$Z_1{}^7$ stopping-power formula for fast heavy ions

S. P. Ahlen
Department of Physics, University of California, Berkeley, California 94720
(Received 22 September 1977)

A theoretical stopping-power formula for nuclei and antinuclei is presented. It is estimated to be valid in the regime specified by $\beta > 0.3$, $\gamma < 50$, and $|Z_1/\beta| < 100$ at the $\pm 3\%$ level of accuracy.

The first-order Born approximation upon which the Bethe-Bloch formula is based[1] is inadequate to the task of accurately describing the energy-loss rate of highly charged particles. Higher-order corrections to the distant collisions have been discussed by Ashley, Ritchie, and Brandt,[2,3] Jackson and McCarthy,[4] and Hill and Merzbacher.[5] Jackson and McCarthy[4] have also considered the second-Born Mott-cross-section correction to the stopping power and Eby and Morgan[6] have used the exact Mott cross section to numerically evaluate the stopping power for several values of Z_1 and β ($Z_1 e$ is the effective charge of the heavy particle and βc is its velocity). Eby and Morgan,[6] Hill and Merzbacher,[5] and Lindhard[7] have pointed out the importance of the Bloch correction.[8] Most recently, Lindhard[7] has discussed the low-velocity polarization corrections in terms of a plasma absorbing medium. Low-velocity experimental evidence[9] seems to support Lindhard's approach although data with fully stripped channeled ions[10] are inconsistent with any theoretical technique developed thus far (this is quite possibly due to channeling effects which one would not experience with amorphous absorbers). Experimental information is lacking in the relativistic regime, but such information is probably imminent in view of recent developments of high-energy heavy-ion accelerators (the Lawrence Berkeley Laboratory Bevalac has recently achieved a 2-GeV/amu ^{56}Fe beam).

It is difficult to overemphasize the practical importance of a closed-form expression for the stopping power, for both the design and analysis of experiments. Until the present time, the Bethe-Bloch formula has been the only available expression, and it is limited to those charges and velocities for which the first-Born approximation is valid, namely, for $|Z_1\alpha/\beta| \ll 1$, where α is the fine-structure constant. For a singly charged particle, this condition begins to fail at velocities below which inner-shell corrections become important and, if the particle is positively charged, at those velocities for which electron-capture and loss processes predominate. These effects are difficult to treat theoretically, so it seems that a closed-form expression is available only in that regime for which the separation of distant and close collisions is valid, namely, at large velocities compared to atomic velocities. At lower velocities, the absence of a reliable closed-form expression can be blamed on the onset of shell corrections, electron-capture and -loss processes, and failure of the first-Born approximation, the relative importance of which is difficult to assess. For a multiply charged particle, $|Z_1\alpha/\beta|$ can approach unity for velocities large enough so that shell corrections can be ignored. One should be able to separate the shell corrections from the higher-Born terms. (One still is subject to capture and loss problems if the particle is a positively charged nucleus; however, the relative importance of these for a given $Z_1\alpha/\beta$ becomes smaller as Z_1 increases.) In this paper I will present a stopping formula for this high-velocity regime for which shell corrections can be ignored. In particular, the results will not apply for velocities less than those indicated in Fig. 1 as a function of the atomic weight Z_2 of the absorbing material. At these velocities, shell corrections reduce the stopping power by 1% for protons. Values for Fig. 1 were obtained from Ref. 1.

The distant-collision corrections of Refs. 2–5 can be thought of as arising from the classical in-

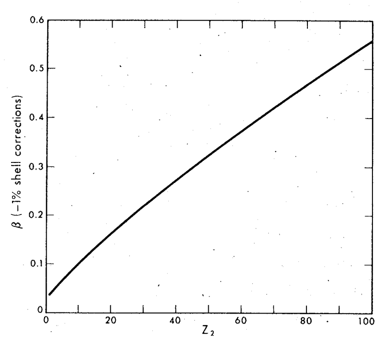

FIG. 1. Velocity, in units of the speed of light, for which shell corrections reduce the stopping power for protons by 1% as a function of the atomic number Z_2 of the absorbing medium.

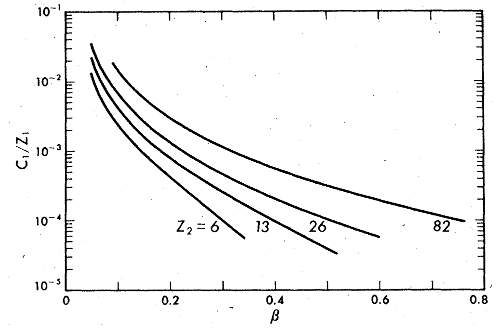

FIG. 2. Fractional correction of Jackson and McCarthy[4] to the total energy loss per effective charge Z_1.

teraction of the gradient of the electric field of the incident particle with the induced dipole moment of the atom (the first-order interaction, as it appears in the first-Born formalism, can be thought of as the classical interaction of the electric field with the charge). Jackson and McCarthy[4] represent the fractional correction to total energy loss as the universal function

$$C_1 = Z_1 F(V)/Z_2^{1/2} , \tag{1}$$

where $V = 137\beta/Z_2^{1/2}$. Equation (1) is the result of a nonrelativistic analysis. C_1/Z_1 is plotted as a function of β for various values of Z_2 in Fig. 2.

While Eq. (1) is sufficient to explain the deviation from quadratic charge scaling for low-energy He data, it does not fit that for low-energy Li data.[9] However, by including close-collision polarization effects along with the Bloch correction,[7] the agreement between theory and experiment is reasonable for both He and Li.[9] Experimental data involving differences between the stopping power of positive and negative pions is also better accounted for by the Lindhard theory.[7] Lindhard finds that the close-collision polarization correction is comparable to C_1, so that the total polarization correction is $2C_1$. Lindhard[7] also gives the Bloch correction

$$C_2 = -1.202 Z_1^2 \frac{\alpha^2}{\beta^2} L_0 , \tag{2}$$

where $L_0 \approx \ln 2mv^2/I$, $\alpha = 1/137$, m is the electron mass, $v = \beta c$, and I is the logarithmic-mean ionization potential of the medium.

It can scarcely be claimed that the low-energy Z_1^3 corrections are thoroughly understood. The measurements which support the conclusion that $2C_1 + C_2$ gives the appropriate correction to the stopping-power formula were done at velocities well below those for which shell corrections become important. It is generally assumed that the shell corrections are charge independent, but this may be interpreted as a definition of the shell corrections. In any case, the value $2C_1$ will be used to gauge the importance of the polarization phenomenon (the Bloch correction is something different and will be considered separately). As an example, it is seen that, for an aluminum absorber, $2|C_1|$ is less than 1% for $|Z_1| < 100$ as long as $\beta > 0.45$. For these values, $|Z_1|/\beta = 222$ and the absence of a closed-form expression for the close-collision energy loss is the dominant factor in the problem, as will be seen below.

In an attempt to establish the connection between Bohr's[11] classical stopping-power formula and Bethe's[12] quantum-mechanical version, Bloch[8] calculated the stopping power by taking into account the finite lateral spread of the close-collision electron "beam." The connection is found to be described by the digamma function[13] ψ and the Bohr-Bethe-Bloch formula can be written

$$\frac{dE}{dx} = \frac{4\pi N Z_1^2 e^4}{mv^2}\left[\ln \frac{2mv^2\gamma^2}{I} - \beta^2 + \psi(1) - \mathrm{Re}\,\psi(1 + iZ_1\alpha/\beta)\right] , \tag{3}$$

where N is the electron density and $\gamma = 1/(1-\beta^2)^{1/2}$. As $Z_1\alpha/\beta \to 0$, the usual Bethe-Bloch formula obtains and, as $|Z_1|\alpha/\beta$ becomes much larger than 1, Bohr's formula obtains. The digamma function is the logarithmic derivative of the gamma function and can be expressed as an alternating series [13]:

$$\mathrm{Re}\,\psi(1+iy) = 1 + \psi(1) - 1/(1+y^2) + \sum_{n=1}^{\infty} (-1)^{n+1}[\zeta(2n+1) - 1]\, y^{2n}, \quad |y| < 2 \tag{4}$$

where $\psi(1) = -0.577\,21\ldots$ is the negative of Euler's constant and $\zeta(n)$ is the Riemann zeta function

$$\zeta(n) = \sum_{k=1}^{\infty} k^{-n} . \tag{5}$$

The error of the mth partial sum of the above alternating series is given by $[\zeta(2m+3) - 1]y^{2m+2}$. For $|y| < 1$, this error is less than $\zeta(2m+3) - 1$. Since $\zeta(5) - 1 = 0.0369\ldots$, and since the logarithmic term is generally of order 10, we will be in error by less than 1% for total dE/dx for $|Z_1|\alpha/\beta < 1$ if we keep only the first term of the series. Hence, we can write

$$\psi(1) - \mathrm{Re}\,\psi(1 + iZ_1\alpha/\beta) = -1 - 0.202(Z_1\alpha/\beta)^2 + 1/[1 + (Z_1\alpha/\beta)^2] . \tag{6}$$

Lindhard's expression, Eq. (2), is obtained in the limit $|Z_1|\alpha/\beta \ll 1$.

Even when Eq. (3) is corrected by Eq. (1) it is in error due to omission of the density-effect correction[14] and due to failure of the first-Born Mott cross section to adequately describe the close collisions.[4,6] These close-collision corrections tend to become more important at large velocities. It is difficult to evaluate the magnitude of their effect due to the slowly converging Legendre expansions which are necessary for an evaluation of the exact Mott cross section. These expansions have been summed numerically by Doggett and Spencer,[15] among others (see Ref. 15 for references to previous work on the subject). However, the tabulated cross sections are not easily incorporated into the close-collision energy-loss formula since an integration over c.m. scattering angles is required (refer to the work of Eby and Morgan,[6] who have performed such numerical calculations for several cases). To compound this practical hindrance is the more fundamental problem of extremely slow convergence for very small c.m. scattering angles, where the cross section is largest (the tabulations of Ref. 15 extend from 180° to 15°). For these reasons I have adopted the Z_1^7 expansion derived by Curr[16] for the Mott cross section which is valid for c.m. angles above 30° and the expression from Bartlett and Watson[15,17] for the small c.m.-angle cross section. By performing the appropriate integration over angles it is found that

$$\frac{dE}{dx}=\frac{4\pi N Z_1^2 e^4}{mv^2}\left[\ln\frac{2mv^2\gamma^2}{I}-\beta^2-1-0.202\left(\frac{Z_1\alpha}{\beta}\right)^2+\frac{1}{1+(Z_1\alpha/\beta)^2}+\tfrac{1}{2}G(Z_1,\beta)-\tfrac{1}{2}\delta(\beta)\right]\times[1+2Z_1F(V)/Z_2^{1/2}], \quad (7)$$

where values of I can be found in Refs. 14 and 18, and a general expression for $\delta(\beta)$, the density-effect correction, is given by Sternheimer and Peierls.[14] G is the close-collision correction which is given by

$$\begin{aligned}G(Z_1,\beta)=&(Z_1\alpha\beta)[1.725+0.52\pi\cos\chi-2(\epsilon_0/\epsilon_m)^{1/2}\pi\cos\chi]\\&+(Z_1\alpha)^2(3.246-0.451\beta^2)\\&+(Z_1\alpha)^3(1.522\beta+0.987/\beta)\\&+(Z_1\alpha)^4(4.569-0.494\beta^2-2.696/\beta^2)\\&+(Z_1\alpha)^5(1.254\beta+0.222\beta-1.170/\beta^3)\,, \quad (8)\end{aligned}$$

where $\epsilon_m=2mv^2\gamma^2$, $\sqrt{\epsilon_0}=\sum_i f_i\sqrt{\hbar\omega_i}$ is the mean-square-root ionization potential (approximate values for the oscillator strengths f_i and the ionization potentials $\hbar\omega_i$ can be found in Sternheimer's work,[19-23] and $\cos\chi$ is defined in Ref. 15. Values of $\cos\chi$ as a function of $|Z_1|\alpha/\beta$ are given in Table I. For ordinary nuclei with Z_0 protons, Z_1 is taken to be[24]

$$Z_1=Z_0[1-\exp(-130\beta/Z_0^{2/3})]\,. \quad (9)$$

TABLE I. Values of $\cos\chi$ as a function of $|Z_1|\alpha/\beta$.

$\|Z_1\alpha/\beta\|$	$\cos\chi$
0	1.000
0.05	0.9905
0.10	0.9631
0.15	0.9208
0.20	0.8680
0.30	0.7478
0.40	0.6303
0.50	0.5290
0.60	0.4471
0.80	0.3323
1.00	0.2610
1.20	0.2145
1.50	0.1696
2.00	0.1261

The error of Eq. (7) is estimated to be less than 1% due to incorrect connection of the Bartlett-Watson cross section to the Curr cross section in the 10°–30° c.m. angular interval. Based on Curr's estimates for the accuracy of his expression, I expect Eq. (7) to have a fractional error of $(Z_1\alpha)^9/6\beta^9$ due to deviation from the true Mott cross section. For $|Z_1|/\beta<100$ this error will be less than 1%. At extreme relativistic energies, Eq. (7) will fail for a number of reasons, such as (a) the effect of the spin and internal structure of the nucleus, (b) emission of bremsstrahlung radiation by the nucleus, (c) the onset of radiative corrections to the close-collision cross section, (d) kinematical complications. For heavy nuclei, the most severe restriction is the point-charge approximation. Form-factor effects can become important for γ as small as 10 for very heavy nuclei.

The size of the various corrections to the Bethe-Bloch formula are indicated in Fig. 3 for the case of an aluminum absorber. The density-effect correction was taken from Ref. 14:

$$\delta=\begin{cases}4.606X+C+a(X_1-X)^m, & X_0<X<X_1\,,\\ 0, & X<X_0\,,\end{cases} \quad (10)$$

where, for aluminum, $C=-4.19$, $X_0=0.37$, $X_1=3.0$, $m=3.0$, $a=0.137$, and $X=\log\beta\gamma$. ϵ_0 was calculated using the ionization potentials listed in Ref. 19; $\epsilon_0=176$ eV (if ϵ_0 is in error by 100%, the error in dE/dx is much less than 1% for most cases; in fact, ϵ_0 can be set equal to 0 with negligible error for all but the heaviest absorbers).

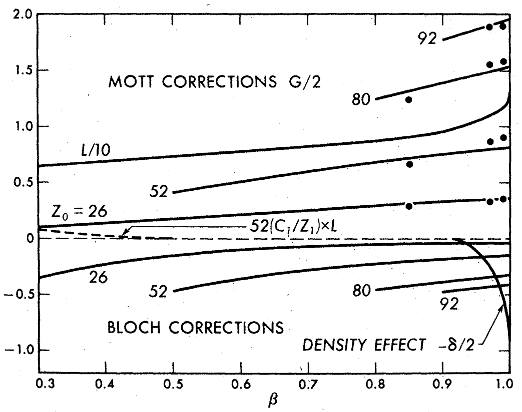

FIG. 3. Corrections to the Bethe-Bloch stopping-power formula for an aluminum absorber as a function of the atomic number Z_0 (from which the effective charge Z_1 was derived) and velocity. See text for a discussion of these corrections.

The value used for I, namely, 163 eV, was taken from Ref. 14.

The corrections were calculated for $Z_0 = 26$, 52, 80, and 92. The low-velocity limits correspond to $Z_1/\beta = 100$. It is seen that for most velocities of interest to cosmic-ray workers and Bevalac users the Bloch and Mott corrections dominate. Shell corrections are much less than 1% of L, where L is the term from the Bethe-Bloch equation:

$$L = \ln(2mv^2\gamma^2/I) - \beta^2 . \qquad (11)$$

The polarization correction for $Z_0 = 26$ is illustrated and is seen to become insignificant at $\beta = 0.5$. This correction is negligible for the range of velocities considered for the larger charges.

The Mott corrections from Eby and Morgan[6] are shown as the large solid circles. The difference between these exact values and those obtained from Eq. (8) are always small enough so that the value of dE/dx calculated using Eq. (7) is in error by less than 1%.

If Eq. (7) is to used for negatively charged particles, then $Z_1 = Q/e$, where Q is the charge of the particle (so Z_1 is then negative). The Bloch correction only depends on the magnitude of Z_1 but the low-velocity polarization corrections and the Mott corrections depend on the polarity as well as the magnitude.

For $|Z_1|/\beta \leq 100$, Eq. (7) is probably accurate to better than ±3% in an absolute sense. Much better accuracy can be expected for describing relative behavior for charged-particle energy loss. This equation should prove quite useful to cosmic-ray and relativistic-heavy-ion researchers by extending the regime of applicability of the Bethe-Bloch formula, which is limited to $|Z_1|\alpha/\beta \ll 1$.

ACKNOWLEDGMENT

This work was supported by ERDA Contract No. AT(04-3)-34.

[1] U. Fano, Ann. Rev. Nucl. Sci. 13, 1 (1963).
[2] J. C. Ashley, R. H. Ritchie, and W. Brandt, Phys. Rev. B 5, 2393 (1972).
[3] J. C. Ashley, R. H. Ritchie, and W. Brandt, Phys. Rev. A 8, 2402 (1973).
[4] J. D. Jackson and R. L. McCarthy, Phys. Rev. B 6, 4131 (1972).
[5] K. W. Hill and E. Merzbacher, Phys. Rev. A 9, 156 (1974).
[6] P. B. Eby and S. H. Morgan, Phys. Rev. A 5, 2536 (1972).
[7] J. Lindhard, Nucl. Instrum. Methods 132, 1 (1976).
[8] F. Bloch, Ann. Phys. (Leipz.) 16, 285 (1933).
[9] H. H. Anderson, J. F. Bak, H. Knudsen, P. Møller Peterson, and B. R. Nielsen, Nucl. Instrum. Methods 140, 537 (1977).
[10] S. Datz, J. Gomez del Campo, P. F. Dittner, P.D. Miller, and J. A. Biggerstaff, Phys. Rev. Lett. 38, 1145 (1977).
[11] N. Bohr, Philos. Mag. 25, 10 (1913).
[12] H. Bethe, Ann. Phys. (Leipz.) 5 325 (1930).
[13] M. Abromowitz and I. A. Stegun, *Handbook of Mathematical Function*, NBS Appl. Math Series, No. 55 (1970).
[14] R. M. Sternheimer and R. R. Peierls, Phys. Rev. B 3, 3681 (1971).
[15] J. A. Doggett and L. V. Spencer, Phys. Rev. 103, 1597 (1956).
[16] R. M. Curr, Proc. Phys. Soc. (Lond.) A 68, 156 (1955).
[17] J. H. Bartlett and R. E. Watson, Proc. Am. Acad. Arts Sci. 74, 53 (1940).
[18] E. V. Benton and R. P. Henke, Nucl. Instrum. Methods 67, 87 (1969).
[19] R. M. Sternheimer, Phys. Rev. 88, 851 (1952).
[20] R. M. Sternheimer, Phys. Rev. 89, 1148 (1953).
[21] R. M. Sternheimer, Phys. Rev. 103, 511 (1956).
[22] R. M. Sternheimer, Phys. Rev. 145, 247 (1966).
[23] R. M. Sternheimer, Phys. Rev. 164, 349 (1967).
[24] T. E. Pierce and M. Blann, Phys. Rev. 173, 390 (1968).