Comments and Addenda

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Comment on radiative magnetic energy shifts in hydrogen

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It is shown that the magnetic radiative energy shift derived from the relativistic-Lamb-shift expression of Erickson and Yennie reduces in the nonrelativistic limit to a formula given by Grotch and Hegstrom, which was derived starting from the nonrelativistic theory. This clears up a discrepancy between those two approaches. The corresponding correction to the g factor, which exists only for states with $l \neq 0$, is estimated to be $-0.24\alpha^3$ for the $2P$ state of hydrogen.

A number of years ago Brodsky and Parsons[1] studied the Zeeman theory in the excited states of hydrogen. In their work, especially Appendix A, they called attention to Lamb-shift-type corrections to the magnetic interactions, and also pointed out that these were of relative order α^3, although they did not explicitly calculate them. Subsequently Grotch and Hegstrom[2] examined relativistic and radiative interactions of many-electron atoms in an external magnetic field, and also noted the presence of these Lamb-type corrections which arose from emission and absorption of low-frequency photons. The results contained in Ref. 2 deviated from somewhat earlier work[3] based on the relativistic expression extracted from the Lamb-shift work of Erickson and Yennie.[4]

The present Comment considers this expression for hydrogen in greater detail starting from the relativistic expression; it shows that the appropriate reduction does lead to agreement with the nonrelativistic calculation of Ref. 2 and thereby corrects some omissions in Ref. 3. Starting with the derived expression we then systematically calculate this correction since order α^3 terms could possibly lead to g-factor corrections at the level of 1 ppm. We have calculated these approximately for the $2P$ state of hydrogen and find that they lead to a g-factor correction of about $-0.24\alpha^3$, which is smaller than 1 ppm.

The relativistic expression extracted from the Erickson-Yennie[4] paper is

$$\Delta E_n(L) = -\frac{2\alpha}{3\pi m^2}\langle n|\Pi_\mu\left(\ln\frac{m}{2(H-E_n)}+\frac{11}{24}\right)\gamma_0[\Pi^\mu, \not\Pi]|n\rangle , \qquad (1)$$

where $(\not\Pi - m)|n\rangle = 0$, $\Pi_\mu = p_\mu - eA_\mu$, and $H - E_n = (m^2 - \not\Pi^2)/2m$. The four-vector potential has components $eA_0 = V$ and $e\vec{A} = e\frac{1}{2}\vec{B}\times\vec{r}$, with $V = -Z\alpha/r$. Terms linear in $\vec{B}$ can arise within $|n\rangle$, $H - E_n$, or Π_μ. The four vector p_μ has the constant zeroth component $p_0 = E_n$ and the spatial component $\vec{p}$.

To begin the nonrelativistic reduction of Eq. (1) in order to extract the leading term we write

$$\gamma_0[\Pi^\mu, \not\Pi] = [\Pi^\mu, \gamma_0(\gamma_0(E_n - V) - \vec{\gamma}\cdot\vec{\Pi})] = -[\Pi^\mu, H_D] , \qquad (2)$$

where H_D is the Dirac Hamiltonian. We now insert a complete set of intermediate-energy eigenstates for which, in the nonrelativistic limit, only the positive-energy contributions are retained. It may then be shown that Eq. (1) leads to the approximate expression

$$-\frac{2\alpha}{3\pi m^2}\int_0^\infty \frac{dk}{1+2k/m}\sum_{n'}\langle n|\Pi_\mu \frac{1}{k+\epsilon_{n'}-\epsilon_n}|n'\rangle \times\langle n'|[H_D,\Pi^\mu]|n\rangle - \frac{2\alpha}{3\pi m^2}\frac{11}{24}\langle n|\Pi_\mu[H_D,\Pi^\mu]|n\rangle . \quad (3)$$

The factor $(1+2k/m)^{-1}$ cuts off the k integration in an appropriate way to recover the logarithm in Eq. (1), and the factors $\epsilon_{n'}$ and ϵ_n denote the nonrelativistic energy levels.

The commutators needed are

$$[H_D,\Pi_0]=i(Z\alpha/r^3)\,\vec{\alpha}\cdot\vec{r}$$

and

$$[H_D,\Pi_i]=-ie(\vec{\alpha}\times\vec{B})_i+(iZ\alpha/r^3)r_i . \quad (4)$$

Consider now the second term of Eq. (3)

$$-\frac{2\alpha}{3\pi m^2}\frac{11}{24}\langle n|\left[\Pi_0\frac{iZ\alpha}{r^3}\,\vec{\alpha}\cdot\vec{r}-\vec{\Pi}\cdot(-ie)\,\vec{\alpha}\times\vec{B} -\frac{iZ\alpha}{r^3}\,\vec{\Pi}\cdot\vec{r}\right]|n\rangle . \quad (5)$$

The terms proportional to $\vec{\alpha}\cdot\vec{r}$ and $\vec{\alpha}\times\vec{B}$ couple the upper and lower components of the Dirac wave function. The $\vec{\alpha}\times\vec{B}$ term will immediately lead to a term proportional to $(\vec{p}\times\vec{p})\cdot\vec{B}$ and is therefore zero. The even operator $\vec{\Pi}\cdot\vec{r}$ can only bring in the lower components of the wave function by coupling lower components to lower components, but this can be seen to be of relative order α^5. The remaining term, i.e., $\vec{\alpha}\cdot\vec{r}$, gives

$$-\frac{2\alpha}{3\pi m^2}\frac{11}{24}\frac{i}{2m}(n|\Pi_0\frac{Z\alpha}{r^3}(\vec{r}\cdot\vec{\Pi}+i\vec{\sigma}\cdot\vec{r}\times\vec{\Pi}) +(\vec{\Pi}\cdot\vec{r}+i\vec{\sigma}\cdot\vec{\Pi}\times\vec{r})\Pi_0\frac{Z\alpha}{r^3}|n) \quad (6)$$

with $|n)$ the nonrelativistic state vector. The $\vec{B}$ dependence coming from Π_0 leads to terms which are also of order α^5 while other $\vec{B}$-dependent factors cancel. Thus only the first term of Eq. (3) leads to an order α^3 contribution. We have

$$-\frac{2\alpha}{3\pi m^2}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}\langle n|\Pi_0\left|\frac{|n'\rangle\langle n'|}{k+\epsilon_{n'}-\epsilon_n}\right|\frac{iZ\alpha}{r^3}\,\vec{\alpha}\cdot\vec{r}|n\rangle + \frac{2\alpha}{3\pi m^2}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}\langle n|\Pi_i\left|\frac{|n'\rangle\langle n'|}{k+\epsilon_{n'}-\epsilon_n}\right|-ie(\vec{\alpha}\times\vec{B})_i+\frac{iZ\alpha}{r^3}r_i|n\rangle . \quad (7)$$

The first term gives contributions which are order α^5 or smaller. In the second term we make the nonrelativistic reduction. This gives rise to

$$\begin{aligned}&\frac{2\alpha}{3\pi m^3}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}(n|p_i\frac{|n')(n'|}{k+\epsilon_{n'}-\epsilon_n}(-ie)(\vec{p}\times\vec{B})_i|n) + \frac{2\alpha}{3\pi m^2}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}(n|\Pi_i\frac{|n')(n'|}{k+\epsilon_{n'}-\epsilon_n}\left[V+\frac{p^2}{2m},p_i\right]|n)\\ &=\frac{2\alpha}{3\pi m^3}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}(n|p_i\frac{|n')(n'|}{k+\epsilon_{n'}-\epsilon_n}(-ie)(\vec{p}\times\vec{B})_i|n) + \frac{2\alpha}{3\pi m^2}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}(n|\left[\Pi_i,V+\frac{p^2}{2m}\right]\frac{|n')(n'|}{k+\epsilon_{n'}-\epsilon_n}p_i|n) .\end{aligned} \quad (8)$$

But since $\vec{\Pi}_i=\vec{p}_i-\frac{1}{2}e(\vec{B}\times\vec{r})_i$ and $[-\frac{1}{2}e(\vec{B}\times\vec{r})_i,p^2/2m]=-(ei/2m)(\vec{B}\times\vec{p})_i$, we find

$$\frac{2\alpha}{3\pi m^3}(-ie)\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}\vec{B}\cdot\frac{(n|\vec{p}|n')\times(n'|\vec{p}|n)}{k+\epsilon_{n'}-\epsilon_n} + \frac{2\alpha}{3\pi m^3}\frac{(-ie)}{2}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}\vec{B}\cdot\frac{(n|\vec{p}|n')\times(n'|\vec{p}|n)}{k+\epsilon_{n'}-\epsilon_n} = -\frac{ie\alpha}{\pi m^3}\int_0^\infty\frac{dk}{1+2k/m}\sum_{n'}\vec{B}\cdot\frac{(n|\vec{p}|n')\times(n'|\vec{p}|n)}{k+\epsilon_{n'}-\epsilon_n} . \quad (9)$$

This is the expression given in Ref. 2, where it was derived from the nonrelativistic theory. We therefore see that the reduction proceeding from Eq. (1) above leads to Eq. (9) and that in previous publications[3] the coefficient was not correctly given.

In the above discussion we have tacitly ignored any magnetic field dependence which might arise in the difference $\epsilon_{n'} - \epsilon_n$ appearing in the energy denominator. Arguments justifying this have been given elsewhere[2] and will not therefore, be repeated here. It should also be mentioned that the extension of Eq. (9) to the many-electron atom has been discussed in Ref. 2, where it is shown that $\vec{p}$ is replaced by the sum of the electron momenta. The expression so obtained includes the effect of retardation in the electron-electron interaction.

It is clear from the structure of Eq. (9) that this magnetic interaction will yield g-factor corrections of order α^3 for states with nonzero orbital angular momentum. We have been unable, thus far, to evaluate this expression analytically. To obtain a reasonable approximation we shall sum a number of contributions to Eq. (9). This will be carried out for the $2P$ states of hydrogen. A more accurate answer can be obtained by including more terms in the series.

We can put Eq. (9) in a more convenient form by expressing $\vec{p}$ as a commutator of the Hamiltonian with $\vec{r}$ and by choosing $\vec{B}$ along the z axis. Matrix elements of x and y then arise, and these operators can each be expressed in terms of raising and lowering operators for the z component of angular momentum. We shall choose that state $|n)$ to have $l=1, m=1$, and therefore the matrix elements will connect the state $|n)$ to states with $l=2$ and with $l=0$. The angular integrations are very easily carried out. The details will not be given, but the result obtained from Eq. (9) is

$$-\frac{2}{3}\frac{\alpha eB}{\pi m}\int_0^\infty \frac{dk}{1+2k/m}\sum_{n'\geq 3}(\epsilon_{n'}-\epsilon_n)^2 \times\frac{1}{k+\epsilon_{n'}-\epsilon_n}|R_{21}^{n'2}|^2 + \frac{2}{3}\frac{\alpha eB}{\pi m}\int_0^\infty \frac{dk}{1+2k/m}\sum_{n'}(\epsilon_{n'}-\epsilon_n)^2 \times\frac{1}{k+\epsilon_{n'}-\epsilon_n}|R_{21}^{n'0}|^2, \quad (10)$$

where

$$R_{nl}^{n'l'} = \int_0^\infty R_{n'l'}(r)R_{nl}(r)r^3\,dr \quad (11)$$

with $R_{nl}(r)$ the radial wave function for the hydrogenic state of principal quantum number n and orbital angular momentum l. In our units the binding energies needed in Eq. (10) are given by $\epsilon_{n'} = -m\alpha^2/2n'^2$.

The integration on k may readily be carried out. Ignoring the energy difference $\epsilon_{n'} - \epsilon_n$ as compared to $\frac{1}{2}m$, we find

$$\frac{2}{3}\frac{\alpha eB}{\pi m}\sum_{n'\geq 3}\frac{m^2\alpha^4}{4}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2 \times\ln\left(\alpha^2\left|\frac{1}{n'^2}-\frac{1}{n^2}\right|\right)(R_{21}^{n'2})^2 - \frac{2}{3}\frac{\alpha eB}{\pi m}\sum_{n'}\frac{m^2\alpha^4}{4}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2 \times\ln\left(\alpha^2\left|\frac{1}{n'^2}-\frac{1}{n^2}\right|\right)(R_{21}^{n'0})^2. \quad (12)$$

Finally, by factoring out the Bohr radius by writing $R_{21}^{n'2} = \mathcal{R}_{21}^{n'2}/m\alpha$ and $R_{21}^{n'0} = \mathcal{R}_{21}^{n'0}/m\alpha$ we find

$$\frac{2}{3}\frac{\alpha eB}{4\pi m}\alpha^2\sum_{n'\geq 3}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2 \times\ln\left(\alpha^2\left|\frac{1}{n'^2}-\frac{1}{n^2}\right|\right)(\mathcal{R}_{21}^{n'2})^2 - \frac{2}{3}\frac{\alpha eB}{4\pi m}\alpha^2\sum_{n'}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2 \times\ln\left(\alpha^2\left|\frac{1}{n'^2}-\frac{1}{n^2}\right|\right)(\mathcal{R}_{21}^{n'0})^2. \quad (13)$$

Since this energy shift leads to a g-factor correction of $[-(e/2m)B]\,\Delta g$, we have

$$\Delta g = -\frac{1}{3\pi}\alpha^3\sum_{n'\geq 3}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2 \times\ln\left(\alpha^2\left|\frac{1}{n'^2}-\frac{1}{n^2}\right|\right)(\mathcal{R}_{21}^{n'2})^2 + \frac{1}{3\pi}\alpha^3\sum_{n'}\left(\frac{1}{n'^2}-\frac{1}{n^2}\right)^2 \times\ln\left(\alpha^2\left|\frac{1}{n'^2}-\frac{1}{n^2}\right|\right)(\mathcal{R}_{21}^{n'0})^2. \quad (14)$$

The integrals of the radial matrix elements appearing in Eq. (15) are quite complicated and are expressed in terms of several hypergeometric functions.[5] Condon and Shortley[6] have prepared a convenient tabulation of $(\mathcal{R}_{21}^{n'2})^2$ for n' values of 3 through 8 and for $(\mathcal{R}_{21}^{n'0})^2$ for values from 1 to 8. We have extended these to $n'=12$ to obtain a better estimate of the value of Δg. If we sum Eq. (14) through $n'=8$ we obtain $\Delta g = -0.265\alpha^3$; for $n'=9$ we find $-0.259\alpha^3$; for $n'=10, -0.254\alpha^3$; for $n'=11, -0.251\alpha^3$; and for $n'=12, -0.249\alpha^3$. Based on these results we would estimate the actual value at about $-0.24\alpha^3$ or -0.1 ppm.

The contribution calculated here is of the same order as the binding corrections to the anomalous moment, but as mentioned earlier the above correction arises only for $l \neq 0$. At present, experiments are probably not sufficiently sensitive to the presence of such small-g factor corrections, but nevertheless it is worthwhile to be aware of the origin of various contributions to g factors, especially since experimental accuracy seems to improve continually.

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