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# Multipole polarizabilities and shielding factors of the hydrogen atom from the hydrodynamic analogy to quantum mechanics. II

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Expressions for multipole polarizabilities and shielding of the hydrogen atom have been obtained by generalizing the scheme worked out by Weiner and Askar based on the hydrodynamic analogy to quantum mechanics. Our expressions, for the dipole case, reduce to those of Weiner and Askar and for the zero-frequency limit to those of Dalgarno. Our calculated resonance frequencies are in much better agreement with the exact values than those reported by the variational approach.

Recently[1] (paper I) we formulated a differential equation and its variational analog using the hydrodynamic analogy to quantum mechanics in order to study the multipoles of the hydrogen atom. The differential equation for the multipole $L$ [Eq. (3.3) of paper I] is

$$(D^2_{R,L} - \omega^2)F_L(R) = ELR^{L-1}\,, \tag{1}$$

$$D_{R,L} = \frac{1}{2R^2}\frac{d}{dR}R^2\frac{d}{dR} - \frac{d}{dR} - \frac{L(L+1)}{2R^2}\,. \tag{2}$$

In paper I, we solved the variational analog of Eq. (1) [Eq. (3.10) of paper I] and found expressions for the multipole polarizabilities and shielding factors of the hydrogen atom following Askar and Demiralp.[2] In the present paper, we generalize the scheme developed by Weiner and Askar[3] to study the multipole polarizabilities and shielding factors of the hydrogen atom.

The scheme essentially consists in solving differential equation (1) by considering a solution of the type

$$F_L(R) = E\sum_{k=0}^{\infty}\omega^{2k}F_k\,, \tag{3}$$

and requiring the solution to hold for all values of $\omega^2$. One obtains the following set of equations:

$$D^2_{R,L}F_0 = LR^{L-1}$$

and

$$D^2_{R,L}F_k = F_{k-1}, \quad k = 1, 2, \ldots\,. \tag{4}$$

We now consider a solution of the Frobenius type,

$$F_k = R^S\sum_{n=1}^{\infty}a_{k,n}R^n, \quad k = 0, 1, 2, \ldots, \tag{5}$$

$S$ being the index of the power series. The indicial equation gives four values for $S$: $L-1$, $-L-2$, $L+1$, $-L$. Out of these only $S = L-1$ yields a physically appropriate solution consistent with the boundary condition that $\Psi_0 F_L(R)$ remains bounded at $R = 0$ and vanishes at $R = \infty$.

The following set of algebraic equations result by substitution of Eq. (5) into Eq. (4) with $S = L-1$:

$$\begin{aligned}
&B_2a_{k,2} - C_1a_{k,1} = 0\,,\\
&B_3a_{k,3} - C_2a_{k,2} + D_1a_{k,1} = 0\,,\\
&B_4a_{k,4} - C_3a_{k,3} + D_2a_{k,2} = 0\,,\\
&B_{n+4}a_{k,n+4} - C_{n+3}a_{k,n+3} + D_{n+2}a_{k,n+2} = a_{k-1,n}\,,\\
&n = 1, 2, \ldots, N_k - 4
\end{aligned}$$

where

$$\begin{aligned}
B_n &= \tfrac{1}{4}\left[(n+L-1)(n+L) - L(L+1)\right]\\
&\quad\times\left[(n+L-2)(n+L-3) - L(L+1)\right],\\
C_n &= \tfrac{1}{2}(n+L-1)\\
&\quad\times\left[(n+L-1)(n+L-2) + (n+L)(n+L-3)\right]\\
&\quad - L(L+1)(n+L-2)\,,\\
D_n &= (n+L-1)(n+L-2)\,,
\end{aligned} \tag{6}$$

and

$$N_k = 2k+2\,. \tag{7}$$

The solution for $F_k$ is a finite series[3] with the highest power of $R$ equal to $2k+2$. To calculate $a_{k,n}$, we use the recursion relationship

$$\begin{aligned}
&a_{k,n} = (a_{k-1,n-2} + C_{n+1}a_{k,n+1} - B_{n+2}a_{k,n-2})/D_n\,,\\
&n = N_k, N_{k-1}, \ldots, 2
\end{aligned}$$

with

$$\begin{aligned}
&a_{k,n} = 0, \quad n > N_k, \quad a_{k-1,n-2} = 0, \quad n = 2\\
&a_{k,1} = (B_2/C_1)a_{k,2}\,.
\end{aligned} \tag{8}$$

For $k = 0$, we get

$$D_2a_{0,2} = L \tag{9}$$

and

$$a_{0,1} = (B_2 | C_1)a_{0,2}\,,$$

yielding

$$a_{0,1}=1/L,\ a_{0,2}=1/(L+1). \qquad (10)$$

Hence for $k=0$, we have

$$F_0=(1/L)R^L+[1/(L+1)]R^{L+1}. \qquad (11)$$

It can be easily seen that above general expressions, for multipole $L$, reduce to those of Weiner and Askar[3] for the special case of a dipole ($L=1$). If we substitute Eq. (11) in Eq. (3), we get the same expression for $F_L(R)$ as was obtained earlier by a variational approach [Eq. (4.7) of paper I].

The multipole polarizabilities $\alpha_{2L}$ and shielding factors $Y_{2L}$ are given by[1]

$$\alpha_{2L}=\frac{8}{2L+1}\int_0^{\infty} e^{-2R}F_L(R)R^{L+2}dR\,, \qquad (12)$$

and

$$Y_{2L}=\frac{8}{2L+1}\int_0^{\infty} e^{-2R}\frac{F_L(R)}{R^{L-1}}\,dR\,.$$

Substituting for $F_L(R)$ and integrating we get

$$\alpha_{2L}(\omega^2)=\sum_{k=0}^{\infty}\alpha_k\omega^{2k}\,,$$

and

$$Y_{2L}(\omega^2)=\sum_{k=0}^{\infty}\gamma_k\omega^{2k}\,, \qquad (13)$$

where $\alpha_k$ and $\gamma_k$ are Cauchy's coefficients and are given as

$$\alpha_k=\frac{8}{2L+1}\sum_{n=1}^{N_k}a_{k,n}I_n,\quad k=0,1,2,\ldots$$

with

$$I_n=(n+2L+1)!/2^{n+2L+2}\,, \qquad (14)$$

and

$$\gamma_k=\frac{8}{2L+1}\sum_{n=1}^{N_k}a_{k,n}H_n,\quad k=0,1,2,\ldots \qquad (15)$$

with

$$H_n=n!/2^{n+1}\,.$$

Using the above equations, the static multipole polarizability $\alpha_{2L}(0)$ and the static shielding factor $Y_{2L}(0)$ become

$$\alpha_{2L}(0)=\frac{8}{2L+1}(a_{0,1}I_1+a_{0,2}I_2)=\frac{(2L+2)!(L+2)}{2^{2L+1}L(L+1)} \qquad (16)$$

and

TABLE I. Numerical values for Cauchy moments of multipole polarizabilities $\alpha_k$, multipole shielding factors $\gamma_k$, and calculated resonance frequencies for $L=1$ to 6. The notation $a\ b$ represents $a\times 10^b$.

| | $L=1$; exact $\omega_r=0.375\,000$ | | | | | | $L=2$; exact $\omega_r=0.444\,444$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\alpha_k$ | | $\gamma_k$ | | $\omega_{\text{Calc}}$ | $k$ | $\alpha_k$ | | $\gamma_k$ | | $\omega_{\text{Calc}}$ |
| 0 | 4.5 | | 1 | | ... | 0 | 15 | | 0.333 33 | | ... |
| 1 | 2.658 33 | 1 | 4.5 | | 0.411 435 | 1 | 4.997 91 | 1 | 6.120 37 | −1 | 0.547 837 |
| 2 | 1.721 88 | 2 | 2.658 33 | 1 | 0.392 919 | 2 | 1.914 22 | 2 | 1.804 80 | 0 | 0.510 972 |
| 3 | 1.162 09 | 3 | 1.721 88 | 2 | 0.384 930 | 3 | 7.941 55 | 2 | 6.490 67 | 0 | 0.490 957 |
| 4 | 8.010 16 | 3 | 1.162 09 | 3 | 0.380 890 | 4 | 3.465 78 | 3 | 2.594 13 | 1 | 0.478 687 |
| 5 | 5.586 84 | 4 | 8.010 16 | 3 | 0.378 649 | 5 | 1.565 19 | 4 | 1.105 11 | 2 | 0.470 561 |
| 6 | 3.923 95 | 5 | 5.586 84 | 4 | 0.377 330 | 6 | 7.242 40 | 4 | 4.908 92 | 2 | 0.464 882 |
| 7 | 2.767 86 | 6 | 3.923 95 | 5 | 0.376 521 | 7 | 3.411 46 | 5 | 2.244 64 | 3 | 0.460 755 |
| 8 | 1.957 71 | 7 | 2.767 86 | 6 | 0.376 009 | 8 | 1.628 72 | 6 | 1.048 07 | 4 | 0.457 664 |
| 9 | 1.387 13 | 8 | 1.957 71 | 7 | 0.375 678 | 9 | 7.857 12 | 6 | 4.970 52 | 4 | 0.455 293 |
| 10 | 9.839 90 | 8 | 1.387 13 | 8 | 0.375 460 | 10 | 3.821 42 | 7 | 2.385 47 | 5 | 0.453 439 |
| 11 | 6.985 54 | 9 | 9.839 90 | 8 | 0.375 314 | 11 | 1.870 73 | 8 | 1.155 48 | 6 | 0.451 966 |
| 12 | 4.961 77 | 10 | 6.985 54 | 9 | 0.375 216 | 12 | 9.206 23 | 8 | 5.638 11 | 6 | 0.450 780 |
| 13 | 3.525 56 | 11 | 4.961 77 | 10 | 0.375 150 | 13 | 4.550 01 | 9 | 2.767 29 | 7 | 0.449 815 |
| 14 | 2.505 67 | 12 | 3.525 56 | 11 | 0.375 104 | 14 | 2.256 72 | 10 | 1.364 73 | 8 | 0.449 022 |
| 15 | 1.781 12 | 13 | 2.505 67 | 12 | 0.375 072 | 15 | 1.122 57 | 11 | 6.756 74 | 8 | 0.448 364 |
| 16 | 1.266 23 | 14 | 1.781 12 | 13 | 0.375 051 | 16 | 5.597 80 | 11 | 3.356 06 | 9 | 0.447 815 |
| 17 | 9.002 63 | 14 | 1.266 23 | 14 | 0.375 035 | 17 | 2.797 15 | 12 | 1.671 45 | 10 | 0.447 353 |
| 18 | 6.401 02 | 15 | 9.002 63 | 14 | 0.375 025 | 18 | 1.400 14 | 13 | 8.343 27 | 10 | 0.446 963 |
| 19 | 4.551 41 | 16 | 6.401 02 | 15 | 0.375 017 | 19 | 7.018 92 | 13 | 4.172 60 | 11 | 0.446 632 |
| 20 | 3.236 34 | 17 | 4.551 41 | 16 | 0.375 012 | 20 | 3.523 06 | 14 | 2.090 16 | 12 | 0.446 349 |
| 21 | 2.301 29 | 18 | 3.236 34 | 17 | 0.375 009 | 21 | 1.770 28 | 15 | 1.048 45 | 13 | 0.446 107 |
| 22 | 1.636 42 | 19 | 2.301 29 | 18 | 0.375 006 | 22 | 8.903 65 | 15 | 5.265 37 | 13 | 0.445 899 |
| 23 | 1.163 65 | 20 | 1.636 42 | 19 | 0.375 004 | 23 | 4.481 72 | 16 | 2.646 97 | 14 | 0.445 719 |
| 24 | 8.274 71 | 20 | 1.163 65 | 20 | 0.375 003 | 24 | 2.257 48 | 17 | 1.331 82 | 15 | 0.445 563 |
| 25 | 5.884 17 | 21 | 8.274 71 | 20 | 0.375 002 | 25 | 1.137 80 | 18 | 6.706 12 | 15 | 0.445 429 |

TABLE I. (*Continued*)

| | $L=3$; exact $\omega_r=0.468\,750$ | | | | | | $L=5$; exact $\omega_r=0.486\,111$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $\alpha_k$ | | $\gamma_k$ | | $\omega_{\text{Calc}}$ | $k$ | $\alpha_k$ | | $\gamma_k$ | | $\omega_{\text{Calc}}$ |
| 0 | 1.312 50 | 2 | 1.666 66 | −1 | ⋯ | 0 | 5.457 36 | 4 | 6.666 66 | −2 | ⋯ |
| 1 | 3.328 17 | 2 | 1.744 11 | −1 | 0.627 981 | 1 | 1.044 40 | 5 | 3.241 27 | −2 | 0.722 867 |
| 2 | 9.816 56 | 2 | 3.329 48 | −1 | 0.582 268 | 2 | 2.312 23 | 5 | 3.354 67 | −2 | 0.672 077 |
| 3 | 3.188 89 | 3 | 8.360 27 | −1 | 0.554 829 | 3 | 5.681 72 | 5 | 5.052 80 | −2 | 0.637 934 |
| 4 | 1.106 99 | 4 | 2.451 12 | 0 | 0.536 718 | 4 | 1.509 61 | 6 | 9.533 03 | −2 | 0.613 489 |
| 5 | 4.031 983 | 4 | 7.930 65 | 0 | 0.523 979 | 5 | 4.261 61 | 6 | 2.089 78 | −1 | 0.595 176 |
| 6 | 1.522 58 | 5 | 2.745 27 | 1 | 0.514 599 | 6 | 1.265 56 | 7 | 5.098 12 | −1 | 0.580 980 |
| 7 | 5.912 78 | 5 | 9.978 12 | 1 | 0.507 450 | 7 | 3.890 38 | 7 | 1.347 12 | 0 | 0.569 677 |
| 8 | 2.347 66 | 6 | 3.761 98 | 2 | 0.501 854 | 8 | 1.238 42 | 8 | 3.786 52 | 0 | 0.560 482 |
| 9 | 9.489 96 | 6 | 1.459 09 | 3 | 0.497 377 | 9 | 4.051 57 | 8 | 1.117 92 | 1 | 0.552 869 |
| 10 | 3.892 98 | 7 | 5.787 54 | 3 | 0.493 731 | 10 | 1.356 71 | 9 | 3.434 94 | 1 | 0.546 471 |
| 11 | 1.616 65 | 8 | 2.337 58 | 4 | 0.490 718 | 11 | 4.634 98 | 9 | 1.090 85 | 2 | 0.541 028 |
| 12 | 6.783 06 | 8 | 9.582 80 | 4 | 0.488 197 | 12 | 1.611 23 | 10 | 3.561 67 | 2 | 0.536 346 |
| 13 | 2.871 03 | 9 | 3.977 24 | 5 | 0.486 064 | 13 | 5.686 86 | 10 | 1.190 63 | 3 | 0.532 282 |
| 14 | 1.224 37 | 10 | 1.667 95 | 6 | 0.484 242 | 14 | 2.034 29 | 11 | 4.061 59 | 3 | 0.528 724 |
| 15 | 5.255 36 | 10 | 7.056 99 | 6 | 0.482 674 | 15 | 7.364 14 | 11 | 1.410 09 | 4 | 0.525 588 |
| 16 | 2.268 53 | 11 | 3.008 43 | 7 | 0.481 315 | 16 | 2.694 28 | 12 | 4.971 34 | 4 | 0.522 804 |
| 17 | 8.840 77 | 11 | 1.290 91 | 8 | 0.480 128 | 17 | 9.951 81 | 12 | 1.776 57 | 5 | 0.520 320 |
| 18 | 4.287 47 | 12 | 5.570 89 | 8 | 0.479 086 | 18 | 3.707 57 | 13 | 6.425 50 | 5 | 0.518 091 |
| 19 | 1.875 18 | 13 | 2.416 07 | 9 | 0.478 167 | 19 | 1.392 04 | 14 | 2.349 01 | 6 | 0.516 081 |
| 20 | 8.229 33 | 13 | 1.052 43 | 10 | 0.477 352 | 20 | 5.263 61 | 14 | 8.670 31 | 6 | 0.514 262 |
| 21 | 3.622 50 | 14 | 4.602 09 | 10 | 0.476 626 | 21 | 2.003 14 | 15 | 3.228 07 | 7 | 0.512 608 |
| 22 | 1.598 95 | 15 | 2.019 33 | 11 | 0.475 977 | 22 | 7.668 29 | 15 | 1.211 30 | 8 | 0.511 100 |
| 23 | 7.074 95 | 15 | 8.887 67 | 11 | 0.475 396 | 23 | 2.951 45 | 16 | 4.577 77 | 8 | 0.509 719 |
| 24 | 3.137 39 | 16 | 3.922 46 | 12 | 0.474 872 | 24 | 1.141 65 | 17 | 1.741 29 | 9 | 0.508 452 |
| 25 | 1.394 05 | 17 | 1.735 39 | 13 | 0.474 400 | 25 | 4.436 38 | 17 | 6.662 96 | 9 | 0.507 285 |
| | $L=4$; exact $\omega_r=0.480\,000$ | | | | | | $L=6$; exact $\omega_r=0.489\,795$ | | | | |
| 0 | 2.126 25 | 3 | 0.1 | | ⋯ | 0 | 2.027 02 | 6 | 4.761 90 | −2 | ⋯ |
| 1 | 4.562 31 | 3 | 6.853 04 | −2 | 0.682 676 | 1 | 3.567 35 | 6 | 1.732 26 | −2 | 0.753 800 |
| 2 | 1.138 14 | 4 | 9.361 79 | −2 | 0.633 130 | 2 | 7.215 29 | 6 | 1.411 35 | −2 | 0.703 146 |
| 3 | 3.146 94 | 4 | 1.780 31 | −1 | 0.601 388 | 3 | 1.618 03 | 7 | 1.734 53 | −2 | 0.667 780 |
| 4 | 9.372 92 | 4 | 4.109 41 | −1 | 0.579 437 | 4 | 3.928 79 | 7 | 2.741 10 | −2 | 0.641 747 |
| 5 | 2.952 56 | 5 | 1.076 64 | 0 | 0.563 426 | 5 | 1.016 07 | 8 | 5.134 23 | −2 | 0.621 821 |
| 6 | 9.715 26 | 5 | 3.082 86 | 0 | 0.551 280 | 6 | 2.765 85 | 8 | 1.087 02 | −1 | 0.606 106 |
| 7 | 3.309 81 | 6 | 9.425 09 | 0 | 0.541 783 | 7 | 7.854 43 | 8 | 2.524 32 | −1 | 0.593 412 |
| 8 | 1.159 93 | 7 | 3.029 49 | 1 | 0.534 176 | 8 | 2.311 20 | 9 | 6.300 50 | −1 | 0.582 959 |
| 9 | 4.161 23 | 7 | 1.012 90 | 2 | 0.527 965 | 9 | 7.009 63 | 9 | 1.666 08 | 0 | 0.574 211 |
| 10 | 1.522 41 | 8 | 3.495 77 | 2 | 0.522 810 | 10 | 2.181 98 | 10 | 4.619 02 | 0 | 0.566 790 |
| 11 | 5.663 45 | 8 | 1.238 24 | 3 | 0.518 473 | 11 | 6.947 35 | 10 | 1.331 95 | 1 | 0.560 422 |
| 12 | 2.137 15 | 9 | 4.481 98 | 3 | 0.514 781 | 12 | 2.256 23 | 11 | 3.970 57 | 1 | 0.554 903 |
| 13 | 8.165 10 | 9 | 1.652 19 | 4 | 0.511 607 | 13 | 7.456 53 | 11 | 1.217 73 | 2 | 0.550 077 |
| 14 | 3.153 36 | 10 | 6.186 09 | 4 | 0.508 855 | 14 | 2.502 81 | 12 | 3.827 39 | 2 | 0.545 825 |
| 15 | 1.229 43 | 11 | 2.347 43 | 5 | 0.506 449 | 15 | 8.518 12 | 12 | 1.228 96 | 3 | 0.542 053 |
| 16 | 4.833 59 | 11 | 9.012 18 | 5 | 0.504 331 | 16 | 2.935 42 | 13 | 4.021 03 | 3 | 0.538 687 |
| 17 | 1.914 57 | 12 | 3.495 36 | 6 | 0.502 456 | 17 | 1.023 02 | 14 | 1.337 69 | 4 | 0.535 666 |
| 18 | 7.634 22 | 12 | 1.367 89 | 7 | 0.500 787 | 18 | 3.601 83 | 14 | 4.516 55 | 4 | 0.532 941 |
| 19 | 3.062 33 | 13 | 5.395 98 | 7 | 0.499 294 | 19 | 1.279 96 | 15 | 1.545 31 | 5 | 0.530 472 |
| 20 | 1.235 03 | 14 | 2.143 71 | 8 | 0.497 951 | 20 | 4.587 29 | 15 | 5.350 74 | 5 | 0.528 227 |
| 21 | 5.005 17 | 14 | 8.570 76 | 8 | 0.496 740 | 21 | 1.656 89 | 16 | 1.872 84 | 6 | 0.526 176 |
| 22 | 2.037 43 | 15 | 3.446 28 | 9 | 0.495 642 | 22 | 6.027 40 | 16 | 6.619 84 | 6 | 0.524 297 |
| 23 | 8.327 13 | 15 | 1.392 90 | 10 | 0.494 644 | 23 | 2.207 22 | 17 | 2.360 90 | 7 | 0.522 571 |
| 24 | 3.415 92 | 16 | 5.656 12 | 10 | 0.493 734 | 24 | 8.132 15 | 17 | 8.489 13 | 7 | 0.520 978 |
| 25 | 1.406 00 | 17 | 2.306 54 | 11 | 0.492 902 | 25 | 3.013 16 | 18 | 3.075 46 | 8 | 0.519 507 |

$$Y_{2L}(0)=2/L(L+1)\,. \tag{17}$$

Equations (16) and (17) represent the exact values for the static multipole polarizabilities and shielding factor of the hydrogen atom. These formulas are identical to those obtained by Dalgarno[4] by the usual perturbation method and also to Eqs. (4.18) and (4.19) of paper I.

We have computed the values of $a_{k,n}$ for $k = 0, 1, \ldots, 25$ and $n = 1, 2, \ldots, 52$ for the first six multipoles,[5] $L = 1$ to $L = 6$. Our values of $a_{k,n}$ for the dipole case for $k = 0, \ldots, 14$ are the same as those given by Weiner and Askar.[3] In Table I, we give the computed values of the Cauchy coefficients $\gamma_k$ and $\alpha_k$, and of the resonance frequency $\omega_r$ along with its exact values for all six cases, $L = 1$ to 6. The resonance frequencies are calculated by utilizing the radius-of-convergence criterion for the power series for $\alpha_{2L}(\omega^2)$:

$$\omega_r^2 = \lim_{k \to \infty} (\alpha_{k-1})/\alpha_k . \tag{18}$$

We can compute $\omega_r$ to arbitrary accuracy by utilizing the values of $\alpha_k$ and $\alpha_{k-1}$ for different values of $k$. For the dipole case, by taking $k$ as large as 25, we have been able to get better than 0.001% agreement for the resonance frequency. For the quadrupole resonance frequency, the agreement is within 1%, whereas for $L = 6$ our calculated value is as much as 6% off. Obviously, for higher multipoles one needs to include $a_{k,n}$ far beyond $k = 25$ to get good agreement. However, our calculated values of resonance frequencies for all multipoles are much closer to the exact values than those obtained by a variational approach.[1,2,6] The reason lies in realizing that the resonance frequency is governed by the choice of $G$ and convergence in it. Earlier[1,2,6] we had taken a simple choice for $G$.

TABLE II. Comparison of multipole resonance frequencies for the hydrogen atom (in a.u.).

| $L$ | Present | Exact | Previous (Refs. 1 and 6) |
|---|---|---|---|
| 1 | 0.375 002 | 0.375 000 | 0.471 404 |
| 2 | 0.445 429 | 0.444 444 | 0.632 455 |
| 3 | 0.474 400 | 0.468 750 | 0.717 137 |
| 4 | 0.492 902 | 0.480 000 | 0.769 800 |
| 5 | 0.507 285 | 0.486 111 | 0.805 822 |
| 6 | 0.519 507 | 0.489 795 | 0.832 050 |

In Table II we have tabulated resonance frequencies for such a choice: they are inferior to present estimates. It should be added that recently[7] we have used a better $G$ function, obtaining resonance character in it. However, convergence is slow because for the dipole case, even with solving an $8 \times 8$ determinantal equation, we get a dipole resonance frequency of 0.398. Moreover, in the present case, we have a clear-cut prescription for computing $\omega_r$ for all multipoles to any arbitrary accuracy.

Finally, we add that calculations of Cauchy's coefficients are important in the study of multipole-multipole dispersion interactions[8] when one member of the interacting pair or triplet is the hydrogen atom or another neutral atom or molecule.

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[1] G. C. Shukla and M. Barbaro, Phys. Rev. A 15, 23 (1977). There are three misprints in this paper; namely, in Eq. (2.7) the last term is $-1/R^2$ (not $1/R^2$), in the integral of Eq. (4.15) there should be division by $R^{L-1}$ (not $R^{L+1}$), and $\omega_L = [(1+2/L)\,(1+1/2L)]^{-1/2}$ [Eq. (5.10)].

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[5] The complete tables for $a_{k,n}$ for $L = 1$ to $L = 6$ are not presented here for economy of space. These will be given in M.Sc. thesis of one of the authors (F.N.Y).

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