms are finite as $\Lambda \to \infty$. The M^2 and M^4 parts of these terms are canceled by more subtle consideration of the nonlinear trajectories. Of course, for the problem at hand, $\Lambda = 1$. The $\Lambda \to \infty$ limit is useful only in removing analytic background terms from the calculated free energy, leaving the "singular part."

We now turn to the $\Theta = 3$ free energy; for k^2 propagators, this is a $\lambda_6 \propto 3-d$ expansion. The calculation to $O(\lambda_6)$ requires two iterations of (4.3), and for the general n-component model involves a quadrature. Dropping terms which are removed by rescaling or the use of simple nonlinear improvements on the bare eigenfunction trajectories, we take the $\Lambda \to \infty$ limit and find

$$\begin{aligned} A = {} & \tfrac{1}{2}tM^2 + \tfrac{1}{4}uM^4 + \tfrac{1}{6}M^6 - \frac{\pi(\lambda_6/\lambda_2)^{1/2}}{12[3(3n+22)]^{1/2}}[(t+3uM^2+5M^4)^{3/2} + (n-1)(t+uM^2+M^4)^{3/2}] \\ & + \frac{\lambda_6/\lambda_2}{24(3n+22)}\Big\{M^2(3u+10M^2)^2\ln(t+3uM^2+5M^4) + (n-1)(u+2M^2) \\ & \times\Big[3M^2(u+2M^2)\ln(t+3uM^2+5M^4) + K - M^2(u+2M^2)\frac{\partial K(A'/M, A'')}{\partial(A'/M)}\Big] \\ & + \tfrac{1}{4}(n-1)(3u+10M^2)[(t+3uM^2+5M^4)(t+uM^2+M^4)]^{1/2}\pi^2\Big\}. \end{aligned} \quad (4.7)$$

The function K in (4.7) is given by

$$K(p,q) = (pq)^{1/2}\int_0^{(p/q)^{1/2}} \frac{d\alpha\,\alpha^2}{1-\alpha^2}\ln\alpha^2. \quad (4.8)$$

K behaves as p^2 for p small and therefore does not contribute on the coexistence surface. The correct values of γ, β, and δ to $O(\lambda_6)$ and $\alpha(\cong\frac{1}{2})$ to $O(1)$ can be read off from (4.7). This expression also gives correctly the eigenvalue for the u (the M^4) perturbation (thermodynamic field) $\lambda_4' = \lambda_4 - 2(n+4)\lambda_6/(3n+22)$. In Sec. V, we will show that for $\lambda_6 = 0$, the approximate generator gives the correct logarithmic behavior by using a nonlinear trajectory calculation.

The general Θ case to $(\Theta - 1)$ loops is beyond the reach of this method as formulated above. In Sec. V, an operator technique is used to extract and exponentiate the leading logarithmic terms from the $(\Theta - 1)$th loop while ignoring the contributions from the intervening lower-order loops.

V. HIGHER-ORDER CRITICAL POINTS: NONLINEAR CROSSOVER AND LOGARITHMIC CORRECTIONS AT BORDERLINE DIMENSIONS

In Sec. IV. higher-order equations of state were calculated by straightforward iterations of the generator. For the isotropic case, $\Theta - 1$ iterations are needed to reach the first logarithmic corrections which are associated with the shift from mean-field exponents. In this section, we will use operator (eigenfunction) trajectories to bypass the intermediate stages. This is, of course, not a complete description of the equation of state since the nonlogarithmic terms are not generally analytic [cf. the $O(\lambda_6^{1/2})$ part of Eq. (4.7)]. However, the logarithmic terms do represent an important part of the equation of state, especially at borderline dimensions. There has been an impression in the literature that approximate renormalization groups do not correctly give these logarithmic corrections at $d=3$ for the k^2 propagator tricritical point.[22] We will show by explicit calculation that this is incorrect. The general Θ isotropic point is also calculated within a simple quadrature, with the crossover between the Gaussian (mean-field) and nontrivial (true critical) fixed point incorporated as in previous sections.

We work in the disordered phase for simplicity and calculate the nonlinear renormalization-group trajectories for all the relevant isotropic operators. To automatically incorporate the high-temperature (infinite Gaussian) fixed point we define operators with built-in propagators [cf. Eq. (4.6)]. If $Q_{2j}(\vec{s}^2)$ is the eigenfunction of degree $2j$, we define

$$\tilde{Q}_j = (1+t)^{-j} Q_{2j}[(1+t)\vec{s}^2]. \tag{5.1}$$

Here t is the reduced temperature [the coefficient of Q_2 in H].

We expand the Hamiltonian H in terms of the $\tilde{Q}_j$,[21]

$$H = \sum_{p=1}^{\Theta} a_p(l)\tilde{Q}_p. \tag{5.2}$$

The trajectories are

$$\dot{a}_p = \lambda_{2p}a_p - \frac{\sum \langle ij|p\rangle a_i a_j}{(1+t)^{i+j-p}} - \frac{\sum \langle ijk|p\rangle a_i a_j a_k}{(1+t)^{i+j+k-p}} - \frac{\dot{t}}{(1+t)^2}\{p+1,p\}a_{p+1} + (\text{quartic terms}) + \cdots. \tag{5.3}$$

The sums are over $i,j,k > 1$; $\langle ij|p\rangle$ is the projection of the quadratic part of the generator applied to Q_{2i} and Q_{2j} onto Q_{2p}, $\langle ijk|p\rangle$ is the analogous projection of the cubic terms, etc.; $\{p+1,p\}$ is defined by

$$[\vec{s}\cdot\vec{\nabla} - 2(j+1)]Q_{2j+2} = 2\{j+1,j\}Q_{2j}.$$

Logarithmic terms at the borderline dimension arise from the generalized Wegner conditions $i+j-p=\Theta, i+j+k-p=2\Theta$, and so forth. For finite $\lambda_{2\Theta}$, these terms give the leading power-law corrections. It can be shown that the cubic and higher degree terms in (5.3) give contributions smaller by factors of $\lambda_{2\Theta}$ (for $\lambda_{2\Theta}>0$) or by powers of $|\ln t|^{-1}$ (for $\lambda_{2\Theta}=0$) and will therefore be dropped. Similarly for $\Theta \geq 3$, the $\dot{t}$ term can be ignored. The resulting system of equations is

$$\dot{a}_p = \lambda_{2p}a_p - \sum_{i+j-p=\Theta} \frac{\langle ij|p\rangle a_i a_j}{(1+t)^{\Theta}}. \tag{5.4}$$

Because of the constraint $i+j-p=\Theta$, (5.4) is triangular in the a_p and can be solved exactly with a single quadrature. We define Y by

$$\dot{Y} = -\frac{\langle\Theta\Theta|\Theta\rangle a_\Theta(l)}{[1+t(l)]^{\Theta}}\, Y \tag{5.5}$$

with the boundary condition $Y(l=0)=1$. As in Sec. III, an explicit expression for Y^{-1} is

$$Y^{-1} = 1 + \mu_{2\Theta}\langle\Theta\Theta|\Theta\rangle \int_0^l \frac{e^{\lambda_{2\Theta}l'}}{[1+t(l')]^{\Theta}}\,dl', \tag{5.6}$$

where we define $\mu_{2p} \equiv a_p(l=0)$. We also define the $2p$-point Green's function

$$\Gamma_{2p} = \lim_{l\to\infty} e^{-\lambda_{2p}l}a_p(l). \tag{5.7}$$

The solutions of (5.4) have the form

$$a_p(l) = e^{\lambda_p l}Y^{\Delta_p}[\mu_{2p} + g_p(Y)], \tag{5.8a}$$

where

$$\Delta_p \equiv 2\langle p\Theta|\Theta\rangle/\langle\Theta\Theta|\Theta\rangle(1+\delta_{p,\Theta}). \tag{5.8b}$$

The ratios Δ_p determine the eigenvalues at the nontrivial fixed point[1] for $\lambda_{2\Theta}>0$,

$$\lambda'_{2p} = \lambda_{2p} - \lambda_{2\Theta}\Delta_p(1+\delta_{p,\Theta}). \tag{5.8c}$$

The functions $g_p(Y)$ are computed from

$$g_p(Y) = \frac{\sum'\langle ij|p\rangle}{\mu_{2\Theta}\langle\Theta\Theta|\Theta\rangle}\int_1^Y dY\, Y^{\Delta_i+\Delta_j-\Delta_p-2}(\mu_{2i}+g_i) \times (\mu_{2j}+g_j), \tag{5.9}$$

where $\sum'$ denotes the sum over $1<i,\ j<\Theta$, $i,j\neq p$. This is a triangular system of algebraic equations; each g_p is a sum of powers of Y.

The results for the first four Γ's are

$$\Gamma_{2\Theta} = \mu_{2\Theta}Y, \tag{5.10a}$$

$$\Gamma_{2\Theta-2} = \mu_{2\Theta-2}Y^{\Delta_{\Theta-1}}, \tag{5.10b}$$

$$\Gamma_{2\mathcal{O}-4}=Y^{\Delta_{\mathcal{O}-2}}\left(\mu_{2\mathcal{O}-4}+\frac{\mu^2_{2\mathcal{O}-2}\langle\mathcal{O}-1,\mathcal{O}-1\,|\,\mathcal{O}-2\rangle(Y^{2\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-2}-1}-1)}{\mu_{2\mathcal{O}}\langle\mathcal{O}\mathcal{O}\,|\,\mathcal{O}\rangle(2\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-2}-1)}\right), \tag{5.10c}$$

$$\Gamma_{2\mathcal{O}-6}=Y^{\Delta_{\mathcal{O}-3}}\Bigg\{\mu_{2\mathcal{O}-6}+\frac{2\mu_{2\mathcal{O}-2}\langle\mathcal{O}-2,\mathcal{O}-1\,|\,\mathcal{O}-3\rangle}{\mu_{2\mathcal{O}}\langle\mathcal{O}\mathcal{O}\,|\,\mathcal{O}\rangle}$$
$$\times\Bigg[\frac{Y^{\Delta_{\mathcal{O}-2}+\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-3}-1}-1}{\Delta_{\mathcal{O}-2}+\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-3}-1}\left(\mu_{2\mathcal{O}-4}-\frac{\mu^2_{2\mathcal{O}-2}\langle\mathcal{O}-1,\mathcal{O}-1\,|\,\mathcal{O}-2\rangle}{\mu_{2\mathcal{O}}\langle\mathcal{O}\mathcal{O}\,|\,\mathcal{O}\rangle(2\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-2}-2)}\right)$$
$$+\frac{\mu^2_{2\mathcal{O}-2}\langle\mathcal{O}-1,\mathcal{O}-1\,|\,\mathcal{O}-2\rangle}{\mu_{2\mathcal{O}}\langle\mathcal{O}\mathcal{O}\,|\,\mathcal{O}\rangle}\left(\frac{Y^{3\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-3}-2}-1}{(3\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-3}-2)(2\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-2}-1)}\right)\Bigg]\Bigg\}. \tag{5.10d}$$

Cases such as $2\Delta_{\mathcal{O}-1}-\Delta_{\mathcal{O}-2}-1=0$ give rise to $\ln Y$ terms and must be handled carefully. The system eventually reaches $\Gamma_2(Y)$ giving an implicit equation for Y; we estimate Y as in Sec. III

$$Y^{-1}(l=\infty)=(\mu_{2\mathcal{O}}/\mu^*_{2\mathcal{O}})(\Gamma_2^{-\lambda_{2\mathcal{O}}/\lambda_2}-1)+1 \tag{5.11}$$
$$(\mu^*_{2\mathcal{O}}\equiv\lambda_{2\mathcal{O}}/\langle\mathcal{O}\mathcal{O}\,|\,\mathcal{O}\rangle).$$

For $\lambda_{2\mathcal{O}}=0$,

$$Y^{-1}=1-(\mu_{2\mathcal{O}}/\lambda_2)\langle\mathcal{O}\mathcal{O}\,|\,\mathcal{O}\rangle\ln\Gamma_2. \tag{5.12}$$

For the case $\mathcal{O}=3$, we can write for the renormalized zero loop $A=\sum\Gamma_{2j}M^{2j}/2j\,!$, or

$$A=\frac{1}{2}\left[\mu_2-\frac{5(n+2)}{6(6-n)}\left(\frac{\mu_4^2}{\mu_6}\right)(Y^{(n-6)/(3n+22)}-1)\right]M^2$$
$$+\frac{1}{4!}\mu_4Y^{2(n+4)/(3n+22)}M^4+\frac{\mu_6}{6!}YM^6. \tag{5.13}$$

We have chosen the normalization $Q_{2j}\sim s^{2j}/(2j)!$ to allow a direct comparison with Ref. 22 with which it agrees exactly for $\lambda_{2\mathcal{O}}=\lambda_6=0$.

We note that the normalization of the fields μ_{2p} differs from Sec. IV where we considered $A_0(M)\sim O(\lambda^0_{2\mathcal{O}})$ and used the fact that $M^{2\mathcal{O}-2}\sim 0(\lambda^{-1}_{2\mathcal{O}})$. Here, since we wished to take the limit $\lambda_{2\mathcal{O}}\to 0$, no scale was assumed for the scaling fields except $\mu_{2p}\ll 1$.[37] To compare (5.13) with (4.13), note that $\mu_4^2\sim\lambda_{2\mathcal{O}}u^2$.

Coupled-order-parameter systems will generally require the solution of a nonlinear problem at the $s^{2\mathcal{O}}$ level and thus the introduction of several quadrature functions. Similarly, the situation below T_c will be complicated by the introduction of the two propagators $\Gamma_2(l)$ and $\Gamma_1(l)/M$. However, the overall structure of the equations and their solution will be essentially the same (cf. Sec. III).

The results in (5.10) were obtained for $\mathcal{O}\geq 3$. However, comparison with the $\mathcal{O}=2$ results show that they hold there as well. We also can extend the system to include "Γ_0"$=A$ if we include a contribution from the trajectory integral. For $\mathcal{O}=2$, this just means allowing $i=\mathcal{O}-1=1$ in (5.10c):

$$\Gamma_0=[t^2/2(u/u^*)](Y^{2\Delta_n-1}-1)/(2\Delta_n-1)\lambda_4$$
$$=-t^2(Y^{(n-4)/(n+8)}-1)/4\alpha(u/u^*). \tag{5.14}$$

For $\mathcal{O}\geq 3$, the trajectory integral contribution is

$$\Delta\Gamma_0\cong-\frac{\lambda_2}{\lambda_0(2\lambda_2-\lambda_0)}\{[t+g_1(Y)]^{\lambda_0/\lambda_2}-t^2\}$$
$$=-\frac{1}{\lambda_0}\,\frac{[t+g_1(Y)]^{2-\alpha}-t^2}{\alpha}. \tag{5.15}$$

The nontrajectory contributions commence for even $\mathcal{O}\geq 4$ and include only the case $i=j=\frac{1}{2}\mathcal{O}$.

We note that the projections used in (5.9) can be calculated with (1.4) or with the approximate Wilson-Kogut generator.[1] They differ by a factor depending only on $\mathcal{O}$ which cancels in the ratios. By the same method as in Ref. 1, we find (for $i+j=k+\mathcal{O}$)

$$\langle ij\,|k\rangle=\frac{i!\,j!\,2k!}{2i!\,2j!\,k!}\sum_m\binom{j+\frac{1}{2}n-1}{m}\binom{k}{j-m}\binom{2j-2m}{\mathcal{O}-2m}. \tag{5.16}$$

VI. HIGHER-ORDER CRITICAL POINTS: GREEN'S OPERATOR TRAJECTORIES

We can combine the Green's-function trajectories of Sec. III with the eigenfunction techniques of Sec. V to give fully exponentiated equations of state.

We begin with the Ising case for which a complete solution is possible. The free energy $A(M)$ can be expanded in a Taylor series around some magnetization M_0:

$$A(M)=\sum_{p=0}^{\infty}\Gamma_p(M_0)\frac{(M-M_0)^p}{p!}. \tag{6.1a}$$

We could now expand the logarithmic terms in the generator in powers of $M-M_0$ to recover the trajectory equations of Sec. III for the Green's functions $\Gamma_p(M_0)$. However, we could also rewrite (6.1) in terms of an eigenfunction expansion

$$A(M)=\sum_p\tilde{\Gamma}_p(M_0)\tilde{Q}_p. \tag{6.1b}$$

The $\tilde{\Gamma}_p$, which are linear combinations of the Γ_p, are what we will term Green's operators. The $\tilde{Q}_p$

are chosen to eliminate linear terms in the equations. In this section, it proves convenient to work entirely with unscaled variables so that M_0 is the (unscaled) physical magnetization. Hence the $\tilde{Q}_p$ have l dependence given by

$$\tilde{Q}_p(l, M-M_0) = [e^{(\lambda_2-\lambda_0)l}(1+\tilde{\Gamma}_2 e^{\lambda_2 l})]^{p/2} \times H_p[(M-M_0)e^{(\lambda_0-\lambda_2)l/2} \times (1+\tilde{\Gamma}_2 e^{\lambda_2 l})^{1/2}] , \quad (6.1c)$$

where H_p are the Hermite polynomials. The flow equations are given by the same prescription as in Sec. V:

$$\dot{\tilde{\Gamma}}_k = -\frac{e^{\lambda_2 \Theta l}}{(1+\tilde{\Gamma}_2 e^{\lambda_2 l})^{\Theta}} \langle p, q \,|\, k\rangle \tilde{\Gamma}_p \tilde{\Gamma}_q , \quad (6.2)$$

where the sum over p and q, $2 \leqslant p$, $q \leqslant 2, p+q=2Q+k$ is implied.[38] Of course, as $l\to\infty, \tilde{\Gamma}_p \to \Gamma_p$ (just as $Q_p \to (M-M_0)^p/p!$).

This system is again triangular in the sense of Section V and could be solved in the same manner. However, for Ising systems with spin-inversion symmetry, we can use the earlier results directly. We found

$$\Gamma_{2i}(M=0) = \gamma_{2i}(\{\mu_{2j}\}, Y) , \quad (6.3)$$

where the γ_{2i} are the solutions given by (5.9). We expect from the work of Sec. III that (6.2) leaves invariant the relationship

$$\tilde{\Gamma}_p(M) = \sum \gamma_{2i}(\{\mu_{2j}\}, Y)\frac{M^{2i-p}}{(2i-p)!} , \quad (6.4a)$$

where we have replaced M_0 by M and where Y is redefined as

$$\frac{\dot{Y}}{Y} = -\frac{\tilde{\Gamma}_{2\Theta}(M)e^{\lambda_{2\Theta} l}}{(1+\tilde{\Gamma}_2 e^{\lambda_2 l})^{\Theta}} \langle 2\Theta, 2\Theta \,|\, 2\Theta\rangle . \quad (6.4b)$$

The function Y will be estimated by

$$Y^{-1} = 1 + \frac{\mu_{2\Theta}}{\lambda_{2\Theta}} \langle 2\Theta, 2\Theta \,|\, 2\Theta\rangle (\Gamma_2^{-\lambda_{2\Theta}/\lambda_2} - 1) , \quad (6.5)$$

where, of course, Γ_2 is $\Gamma_2(M)$.

To show that this simple result holds, we insert (6.4a) into (6.2) to obtain

$$\sum_i \dot{\gamma}_{2i}\frac{M^{2i-p}}{(2i-p)!} = -\frac{e^{\lambda_{2\Theta} l}}{(1+\tilde{\Gamma}_2 e^{\lambda_{2\Theta} l})^{\Theta}} \sum_{s,t,j,k,p} \frac{\langle j, k \,|\, p\rangle}{(2s-j)!(2t-k)!} \times M^{2(s+t)-(j+k)}\gamma_{2s}\gamma_{2t} . \quad (6.6a)$$

We wish to show this is equivalent to

$$\dot{\gamma}_{2p} = \frac{-e^{\lambda_{2\Theta} l}}{(1+\tilde{\Gamma}_2 e^{\lambda_2 l})^{\Theta}} \langle 2s, 2t \,|\, 2p\rangle \gamma_{2s}\gamma_{2t} , \quad (6.6b)$$

This requires the following relation:

$$\frac{\langle 2s, 2t \,|\, 2i\rangle}{(2i-p)!} = \sum \frac{\langle j, k \,|\, p\rangle}{(2s-j)!(2t-k)!} , \quad (6.6c)$$

for $s+t = i+\Theta, j+k = p+2\Theta$. For Ising systems, we can take

$$\langle j, k \,|\, p\rangle = \binom{p}{k-\Theta} .$$

Equation (6.6c) can therefore be written

$$\binom{2i}{p} = \sum_k \binom{2i-2t+\Theta}{p-k+\Theta} \binom{2t-\Theta}{k-\Theta} .$$

Multiplying by z^p and summing over p, we have the desired result

$$(1+z)^{2i} = \sum_k z^{k-\Theta}(1+z)^{2i-2t+\Theta} \binom{2t-\Theta}{k-\Theta} = (1+z)^{2i} .$$

Thus, for the Ising case the results of Sec. V hold with (6.5) for Y. This was noted in the $\Theta = 3$ case for $\lambda_6 = 0$, k^2 propagators by Stephen *et al.*[22(b)]

The case of general n is more complex. Not only are there two propagators g_1 (transverse) and g_2 (longitudinal) but also, there are singularities from terms other then the $O(\lambda_{2\Theta})$ logarithms. For example, the tricritical case given in Sec. IV has a $(\Gamma_1/M)^{3/2}$ singularity at $O(\lambda_6^{1/2})$. This means that a careful order-by-order solution is required to incorporate successive unlinearized singularities in a complete solution.

However, it is of independent interest to consider how the operators may be constructed. One method which preserves the rotational symmetry is to expand A in powers of $x - x_0 \equiv M^2 - M_0^2$,

$$A = \sum_p R_{2p}(x_0)\frac{(x-x_0)^p}{p!} , \quad (6.7a)$$

and then rewrite this in terms of operators:

$$A = \sum_p \tilde{R}_{2p}\tilde{Q}_{2p}(l, x, x_0) . \quad (6.7b)$$

The $\tilde{Q}_{2p}$ are related to the local operators of Wilson and Wegner satisfying (in their scaled form)

$$\left(-x_0\frac{\partial}{\partial x_0} + \mathfrak{L}\right)\tilde{Q}_{2p} = \lambda_{2p}\tilde{Q}_{2p} , \quad (6.8)$$

where $\mathfrak{L}$ is the linear renormalization operator. If we choose the scale of l such that $\lambda_0 - \lambda_2 = 4$, then

$$\tilde{Q}_{2p}(l, x, x_0) = (g_2 e^{4l})^p F_p\left(\frac{xe^{4l}}{g_2}, \frac{x_0 e^{4l}}{g_2}\right) , \quad (6.8a)$$

$$F_p(x, x_0) = \sum L^{\tilde{\alpha}}_{p-j}(x)\frac{(-x_0)^j}{j!} , \quad (6.8b)$$

$$\tilde{\alpha} = \tfrac{1}{2}(n-1)(g_1/g_2) - \tfrac{1}{2} , \quad (6.8c)$$

where $L_p^{\tilde{\alpha}}$ is the Laguerre polynomial, is the appropriate set. It is then straightforward to show that the results of Sec. V can again be used [with (6.4) used for Y] in regions where the two propagators are roughly equal—as in the disordered phase—or where the transverse propagator is much smaller than the longitudinal propagator—near the coexistence surface ($\Theta \geq 3$).

VII. DISCUSSION

We have described a number of differential renormalization-group techniques for the calculation of the free energy and equation of state. These methods are of a sufficiently general character to be applicable to any critical system described by a Landau-Ginzberg-Wilson Hamiltonian. They differ among themselves in the type and amount of nonlinear renormalization information incorporated. In this context, we note that nonlinearity has two different aspects: (i) crossover information involving two or more fixed points; and (ii) exponentiation of logarithmic singularities (which is roughly equivalent to diagram resummation). We feel that the differential technique thereby offers an advantage over purely diagramatic perturbation theories. Just as the generator approach automatically evaluates the diagram weights and integrals necessary to evaluate critical-point exponents (as described in Ref. 1), it also includes the leading singular behavior of the diagrams to all orders in the relatively simple and compact nonlinear structure of the renormalization group trajectories. This is not to suggest that careful diagram resummation cannot reproduce the results given here; moreover, at higher order in exponentiated perturbation theory, the relative advantages of the differential method may diminish. However, the approach is simple and straightforward to apply; it is supported by a great body of knowledge about differential equations which will be increasingly useful as the salient features as identified in a diagram expansion are recognized in the differential formulation.

If we compare the different methods, the iteration method of Sec. IV is the most naive and the least nonlinear. It corresponds closely to the diagram expansion, each iteration representing the contribution of the next higher loop. It does not easily provide crossover or exponentiation information and is therefore confined to exploration of the critical regime of a single fixed point. Its simplicity has the virtue of neglecting no contribution to the free energy and for this reason it can be used to determine the general character of the singularities of the equation of state. It is a first stage in a calculation to provide a simple form with which the results of more sophisticated techniques (which may stress different aspects of the equation of state) may be compared.

The next level of nonlinearity is represented by the Hamiltonian parameter trajectory method of Sec. II. This familiar method contains, in principle, all the crossover information, but in interesting cases the nonlinear equations may be extremely difficult to solve for more than asymptotic information. In its favor, it is partially exponentiated (thus providing the $T > T_c$ crossover Green's functions) and is only slightly more difficult (within quadratures) than the iterative approach.

The most elaborate nonlinear technique is the Green's-function trajectories method of Sec. III. Its virtues are manifest since it exponentiates the order-parameter-dependent logs as well as the nonlinear scaling fields. We also obtain in a lowest-order calculation (such as given in Sec. III) an exponentiated form of the corresponding zero-loop and one-loop calculations of Sec. II or IV. However, the number of quadrature functions (e.g., Y_1 and Y_2) is increased and the nonlinear problem is generally made more difficult.

For higher-order (nonquartic) critical systems, the operator method used in Secs. IV–VI is almost a necessity. For Ising systems (or more generally in situations where a single propagator may be employed such as in the disordered phase of the isotropic n-component case), the method allows one to seek out, for example, the logarithms responsible for changing the critical-point exponents without examining the intervening terms.

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APPENDIX A: SOLUTION OF NONLINEAR RENORMALIZATION-GROUP TRAJECTORIES

Consider a set of three coupled nonlinear equations of the form

$$\begin{aligned} \dot{r} &= \lambda_2 r + (A_1 u_1 + A_2 u_2)/(1+r) , \\ \dot{u}_1 &= u_1[\lambda_4 - (B_1 u_1 + C_2 u_2)/(1+r)^{-2}] , \\ \dot{u}_2 &= u_2[\lambda_4 - (B_2 u_2 + C_1 u_1)/(1+r)^{-2}] . \end{aligned} \tag{A1}$$

Equations of this form arise in the compressible magnet and the hypercubical models discussed in Sec. II. It proves convenient to define[15]

TABLE I. Fixed point values y_1^*, y_2^*, z^* and associated eigenvalues for the four fixed points of the general coupled-order-parameter model, Eq. (A3).

Fixed point	y_1^*	y_2^*	z^*	Temperature eigenvalue	In plane eigenvalues	
(i)	1	0	$1-\delta_1$	$\lambda_2-\lambda_4\Delta_1$	$-\lambda_4$	$\lambda_4(1-\delta_1)$
(ii)	0	1	$-(1-\delta_2)$	$\lambda_2-\lambda_4\Delta_2$	$-\lambda_4$	$\lambda_4(1-\delta_2)$
(iii) z point	$\frac{1-\delta_2}{(1-\delta_1\delta_2)}$	$\frac{1-\delta_1}{(1-\delta_1\delta_2)}$	0	$\lambda_2-\lambda_4\Delta_z$	$-\lambda_4$	$-\lambda_4\frac{(1-\delta_1)(1-\delta_2)}{1-\delta_1\delta_2}$
(iv) Gaussian	0	0	0	λ_2	λ_4	λ_4

$$x=[r+(A_1u_1+A_2u_2)/(\lambda_2-\lambda_4)(1+r)] \;,$$
$$(\lambda_4/B_i)y_i=u_i/(1+r)^2, \quad (\lambda_4/B_i)\bar{u}_i=u_i \;. \tag{A2}$$

There are four fixed points in the $t=0$ critical plane. The eigenvalues at each fixed point are given in Table I. We define

$$\Delta_i \equiv A_i/B_i \;,$$
$$\delta_i \equiv C_i/B_i \;,$$
$$\Delta_z \equiv [\Delta_1(1-\delta_2)+\Delta_2(1-\delta_1)]/(1-\delta_1\delta_2) \;.$$

The critical-point exponent γ is given by, for example, $\gamma_i^{-1}=1-\lambda_4\Delta_i/\lambda_2$, at the $i=1,2$ fixed points. The values of the various parameters and the identification of the variables for the two models of Sec. II are given in Table II.

Proceeding as in Ref. 15, we define the functions Y_i by

$$y_i'=-\lambda_4 y_i Y_i \;. \tag{A3}$$

(For a discussion of the properties of the Y_i cf. Refs. 1 and 15.) Y_1 and Y_2 are not independent:

$$(1-\delta_1)y_1Y_2^{1-\delta_2}-(1-\delta_2)y_2Y_1^{1-\delta_1} = (1-\delta_1)y_1-(1-\delta_2)y_2=z \;. \tag{A4}$$

We now will assume provisionally that the Y_i are known. For the equation of state, we need to compute

$$[t_\infty, u_{i,\infty}, q_i] \equiv \lim_{l\to\infty}[t(l)e^{-\lambda_2 l}, u_i(l)e^{-\lambda_4 l}, Y_i(l)/Y_i(l=0)] \;. \tag{A5}$$

The functions q_iY_i are renormalization invariants as functions of t and u_i. For $\lambda_2>\lambda_4$ $(d_0>\sigma_>)$, $t_\infty=r_\infty$ of text. We have[7]

$$[t_\infty, u_{1,\infty}, u_{2,\infty}]=[tq_1^{\Delta_1}q_2^{\Delta_2}, u_1q_1q_2^{\delta_2}, u_2q_2q_1^{\delta_1}] \;. \tag{A6}$$

Here and elsewhere a subscript ∞ denotes the $l\to\infty$ limit; otherwise a variable or function has its initial $l=0$ value. From (A4), we get a relationship between q_1 and q_2:

$$(1-I)=(Y_2q_2)^{\delta_2-1}-I(q_1Y_1)^{\delta_1-1} \;, \tag{A7}$$

where I is a renormalization invariant

$$I\equiv(1-\delta_2)\bar{u}_2Y_2^{\delta_2-1}/(1-\delta_1)\bar{u}_1Y_1^{\delta_1-1} \;. \tag{A8}$$

In the decoupled case, $\delta_2=0$, we can write

$$(Y_2q_2)^{-1}=1-I[1-(q_1Y_1)^{\delta_1-1}] \;, \tag{A9}$$

and thus obtain q_2 in terms of q_1 directly,

$$q_2=\frac{\bar{z}+\bar{u}_2}{\bar{z}+\bar{u}_2q_1^{\delta_1-1}}=\frac{(1-\delta_1)\bar{u}_1}{(1-\delta_1)\bar{u}_1+\bar{u}_2(q_1^{\delta_1-1}-1)} \;, \tag{A10}$$

where $\bar{z}=(1-\delta_1)\bar{u}_1-\bar{u}_2$. In this decoupled case, we can give explicit solutions for the Y_i and hence for

TABLE II. Identification of the variables and values of the parameters of the general coupled-order-parameter model, Eq. (A3), when applied to the compressible magnet and the nm hypercubical model.

Model \ Parameter	u_1	u_2	$\frac{1}{2}A_1$	$\frac{1}{2}A_2$	$\frac{1}{2}B_1$	$\frac{1}{2}B_2$	$\frac{1}{2}C_1$	$\frac{1}{2}C_2$	Δ_1	Δ_2	δ_1	δ_2
Compressible magnet (2.6)	u	v	$m+2$	m	$m+8$	m	$2(m+2)$	0	$\frac{m+2}{m+8}$	1	$\frac{2(m+2)}{(m+8)}$	0
nm hypercubical (2.11)	c	u	$m+2$	$mn+2$	$m+8$	$nm+8$	$2(m+2)$	12	$\frac{m+2}{m+8}$	$\frac{nm+2}{nm+8}$	$\frac{2(m+2)}{(m+8)}$	$\frac{12}{nm+8}$

all the nonlinear scaling fields. This is the case of the compressible magnet model, for which in Ref. 1 a solution was found good in the region $y_2/y_1 < 1 - \delta_1$. A more complicated solution should be obtainable in the complimentary region. In the large-l limit, we find

$$Y_1 q_1 = [1 - (Y_1 q_1)^{1/\gamma_1} (Y_2 q_2)^{-\lambda_4 \Delta_2/\lambda_2}/DI_t] \times [1 + (\lambda_4/\lambda_2)\tilde{\Delta}_\infty (Y_1 q_1)^{1/\gamma_1} \times (Y_2 q_2)^{-\lambda_4 \Delta_2/\lambda_2}/DI_t] , \tag{A11}$$

where I_t is a renormalization invariant independent of I,

$$I_t = |t|^{\lambda_4/\lambda_2} Y_1^{1/\gamma_1} Y_2^{-\lambda_4 \Delta_2/\lambda_2} \bar{u}_1^{-1} , \tag{A12}$$

and where the factors D and $\tilde{\Delta}_\infty$ are given by

$$D = 1 + (\lambda_4/\lambda_2)(1 - 2\tilde{\Delta}_\infty) , \quad \tilde{\Delta}_\infty = \Delta_1 + \Delta_2[(1 - \delta_1)\bar{u}_1 - \bar{z} q_2]/\bar{u}_1 . \tag{A13}$$

Equation (A11) can be solved to any desired order recursively. However, many of the features of the solution can be extracted by inspection. For the case $\delta_1 > 1$, the (i) fixed point is stabler than the (iii) or z point and we expect it to dominate. As $t \to 0$, we expect q_1 goes to zero as some power of t. We find

$$q_2 \to (\bar{z} + \bar{u}_2)/\bar{z} , \quad \tilde{\Delta}_\infty \to \Delta_1 .$$

This is not essentially different from the pure-(i) fixed-point behavior $q_2 = 1$, $\tilde{\Delta}_\infty = \Delta_1$. [Note that it immediately shows that at $z = 0$, the presumption of (i)-point-dominated behavior cannot hold, as expected.] Furthermore, as $t \to 0$, $Y_1 \to 1 - y_1$ exactly and hence [since Y_2 is known from (A6)] we must have

$$Y_1 q_1 \to I_t^{\gamma_1} (Y_2 q_2)^{\gamma_1 \lambda_4 \Delta_2/\lambda_2} , \quad q_1 \to t^{\lambda_4 \gamma_1/\lambda_2} .$$

On the other hand, for $\delta_1 < 1$, we have an entirely different critical behavior, dominated by the z point (except at $y_2 = 0$). However, the leading order t dependence of q_1 is almost unchanged. Therefore, we can pick an initial guess for $Y_1 q_1$ which is the λ_4-expansion solution of (A11):

$$Y_1 q_1 = I_t/(1 + I_t) . \tag{A14}$$

This is the expression used in Sec. II. More precise forms can be obtained by inserting (A14) into (A11) and iterating. The weak dependence of I_t on Y_2 is dropped to this order in Sec. II.

In the one variable case, an exact though implicit solution can be obtained. If the second square bracket is dropped (and $D = 1$ for convenience), we have

$$(Y_1 q_1)^{-1} = 1 + t_\infty^{\lambda_4/\lambda_2} Y/\bar{u}_1 , \tag{A15}$$

with $Y_1 = 1 - \bar{u}_1$. This is the estimate used in Secs. III–V. The estimates are not as crude as they may appear to be at first glance. The following integral (for A a constant) can be done exactly:

$$\int_0^\infty \frac{e^{\lambda_{20} l}\, dl}{(1 + A e^{\lambda_2 l})^p} = \lambda_{20}^{-1} \left[pB\left(p - \frac{\lambda_{20}}{\lambda_2}, 1 + \frac{\lambda_{20}}{\lambda_2}\right) A^{-\lambda_{20}/\lambda_2} - {}_2F_1\left(p, \frac{\lambda_{20}}{\lambda_2}, \frac{\lambda_{20}}{\lambda_2 + 1}; -A\right)\right] . \tag{A16}$$

Dropping all but the constant term in the hypergeometric function and evaluating the beta function at $\lambda_{20} = 0$ gives (A15) with $Y = 1 - \bar{u}_1$.

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[34]This result is perhaps better than its derivation. The terms dropped in passing from (3.24) to (3.25) can be shown to be $O(\epsilon)$ smaller than terms kept. They contribute $\epsilon^2(x+1)^2 \ln^2(x+1/x+3)$ terms to the equation of state.

[35]The zero of A has been shifted so that $\partial A/\partial l = 0$ at $A=0$. The previous normalization (1.4) was chosen to emphasize the relationship with the shell-momentum integration. The normalization of ℓ is such that $\lambda_{2p} = \lambda_0 - 4p$.

[36]The function f is a solution of $\partial f/\partial z = \lambda_4 v^* z/(z+t)$ which is the propagator factor in this normalization. In (4.5), we used $f(z, 0)$.

[37]If we wished to assign a scale for $\lambda_{2\theta} > 0$ as in Sec. IV we would have $\mu_{2j} \sim O(\lambda_{2\theta}^{(j-1)/(\theta-1)})$. Thus the terms kept in (5.4) represent the $O(\lambda_{2\theta})$ corrections to each μ_{2p}.

[38]To conform with the convention used here, we should have written $\langle 2i, 2j | 2k \rangle$ in Sec. V. We omitted the 2's since, in context, no confusion should arise.