Stimulated emission and absorption in classical systems

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Electromagnetic radiation from a classical charge distribution interacts with an incident plane wave, and the interaction terms in the resulting Poynting vector are shown to give a net contribution to the total energy flux exactly in the forward-scattering direction. A simple relativistically valid expression for the interaction energy in the case of a single particle is derived, and we show that energy is conserved in Thomson scattering if radiation-reaction forces are included. A perturbation theory is developed for a particle having its own given unperturbed motion, and the perturbed interaction energy is shown to be proportional to the intensity of the incident wave. We apply this theory to a "free-electron laser" consisting of a relativistic electron in a linear accelerator, and show that stimulated emission or absorption could both occur. The result contradicts that obtained using the Einstein-coefficient method.

I. INTRODUCTION

This paper presents a contribution to the theory of classical stimulated emission and absorption, in which a plane electromagnetic wave is scattered by a relativistic charged particle (or general charge distribution). The Poynting vector of the resulting radiation is composed of the Poynting vector of the incident wave plus the Poynting vector of the radiation from the charge, plus interaction cross terms. In the far-field approximation, the interaction terms are shown to give a contribution to the total energy flux exactly in the forward-scattering direction "downstream" from the charge, and a simple relativistically valid expression for the interaction energy H' is derived. This result is quite similar to the "optical theorem" in quantum-mechanical scattering theory. A satisfying interpretation of H' is given in terms of an energy-conservation law.

In Sec. III, the concept is applied in a simple way to Thomson scattering, and it is shown in this context how the scattered energy is taken from the wave, the loss showing up in the forward direction from the particle. Radiation reaction forces must be included here, however. Section IV develops a perturbation approach in which the particle is presumed to have a basic unperturbed motion of its own. The incident wave perturbs this motion, and the perturbed interaction energy $\delta H'$, which can be either positive (stimulated emission) or negative (absorption), is shown to be proportional to the intensity of the incident wave. The interpretation is that the particle emits or absorbs photons of the same wave vector as the incident wave, at a rate proportional to the wave intensity, as is characteristic of a stimulated process.

It might be possible to use this formalism to design "free-electron lasers," and in Sec. V, we apply the theory to a particle whose unperturbed motion is in one dimension, collinear with the propagation direction of the incident wave. After taking a phase average, we show that the particle usually gains kinetic energy, corresponding to absorption. However, it loses potential energy, so that stimulated emission might still be possible.

In Sec. VI, we apply the theory to a uniform linear accelerator, assuming that the particle is highly relativistic, that its unperturbed direction of motion is nearly (but not exactly) coincident with the wave-propagation direction, and that the frequency of the wave is not too high. After averaging over the phase of the wave, we obtain an expression for the dimensionless absorption coefficient ("optical thickness") of the accelerator. This coefficient can be either positive or negative, corresponding to absorption or stimulated emission, but the form of the result is not the same as one obtained previously by the Einstein-coefficient method, and reasons for this contradiction are discussed. We suggest that the principle of detailed balance may not apply in this case.

II. INTERACTION ENERGY

Let us consider a charge distribution emitting classical electromagnetic radiation. Let R be the distance from a point near the center of the distribution (taken to be the origin). Then if R is large, we write the electric field from the radiating distribution as $\vec{E}_r = \vec{a}(\vec{R}, t)/R$, where $\vec{R}$ is the vector from the origin to the field point. With $\vec{n} \equiv \vec{R}/R$, the magnetic field is $\vec{B}_r = \vec{n} \times \vec{E}_r$. If we consider a single particle, of charge e, then of course[1]

$$\vec{a}(\vec{R}, t) = e\vec{n} \times \{(\vec{n} - \vec{\beta}) \times \dot{\vec{\beta}}/[c(1 - \vec{n}\cdot\vec{\beta})^3]\}_{\text{ret}}, \quad (1)$$

where $\vec{\beta}(t)$ is the velocity of the particle in units of the velocity of light c, and the dot indicates the time derivate. The subscript ret denotes evaluation at the retarded time $t' = t - |\vec{R} - \vec{r}(t)|/c$, where $\vec{r}(t)$ is the position of the particle as a function of time.

Let us also suppose that there is an incident linearly polarized electromagnetic wave of frequency ω and wave vector $\vec{k}$. We assume that the process takes place in a vacuum, so that $\omega = ck$. We write the incident electric field as

$$\vec{E}_i = \vec{\epsilon} E_i \cos(\vec{k} \cdot \vec{R} - \omega t - \phi),$$

where $\vec{\epsilon}$ is a unit polarization vector. Likewise, $\vec{B}_i = \hat{k} \times \vec{E}_i$, where $\hat{k} \equiv \vec{k}/k$. ϕ is an arbitrary phase angle.

We now consider the total Poynting vector $\vec{S}$ for the radiation field:

$$\begin{aligned}\vec{S} &= c\vec{E} \times \vec{B}/4\pi \\ &= c(\vec{E}_i + \vec{E}_r) \times (\vec{B}_i + \vec{B}_r)/4\pi \\ &= c(\vec{E}_i \times \vec{B}_i + \vec{E}_r \times \vec{B}_r + \vec{E}_i \times \vec{B}_r + \vec{E}_r \times \vec{B}_i)/4\pi \\ &= \vec{S}_i + \vec{S}_r + \vec{S}'. \end{aligned} \tag{2}$$

The first term $\vec{S}_i$ is simply the Poynting vector of the incident wave by itself; and the second $\vec{S}_r$ is the Poynting vector of the energy radiated by the particle (or charge distribution) by itself. We concentrate our efforts on the last term,

$$\vec{S}' \equiv c(\vec{E}_i \times \vec{B}_r + \vec{E}_r \times \vec{B}_i)/4\pi,$$

which represents an interaction between the two radiation fields. In a manner analogous to the treatment by Sargent *et al.*[2] and by Born and Wolf,[3] we integrate it over a sphere of radius R centered on the origin. This gives us a total power output through the sphere

$$P' = \int\int \vec{S}' \cdot d\vec{\sigma} = R^2 \int\int_{4\pi} \vec{S}' \cdot \vec{n}\, d\Omega,$$

where $d\vec{\sigma}$ is the differential surface area and $d\Omega$ the corresponding solid angle. Of course, unless there is some correlation between the motion of the charge distribution and the incident wave, P' averages to zero. We consider applications where such correlations exist later in the paper.

It is now shown that, for any $\vec{a}(\vec{R}, t)$, a contribution to P' occurs only in the forward-scattering direction ($\vec{n} = \hat{k}$), in the limit $kR \to \infty$. The following proof is a generalization of the treatment by Born and Wolf,[3] who assumed $E_r \propto e^{ikR}$ and derived a result very similar to the optical theorem.

Let θ be the angle between $\vec{n}$ and $\hat{k}$, and ψ the usual azimuthal polar angle. Then it follows from the above expressions for the radiation fields that

$$\begin{aligned}\vec{S}' \cdot \vec{n} = cE_i[(1+\cos\theta)\vec{\epsilon} \cdot \vec{a} - (\vec{n} \cdot \vec{\epsilon})(\hat{k} \cdot \vec{a})] \\ \times \cos(kR\cos\theta - \omega t - \phi)/4\pi R, \end{aligned} \tag{3}$$

where we have used the identity $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$.

Then defining $\mu \equiv \cos\theta$ and

$$f[\mu] \equiv \int_0^{2\pi} d\psi[(1+\mu)\vec{\epsilon} \cdot \vec{a} - (\vec{n} \cdot \vec{\epsilon})(\hat{k} \cdot \vec{a})]/4\pi, \tag{4}$$

we obtain

$$\begin{aligned}P' &= R^2 \int\int \vec{S}' \cdot \vec{n}\, d\Omega \\ &= cE_i R \int_{-1}^{+1} f[\mu] \cos(kR\mu - \omega t - \phi)\, d\mu. \end{aligned} \tag{5}$$

In evaluating Eq. (5) in the far-field limit $R \to \infty$, we may use the integration-by-parts relation

$$\int_\alpha^\beta e^{i\chi\mu} F(\mu)\, d\mu = (i\chi)^{-1}[F(\beta)e^{i\chi\beta} - F(\alpha)e^{i\chi\alpha}] - (i\chi)^{-1} \int_\alpha^\beta e^{i\chi\mu} \frac{dF}{d\mu}\, d\mu, \tag{6}$$

where we set $\chi \equiv kR$ and $F(\mu) \equiv e^{-i(\omega t + \phi)} f[\mu]$.

Riemann's lemma states that

$$\lim_{\chi \to \infty} \int_\alpha^\beta e^{i\chi\mu} \frac{dF}{d\mu}\, d\mu = 0,$$

and therefore in the far-field limit Eq. (6) becomes, to first order in χ^{-1},

$$\int_\alpha^\beta e^{i\chi\mu} F(\mu)\, d\mu = (i\chi)^{-1}[F(\beta)e^{i\chi\beta} - F(\alpha)e^{i\chi\alpha}].$$

Then we may take the real part of this equation and set $\beta = +1$ and $\alpha = -1$ to obtain, for $R \to \infty$,

$$P' = cE_i R[\sin(kR\mu - \omega t - \phi) f[\mu](kR)^{-1}]_{-1}^{+1}. \tag{7}$$

Now, $\mu = +1$ means $\vec{R} = R\hat{k}$ and $\vec{n} = \hat{k}$, and therefore $f[+1] = \vec{\epsilon} \cdot \vec{a}(R\hat{k}, t)$; and at $\mu = -1, \vec{n} = -\hat{k}$, and hence $f[-1] = 0$. The integral over ψ is trivial because of the degeneracy at $\theta = 0$. Finally, we have for the interaction power in the far-field limit

$$P' = cE_i k^{-1} \sin(kR - \omega t - \phi)\vec{\epsilon} \cdot \vec{a}(R\hat{k}, t). \tag{8}$$

We see that the contribution to P' occurs only in the forward direction. The reader should compare this equation with the corresponding one in Born and Wolf[3] [their Eq. (109)], where exponential notation is used. Unfortunately, it is not advantageous to use exponential notation in the present work, because of the relativistic nonlinearities which occur. If we evaluate Eq. (3) in the forward direction, we note a 90° phase shift (sine vs cosine) between $\vec{S}' \cdot \vec{n} = \vec{S}' \cdot \hat{k}$ and P'. The difference arises, of course, because of the integration by parts.

We can now examine Eq. (5) to show why P'

depends only on $f[+1]$, and not on the entire structure of $f[\mu]$ over all μ. Consider a detector of fixed solid angle $\Delta\Omega \approx (kR)^{-1}$. In the neighborhood of $\theta = 0$, the argument of the cosine in Eq. (5) is then approximately $kR - \omega t - \frac{1}{2}kR\theta^2 - \phi$, and the detector would see an appreciable flux only for $\theta \lesssim (kR)^{-1/2}$, since otherwise the cosine would oscillate very rapidly over the detector face, as a function of θ. This is basically what Sargent *et al.*[2] conclude in their discussion of a dipole oscillator. We note that there would be a similar contribution near $\theta = \pi$ (backscattering), but $f[-1] = 0$ identically, as we have already seen. Of course, this is only part of the story, since for the detector to see a net flux, the time average of P' must also be nonzero. This depends on whether or not the motion of the particle is correlated with that of the wave, as occurs in the cases discussed below.

The analysis of Thomson scattering in Sec. III illustrates most of the points covered in the above few paragraphs.

For the most part, we discuss applications to a single particle, with $\vec{a}$ given by Eq. (1), and to that end it is convenient to integrate Eq. (8) over time to obtain a total interaction energy

$$H' = \int_{t_1}^{t_2} P'\,dt$$
$$= \frac{eE_i}{k}\int_{t_1}^{t_2}\sin(kR - \omega t - \phi) \times \vec{\epsilon}\cdot\{\hat{k}\times[(\hat{k}-\vec{\beta})\times\dot{\vec{\beta}}]\,\kappa^{-3}\}_{\text{ret}}\,dt\,, \qquad (9)$$

where $\kappa \equiv 1 - \hat{k}\cdot\vec{\beta}$.

This expression can be simplified by an integration by parts,[1] if the variable of integration is changed to the retarded time t' via $dt' = dt/\kappa$. This involves the usual far-field limit expression $t = t' + R/c - \hat{k}\cdot\vec{r}/c$, which we also substitute into the argument of the sine function. The relation $\vec{\epsilon}\cdot\hat{k} = 0$ implies $\vec{\epsilon}\cdot[\hat{k}\times(\hat{k}\times\vec{\beta})] = -\vec{\epsilon}\cdot\vec{\beta}$, and therefore one can show that

$$\frac{d}{dt'}(\kappa^{-1}\vec{\epsilon}\cdot\vec{\beta}) = -\frac{d}{dt'}\{\kappa^{-1}\vec{\epsilon}\cdot[\hat{k}\times(\hat{k}\times\vec{\beta})]\}$$
$$= -\vec{\epsilon}\cdot\{\hat{k}\times[(\hat{k}-\vec{\beta})\times\dot{\vec{\beta}}]\kappa^{-1}\}_{\text{ret}}\,,$$

and the integration by parts of Eq. (9) yields

$$H' \doteq -eE_i\,k^{-1}[\sin(\vec{k}\cdot\vec{r} - \omega t - \phi)\vec{\epsilon}\cdot\vec{\beta}\kappa^{-1}]_{t_1}^{t_2} - ecE_i\int_{t_1}^{t_2}\cos(\vec{k}\cdot\vec{r} - \omega t - \phi)\vec{\epsilon}\cdot\vec{\beta}\,dt\,, \qquad (10)$$

where we have removed the primes from the variable of integration, and $\vec{r}(t)$ and $\vec{\beta}(t)$ are the position and velocity of the particle as a function of the (retarded) time. The boundary terms have been retained since it may be desirable to consider problems in which the acceleration is discontinuous, as in Secs. IV and V.

Equation (10) is quite remarkable in that if we define a "potential energy" $V \equiv eE_i\sin(\vec{k}\cdot\vec{r} - \omega t - \phi) \times\vec{\epsilon}\cdot\vec{\beta}/\kappa k$, we can write

$$H' = -e\int_{t_1}^{t_2}\vec{E}_i\cdot\vec{v}\,dt - [V(t_2) - V(t_1)]\,, \qquad (11)$$

where the integral term is the work done on the particle by the incident wave, and is obviously the change in the particle's kinetic energy due to the presence of the wave. Equation (11) is therefore an energy-conservation law, wherein the radiated interaction energy H' is balanced by the change in the particle's kinetic and potential energies.

It is interesting that such a law exists, for it is well known that conserved energies do not, in general, exist for systems in time-dependent external fields. It does not seem possible to use the ordinary energy-conservation laws of classical electrodynamics in the present case because of the divergences which occur in the neighborhood of a point charge.

This velocity-dependent "potential-energy" term is very similar in form to the spectral energy-distribution function for photons emitted in the radiative beta process (Ref. 1, p. 526). This radiation is associated with the sudden appearance of the electron or positron created by an unstable nucleus during beta decay, and our potential energy term can be viewed as a stimulated analog to this radiative beta process. However, we shall continue to refer to it simply as the "potential energy."

Considerable care should be exercised in interpreting this term, and it might be more appropriate in many cases to delete it and to take the limits on the kinetic-energy integral to be $\pm\infty$. A similar question of choice arises in the interpretation of the usual formula for the "spontaneous" emission from an accelerated charge, where a similar integration by parts has been employed (Ref. 1, p. 671).

Note that a system of many particles could be treated by the method developed here, since the radiation fields are obviously summable. Equations (10) and (11) would then consist of sums over the individual particles.

The fact that H' is an integral of cross terms of $\vec{S}$ composed of the two radiation fields $\vec{E}_i$ and $\vec{E}_r$ indicates the following: Let $\Delta t \equiv t_2 - t_1$. If $\langle P'\rangle \equiv \lim_{\Delta t\to\infty} H'/\Delta t$ is nonzero, then the particle must be emitting or absorbing photons of the same $\vec{k}$ as that of the incident wave. If it were not, $\vec{E}_i$ and $\vec{E}_r$ would be uncorrelated, and $\langle P'\rangle$ would vanish. If the incident radiation field consists of an ensemble of randomly phased photons, then $\langle P'\rangle$ would vanish anyway, unless the motion of the particle were correlated with the photon phases.

III. ENERGY CONSERVATION IN THOMSON SCATTERING

It may be useful at this time to consider an interesting and simple use of Eq. (10), in which we show energy conservation in nonrelativistic Thomson scattering, in the sense that $\langle P'\rangle = \langle \iint \vec{S}_r \cdot d\vec{\sigma}\rangle$, the latter integral being the usual Thomson-scattered power, provided that we include the radiation reaction term in the equation of motion. In other words, it will appear that the Thomson-scattered power has indeed been "stolen" from the incident wave, the loss showing up in the forward direction.

The use of the radiation reaction term in a nonrelativistic situation is not contradictory, and it can be shown in the limit of small E_i, the relativistic corrections to the motion of the particle go to zero faster than the reaction term.

The nonrelativistic radiation reaction force term[4] is $\vec{f}_{\rm rad} \approx mc\tau\ddot{\vec{\beta}}$, where $\tau \equiv 2e^2/3mc^3 \approx 6.26 \times 10^{-24}$ sec, and the complete nonrelativistic equation of motion of the electron in the incident wave is

$$m\dot{\vec{\beta}} = eE_i\vec{\epsilon}\cos(\vec{k}\cdot\vec{r} - \omega t - \phi)/c + m\tau\ddot{\vec{\beta}}\,. \qquad (12)$$

Since $\vec{k}\cdot\vec{\epsilon} = 0$, the appropriate solution of Eq. (12) is, to first order in the small quantity[5] $\tau\omega$,

$$\vec{\beta} = eE_i\vec{\epsilon}[\sin(\omega t + \phi) + \tau\omega\cos(\omega t + \phi)]/mc\omega\,. \qquad (13)$$

Since the phase of the wave is now obviously correlated with $\vec{\beta}$, we set $\phi = 0$. Instead of doing our analysis directly by using Eq. (10), it is instructive to go to Eq. (4) and evaluate P' directly without making use of the far-field integration by parts. In the nonrelativistic (dipole) case,[2] we may approximate

$$\vec{a} = (e/c)\vec{n}\times(\vec{n}\times[\dot{\vec{\beta}}]_{\rm rad})$$
$$= (e^2/mc^2)E_i\vec{n}\times(\vec{n}\times\vec{\epsilon})(\cos\omega t' - \tau\omega\sin\omega t')\,,$$

where $t' \approx t - R/c$ is the retarded time, and Eq. (4) becomes

$$f[\mu] = \frac{e^2E_i}{4\pi mc^2}(\cos\omega t' - \tau\omega\sin\omega t')$$
$$\times\int_0^{2\pi} d\psi\,[(\vec{n}\cdot\vec{\epsilon})^2 - 1 - \vec{n}\cdot\hat{k}]\,.$$

To evaluate the scalar products, define the coordinate system so that $\vec{\epsilon} = (0, 1, 0)$, $\hat{k} = (1, 0, 0)$, and $\vec{n} = (\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi)$. Then one can show that

$$f[\mu] = -e^2E_i(\mu + 1)^2(\cos\omega t' - \tau\omega\sin\omega t')/4mc^2\,.$$

This expression may be substituted into Eq. (5) and the integral over μ may be done exactly. The reader can easily show that the result is

$$P' = e^2E_i^2(m\omega)^{-1}(\cos\omega t' - \tau\omega\sin\omega t')$$
$$\times[\sin\omega t' - (kR)^{-1}\cos\omega t' + (kR)^{-2}\cos\omega t\sin kR]\,,$$

which becomes, in the far-field limit $R\to\infty$,

$$P' = e^2E_i^2(m\omega)^{-1}(\cos\omega t' - \tau\omega\sin\omega t')\sin\omega t'\,. \qquad (14)$$

Taking a time average of Eq. (14), we see that only the second term in the parentheses contributes, so that

$$\langle P'\rangle \equiv \lim_{\Delta t\to\infty}\int_{t_1}^{t_2} P'\frac{dt}{\Delta t} = -e^4E_i^2/3m^2c^3\,. \qquad (15)$$

But, it is well known[1] that the average Thomson-scattered power is, with $\sigma_T \equiv 8\pi(e^2/mc^2)^2/3$,

$$\langle P_T\rangle = \sigma_T cE_i^2/8\pi = -\langle P'\rangle\,.$$

Since $P_T = \iint \vec{S}_r\cdot d\vec{\sigma}$, we see that our formalism conserves energy in Thomson scattering, as advertised.

We now take a quick look at Eq. (10), and we verify that the same result is obtained. Remembering that the variable of integration in Eq. (10) is actually the retarded time, we substitute Eq. (13) with $\phi = 0$ to get

$$H' = e^2E_i^2(m\omega^2)^{-1}$$
$$\times\left([\sin\omega t(\sin\omega t + \tau\omega\cos\omega t)]_{t_1}^{t_2}\right.$$
$$\left. - \omega\int_{t_1}^{t_2}\cos\omega t(\sin\omega t + \tau\omega\cos\omega t)dt\right).$$

Since $\langle P'\rangle \equiv \lim_{\Delta t\to\infty} H'/\Delta t$, only the second term in the integral contributes, and one can easily show that the result is the same as Eq. (15). Note that the potential energy term of Eq. (11) averages to zero, so that only the kinetic-energy term contributes to the scattering in this case.

IV. PERTURBATION THEORY AND STIMULATED EMISSION

We now explore the connection of Eq. (10) with stimulated emission; to do this, we consider the particle to have a basic unperturbed motion $\vec{r}_0(t)$ of its own, such as acceleration in a uniform electric field (Sec. VI), or synchrotron motion. The action of the incident wave is then considered to be a perturbation. This is completely analogous to the situation in quantum mechanics, where stimulated atomic transition probabilities are computed by perturbing the state of an atom (the "unperturbed motion") with the wave.

Let the unperturbed force be given as a function of position and velocity by

$$\vec{f}_0(\vec{r}, \vec{\beta}, t) = e[\vec{E}_0(\vec{r}, t) + \vec{\beta}\times\vec{B}_0(\vec{r}, t)]\,,$$

so that the unperturbed equation of motion is $\dot{\vec{P}}_0 = \vec{f}_0(\vec{r}_0, \vec{\beta}_0, t)$. The exact question of motion is

$$\dot{\vec{P}} = \vec{f}_0(\vec{r}, \vec{\beta}, t) + eE_i[\vec{\epsilon} + \vec{\beta}\times(\hat{k}\times\vec{\epsilon})] \times\cos(\vec{k}\cdot\vec{r} - \omega t - \phi), \tag{16}$$

and if we let $\delta\vec{P} \equiv \vec{P} - \vec{P}_0$, and $\delta\vec{r} \equiv \vec{r} - \vec{r}_0$, the perturbed part of Eq. (16) is, to first order in the δ's and in E_i,

$$\delta\dot{\vec{P}} = (\delta\vec{r}\cdot\vec{\nabla}_r + \delta\vec{\beta}\cdot\vec{\nabla}_\beta)\vec{f}_0(\vec{r}_0, \vec{\beta}_0, t) + eE_i[\vec{\epsilon} + \vec{\beta}_0\times(\hat{k}\times\vec{\epsilon})]\cos\phi_0(t), \tag{17}$$

where $\phi_0(t) \equiv \vec{k}\cdot\vec{r}_0(t) - \omega t - \phi$.

Note that we have omitted any mention of the radiation reaction force, which was necessary in the Thomson-scattering calculation in the previous section. Of course, the reaction force accompanying the unperturbed motion could easily be included in the unperturbed force $\vec{f}_0$. There is also a reaction force which accompanies acceleration due to the perturbing incident wave. However, we assume that[5] $\tau\omega \ll 1$, and therefore this part of the reaction force is small compared to eE_i, the perturbation. This "incident radiation" reaction force assumes importance in the Thomson scattering case only because of the vanishing of the main contribution to $\langle P'\rangle$, which showed promise of being order $e^2E_i^2/m\omega$ [see Eq. (14)].

To solve Eq. (17) and substitute into a perturbed form of Eq. (10), it is necessary to have $\delta\vec{\beta}$ in terms of $\delta\vec{P}$. This may easily be done by noting that $\vec{P} = mc\gamma\vec{\beta}$, where m is the rest mass, and the Lorentz factor γ is $(1 + P^2m^{-2}c^{-2})^{1/2}$. Linearizing yields, with $\hat{P}_0 \equiv \vec{P}_0/|\vec{P}_0|$,

$$\delta\vec{\beta} = (mc\gamma_0)^{-1}[\delta\vec{P} - \hat{P}_0\beta_0^2(\hat{P}_0\cdot\delta\vec{P})]. \tag{18}$$

We must now linearize Eq. (10), realizing that the end-point times t_j may be perturbed also. We obtain, writing $\delta H' \equiv H' - H_0'$ and $\delta H' \equiv \sum_{\nu=1}^{6}\delta H_\nu$,

$$\delta H_1 = -ecE_i\int_{t_1}^{t_2}\vec{\epsilon}\cdot\delta\vec{\beta}\cos\phi_0(t)\,dt\ , \tag{19a}$$

$$\delta H_2 = ecE_i\int_{t_1}^{t_2}\vec{\epsilon}\cdot\vec{\beta}_0\sin\phi_0(t)\vec{k}\cdot\delta\vec{r}\,dt\,, \tag{19b}$$

$$\delta H_3 = -eE_ik^{-1}[\sin\phi_0(t)(\vec{\epsilon}\cdot\vec{\beta}_0)\kappa_0^{-2}(\hat{k}\cdot\delta\vec{\beta})]_{t_1}^{t_2}, \tag{19c}$$

$$\delta H_4 = -eE_ik^{-1}[\cos\phi_0(t)(\vec{\epsilon}\cdot\vec{\beta}_0)\kappa_0^{-1}(\vec{k}\cdot\delta\vec{r})]_{t_1}^{t_2}, \tag{19d}$$

$$\delta H_5 = -eE_ik^{-1}[\sin\phi_0(t)\kappa_0^{-1}\vec{\epsilon}\cdot\delta\vec{\beta}]_{t_1}^{t_2}, \tag{19e}$$

$$\delta H_6 = -eE_ik^{-1}\sum_{j=1}^{2}(-1)^j\{\sin\phi_0(t)\vec{\epsilon}\cdot[\dot{\vec{\beta}}_0\kappa_0^{-1} + \vec{\beta}_0\kappa_0^{-2}(\hat{k}\cdot\dot{\vec{\beta}}_0)]\}_{t=t_j}\delta t_j\,. \tag{19f}$$

δH_6, the part arising from the end-point time variation, is given by $\delta H_6 = (\partial H_0'/\partial t_1)\delta t_1 + (\partial H_0'/\partial t_2)\delta t_2$. It has been simplified by using the relation $\dot{\phi} = \vec{k}\cdot\dot{\vec{r}} - \omega = -\omega\kappa_0$.

The procedure, then, is to solve Eq. (17) for $\delta\vec{\beta}$, with the help of Eq. (18), and then evaluate the δH_ν. Since (17) is linear and inhomogeneous, the appropriate solution is linear in E_i, so that $\delta H'$ is proportional to E_i^2, the incident wave intensity. Therefore, the energy gained or lost by the wave is proportional to the wave intensity and is exactly in the forward direction from the particle. The same remarks would apply to a perturbed version of Eq. (8) for an arbitrary charge distribution, with, of course, its appropriate perturbed equation of motion. If the incident wave field consists of a statistical ensemble of randomly phased waves, then Eqs. (19) should be averaged over ϕ. Indeed, the concept of "optical thickness" or of a gain coefficient normally carries with it the assumption of phase independence. Correspondingly, in the rest of the paper a phase average is always taken.

It is obviously tempting to think of using this perturbation theory to design classical masers, as in Sec. VI, which treats an electron in a linear accelerator.

V. ONE-DIMENSIONAL FREE-ELECTRON LASER

As a simple example of the perturbation theory of Sec. IV, we treat the case in which the unperturbed motion of the particle is one-dimensional, and in which the wave vector $\vec{k}$ of the incident field is aligned with this motion. We choose the x axis to lie in this direction, so that $\hat{k} = \hat{x}$, and $\vec{\beta}_0(t) = \hat{x}\beta_0(t)$. We also assume that the unperturbed force f_0 depends only on x.

Under these conditions, Eq. (17) is

$$\delta\dot{\vec{P}} = \delta x f_0'(x)\hat{x} + eE_i(1 - \beta_0)\vec{\epsilon}\cos\phi_0\,. \tag{20}$$

Note that δx decouples from the rest of the problem, and we may set $\delta x = 0 = \delta P_x$, and $\delta\vec{P} = \delta P\vec{\epsilon}$. Equation (20) may be integrated exactly, since $1 - \beta_0 = -\omega^{-1}d\phi_0/dt$, and we find $\delta\dot{P} = -eE_i\omega^{-1}d(\sin\phi_0)/dt$, or

$$\delta P = -eE_i\omega^{-1}[\sin\phi_0(t) - \sin\phi_0(t_1)] = mc\gamma_0\delta\beta\,. \tag{21}$$

Since $\delta\vec{\beta}$ and $\delta\vec{r}$ are perpendicular to $\vec{k}$ and $\vec{\beta}_0$, the only δH_ν [Eq. (19)] which do not vanish are δH_1 and δH_5. Averaging over ϕ, with

$$\langle f\rangle \equiv \int_0^{2\pi} f(\phi)\,d\phi/2\pi$$

and

$$\phi(t, t_1) \equiv k[x_0(t) - x_0(t_1)] - \omega(t - t_1)\,,$$

one obtains

$$\langle\delta H_1\rangle = \frac{e^2E_i^2}{2m\omega}\int_{t_1}^{t_2}\gamma_0(t)^{-1}\sin\phi(t, t_1)\,dt \tag{22}$$

and

$$\langle \delta H_5 \rangle = e^2 E_i^2 [1 - \cos\phi(t_2, t_1)]/[2m\omega^2 \gamma_0(t_2)\kappa_0(t_2)] \, . \tag{23}$$

$\langle \delta H_1 \rangle$ is, in this case, the negative of the total work done on the particle by the wave, since the unperturbed part of Eq. (10) vanishes when the ϕ average is taken. Note that since $\phi(t, t_1) \le 0$, $\langle \delta H_1 \rangle$ is generally negative. Therefore, the particle usually gains energy from the wave in our one-dimensional case.

However, $\langle \delta H_5 \rangle$, the potential-energy term, is never negative, and therefore there is always some energy gained back by the radiation field. It may be that overall conservation of energy is maintained by work done injecting and removing the particle at the boundaries. Presumably, this is where the energy for the photons emitted in the ordinary radiative beta process comes from.

VI. RELATIVISTIC LINEAR ACCELERATOR

We now apply the perturbation theory developed in Sec. IV to an electron in a linear accelerator. We assume that the electron is everywhere extremely relativistic ($\gamma \gg 1$) and that the acceleration direction, taken to be along the x axis, is nearly coincident with $\hat{k}$, so that the change of wave phase experienced by the electron is small $|\Delta\phi_0(t)| \ll 1$. This problem has been treated by Cocke[6] by the Einstein-coefficient method,[7] and it is interesting that the perturbation method gives a very different result.

In this calculation, we completely neglect the effects of radiation reaction, since this force is very small for one-dimensional motion. In fact, for a uniform electric field E, it can be shown that radiation reaction changes the electron velocity by $\Delta\beta \approx -2r_0/3a\gamma_0^2$, where $r_0 = e^2/mc^2 \approx 2.82 \times 10^{-13}$ cm and $a \equiv mc^2/eE$. a is a length characteristic of the accelerator, and is the distance over which the electron changes its energy by mc^2. For any laboratory or cosmic accelerator[6] surely $r_0/a < 10^{-12}$, and since here we assume $\gamma_0 \gg 1$, we can certainly neglect the effect of including $\Delta\beta$ in computing β_0, ϕ_0, or κ_0 in Eqs. (19).

The basic dynamics of the linear accelerator are well known, and it is necessary only to establish some notation. As in Ref. 6, we let $\zeta \equiv ct/a$, and $\tilde{\omega} \equiv a\omega/c$, where a is defined above. $\tilde{\omega}$ is the frequency measured in units of the electron-acceleration time scale. To allow for deceleration as well as acceleration, we note that a can be either positive or negative. The electron's unperturbed position is given by $x_0(t) = a(\zeta^2+1)^{1/2}$, and its velocity is $\beta_0 = \zeta(\zeta^2+1)^{-1/2}$. The Lorentz factor is then $\gamma_0 = x_0/a = (\zeta^2+1)^{1/2}$, which means that we are interested in t such that $|\zeta| \gg 1$. We can allow for deceleration if we always let $\zeta \approx \gamma_0$ be positive and $t_2 > t_1$, but for deceleration $t_2 < 0$. We see that the electron velocity is always positive, but for deceleration, the motion is along the negative x axis and toward the origin.

Since the unperturbed field is constant, Eq. (17) becomes

$$\delta\dot{\vec{P}} = eE_i[\vec{\epsilon} + \vec{\beta}_0 \times (\hat{k} \times \vec{\epsilon})]\cos(\vec{k}\cdot\vec{r}_0 - \omega t - \phi) \, , \tag{24}$$

and we let $\vec{k}\cdot\vec{r}_0 = kx_0\cos\theta$, where $|\theta| \ll 1$, and $x_0 \approx a\zeta(1 + \frac{1}{2}\zeta^{-2})$. The boundary conditions are defined by supposing that the acceleration region is of fixed length L, and that $\delta\beta(t_1) = \delta r(t_1) = \delta t_1 = 0$. Then it follows that $\delta t_2 = -\delta x(t_2)/v_0(t_2) \approx -\delta x(t_2)/c$.

We have $q \equiv \vec{k}\cdot\vec{r}_0 - \omega t \approx \frac{1}{2}\tilde{\omega}\zeta(\zeta^{-2} - \theta^2)$, and we wish q to be small, which is assured if $\zeta \gg 1$, $|\theta| \ll 1$, and $\tilde{\omega}$ is moderate. The triple vector product in Eq. (24) is approximately $\vec{\beta}_0 \times (\hat{k} \times \vec{\epsilon}) \approx (1 - \frac{1}{2}\zeta^{-2})\vec{s}$, where $\vec{s} \equiv \hat{x} \times (\hat{k} \times \vec{\epsilon})$, $\hat{x}$ being the unit vector in the x direction.

We now define the polarization angles. Let us assume that $\hat{k}$ lies in the x-y plane, so that $\hat{k} = (\cos\theta, \sin\theta, 0)$, and define θ' and α so that $\vec{\epsilon} = (\sin\theta', \cos\theta'\cos\alpha, \cos\theta'\sin\alpha)$. Then $\vec{\epsilon}\cdot\hat{k} = 0$ implies $\cos\theta\sin\theta' = -\sin\theta\cos\theta'\cos\alpha \approx -\theta\cos\theta'\cos\alpha$. Evidently, θ' is small also, and hence $\theta' \approx -\theta\cos\alpha$. One can then approximate the vector $\vec{h} \equiv \vec{\epsilon} + \vec{s}$ as $h_x = \epsilon_x \approx -\theta\cos\alpha$, $h_y \approx -\frac{1}{2}\theta^2\cos\alpha$, and $h_z \approx \frac{1}{2}\theta^2\sin\alpha$. Since $\hat{P}_0 = \hat{x}$, Eq. (18) reads $\delta\beta_x = \delta P_x/mc\gamma_0^3$, $\delta\beta_y = \delta P_y/mc\gamma_0$, and $\delta\beta_z = \delta P_z/mc\gamma_0$, and to first order in q, we have

$$\delta\beta_x \approx eE_i\epsilon_x(mc\gamma_0^3)^{-1}\int_{t_1}^{t}(\cos\phi + q\sin\phi)\,dt' \, ,$$

$$\delta\beta_y \approx eE_i(mc\gamma_0)^{-1}\int_{t_1}^{t}(h_y - \tfrac{1}{2}\zeta^{-2}s_y)(\cos\phi + q\sin\phi)\,dt$$

and a similar expression for $\delta\beta_z$.

The evaluation of the δH_ν is tedious, but straightforward, particularly if the average over ϕ is taken before the integrals are done. First the reader should verify the relations $\epsilon_y h_y + \epsilon_z h_z \approx \theta^2 \times (\frac{1}{2} - \cos^2\alpha) \equiv \theta^2 G(\alpha)$, $\epsilon_y s_y + \epsilon_z s_z \approx -1$, and $1 - \hat{k}\cdot\vec{\beta}_0 \approx \frac{1}{2}(\theta^2 + \gamma_0^{-2})$. Let us define $\gamma_j \equiv \gamma_0(t_j)$, $l \equiv \ln(\gamma_2/\gamma_1)$, $F \equiv (1 + \theta^2\gamma_2^2)^{-1}$, and $C_0 \equiv e^2E_i^2a^2/2mc^2$. Then in the limit of small q, small θ, and large γ_j, the dominant terms are

$$\langle \delta H_1 \rangle \approx -C_0\{G(\alpha)\theta^2[\gamma_2 - \gamma_1(1+l)] + \tfrac{1}{2}[\gamma_1^{-1}(l-1) + \gamma_2^{-1}]\} \, ,$$

$$\langle \delta H_3 \rangle \approx -C_0\cos^2\alpha\,\theta^2\gamma_2^3F^2 \times \{(2\gamma_2^{-2} + \theta^2)[\gamma_2(\gamma_2^{-2} - \theta^2)(\gamma_2 - \gamma_1) - l + \tfrac{1}{2}\theta^2(\gamma_2^2 - \gamma_1^2)] + (\gamma_2^{-2} - \theta^2)(1 - \gamma_2/\gamma_1) - \tfrac{1}{2}(\gamma_2^{-2} - \gamma_1^{-2}) - \theta^2 l\} \, ,$$

$$\langle\delta H_4\rangle \approx C_0 \cos^2\alpha\theta^2 F \times\{3\gamma_2 - 2\gamma_2^2\gamma_1^{-1} - \gamma_1 - \theta^2\gamma_2^2[\gamma_2 - \gamma_1(l+1)] + \gamma_2^2 l/\gamma_1\}\ ,$$

$$\langle\delta H_5\rangle \approx -C_0\gamma_2 F\{G(\alpha)\theta^2[\gamma_2(\gamma_2^{-2} - \theta^2)(\gamma_2-\gamma_1) - l + \tfrac{1}{2}\theta^2(\gamma_2^2 - \gamma_1^2)] + \tfrac{1}{2}(\theta^2 - \gamma_2^{-2})(1-\gamma_2/\gamma_1) + \tfrac{1}{4}(\gamma_2^{-2} - \gamma_1^{-2}) + \tfrac{1}{2}\theta^2 l\}\ .$$

The other two terms are much smaller than the above listed ones, by two powers of θ and $1/\gamma$. The above expressions are valid for deceleration or acceleration. For the former, $\gamma_2 < \gamma_1$. Some simplification can be achieved if the ambient radiation field is unpolarized, so that an average over α can be taken. In that case, $\langle\cos^2\alpha\rangle = \frac{1}{2}$ and $\langle G(\alpha)\rangle = 0$.

It might be possible to construct a maser using such a linear accelerator if values of θ and γ can be found such that $\langle\delta H'\rangle > 0$. The dimensionless absorption coefficient K, or "optical thickness," of such an accelerator would be, except for a geometrical factor,

$$K = -8\pi n_e\langle\delta H'\rangle/E_i^2 = r_0 a^2 n_e f(\theta,\gamma_1,\gamma_2)\ , \qquad (25)$$

where n_e is the electron density, r_0 is the classical electron radius, and f is a dimensionless function. For example, for $\theta = 0$ and $\gamma_2 \gg \gamma_1$, we have the simple expressions

$$\langle\delta H'\rangle \approx r_0 E_i^2 a^2\gamma_2/8\gamma_1^2)\ , \qquad (26)$$

whereupon the "optical depth" is roughly

$$K \approx -\pi n_e r_0 a^2\gamma_2\gamma_1^{-2}\ . \qquad (27)$$

The expression on the right-hand side of Eq. (26) comes from δH_5 and is therefore a "potential-energy" contribution. Since $\theta = 0$ it may be derived from (23) by expanding in powers of γ_j, assuming $\tilde{\omega}$ not large. Equation (22), the kinetic energy term, is negative, but may be neglected in this case. As stated in Sec. II, overall energy conservation is presumably maintained by energy involved in injecting and removing the particle at the accelerator boundaries, as in the ordinary radiative beta process.

Unfortunately, the influence of collective effects[6] has not yet been computed. The restriction that $\tilde{\omega} \equiv \omega a/c$ not be large means that $|K| \lesssim \tilde{\omega}_P^2\omega^{-2}\gamma_2\gamma_1^{-2}$, where $\tilde{\omega}_P \equiv (4\pi e^2 n_e/m)^{1/2}$ is the plasma frequency for a cold plasma. Since for maser action we would like $-K > 1$, i.e., $\omega^2 < \tilde{\omega}_P^2\gamma_2\gamma_1^{-2}$, it is not obvious that propagation effects can be neglected, as has been done in this paper. Thus it seems premature to discuss applications further at this time.

VII. DISCUSSION AND COMPARISON WITH THE EINSTEIN-COEFFICIENT METHOD

We have seen that the Poynting-vector analysis of Sec. II results in an interesting viewpoint on energy conservation in Thomson scattering, and in a perturbation theory which leads to stimulated emission and absorption. Since the power input into the ambient wave is proportional to the wave intensity and results in a gain or loss of photons of the same $\vec{k}$ as the ambient wave, the stimulated character of the effect is clear.

Another way of treating stimulated emission is the Einstein-coefficient method, which has been applied to classical problems with some success, particularly to gyroradiation and synchrotron radiation.[7] This method has also been applied to the relativistic linear accelerator, and the result obtained[6] is quite different from that which appears in the present paper. In our notation, the absorption coefficient derived by the Einstein method is [Ref. 6, Eq. (18)]

$$K_E \approx 4r_0 n_e c^2\omega^{-2}F^3\theta^2\gamma_2^3(3\theta^2\gamma_2^2 - 1)\ . \qquad (28)$$

This expression is much simpler than the sum of our $\langle\delta H_\nu\rangle$, and also differs from it by a factor of $c^2(a\omega)^{-2} = \tilde{\omega}^{-2}$. Equation (28) was derived by assuming that the A coefficient is given by the classical rate of photon emission accompanying the linear acceleration,[1] and the usual *Gedanken-experiment* involving equilibrium with a blackbody radiation field was used to derive the relation between the A and B coefficients. The principal assumption of the Einstein method is the detailed balance hypothesis, which holds for quantum systems in certain circumstances, but might be violated here. Heitler[8] indicates that anisotropies (e.g., nonspherical molecules) can cause detailed balance to fail. The linear accelerator is certainly anisotropic. The "optical theorem" method of the present paper seems more straightforward than the Einstein-coefficient method, which is basically quantum mechanical. However, it is difficult to say *a priori* which of the two methods is really correct.

It is interesting to inquire about the Lorentz-transformation properties of Eqs. (25) and (28). Since an optical thickness is observer independent, one should be able to show that the two dimensionless absorption coefficients are scalars, provided (1) that the transformation velocity is parallel to the accelerator field $\vec{E}$ (otherwise a magnetic field would appear in the new frame), and (2) that the conditions $|\theta| \ll 1 \ll \gamma_j$ are preserved.

Then if Γ is the Lorentz factor of the transformation, one can show that (2) is satisfied provided

$\Gamma \ll \gamma_j$ and $\Gamma \ll \theta^{-1}$. If β is the velocity of the transformation, and if we define $H \equiv [(1-\beta)/(1+\beta)]^{1/2}$, the transformed variables are $\omega' \approx H\omega$, $n_e' \approx Hn_e$, $\gamma_j' \approx H\gamma_j$, $\theta' \approx \theta/H$, and $E' = E$. The scalar character of Eqs. (25) and (28) is then easy to demonstrate.

One might wonder whether the approximation $|q| \ll 1$ is consistent with the emission of photons of the appropriate frequency, since q represents the amount of phase of the wave experienced by the electron as it travels. Thus, a distant observer located in the forward direction would perceive a pulse of radiation from it for a time $\Delta t \ll 2\pi/\omega$. However, this pulse would be essentially a δ function and hence would contain photons of all frequencies $\lesssim 1/\Delta t$. If there were a continual stream of electrons entering the accelerator, our distant observer would measure a nonzero $\langle P' \rangle$, since the $\langle \delta H_\nu \rangle$ represent averages over the phase of the ambient wave; i.e., over the initial time t_1 of entrance of the electron into the accelerator. However, contributions to $\langle P' \rangle$ at frequencies other than the wave frequency would be uncorrelated and randomly phased and would average to zero.

The linear accelerator considered here might be termed a "free-electron laser," a phrase coined by Madey, Schwettmann, and Fairbank,[9] who predicted (and later observed experimentally) lasing by relativistic electrons in a periodic magnetic field. Hopf, Meystre, Scully, and Louisell,[10] using the relativistic Boltzmann equation, conclude that this lasing is in fact a purely classical type of stimulated emission, with a gain coefficient proportional to the first power of the electron density [cf. our Eq. (25)]. This problem could also be attacked with the perturbation theory in our Sec. IV, and it would be interesting to see whether the results were the same.

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[3]M. Born and E. Wolf, *Principles of Optics*, 5th ed. (Pergamon, Oxford, 1975), pp. 657–658. See also a somewhat different approach in R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966). The assumption that $E_r \propto e^{ikR}$ makes their proofs somewhat simpler then the present analysis. Since we wish to develop a perturbation theory, we could adopt this assumption also. However, the ensuing perturbation theory would then become even more complex than it already is.

[4]See Ref. 1, p. 784.

[5]We remind the reader that the reaction term is valid only for small $\tau\omega$.

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