PHYSICAL REVIEW A VOLUME 17, NUMBER 3 MARCH 1978

Memory effects in the linewidth and line shape near the laser threshold

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(Received 21 September 1977)

A mode-coupling approximation and a continued-fraction representation are used to calculate the memory kernels of the dynamical field and photon-number correlation functions near a single-mode laser's threshold. The resulting linewidths of field and intensity fluctuations versus pumping are in good agreement with Risken's numerical calculations as well as with empirical linewidth data. In addition details of the line shape are also reported. The deviations from a Lorentzian are appreciable in the proper threshold region.

I. INTRODUCTION

Consider a homogeneously broadened single-mode laser. It is generally accepted[1,2] that it can be described near its threshold by a Langevin equation for the slowly varying field amplitude $b=b_1+i\,b_2=|b|e^{i\phi}$ (in the rotating-wave approximation and after adiabatic elimination of the atomic variables):

$$\dot{b}=\gamma b-\beta|b|^2 b+\Gamma(t),\quad t\geq 0\ . \tag{1}$$

The real parameters γ and β are determined by the laser properties:

$$\gamma=\frac{g^2}{\kappa+\gamma_\perp}(\sigma-\sigma_{\text{thr}}),\quad \beta=\frac{4g^2\kappa}{\gamma_\parallel(\kappa+\gamma_\perp)}\ . \tag{2}$$

Here κ is the cavity loss, $\gamma_\perp$ the atomic linewidth, $\gamma_\parallel$ the inversion half-width, σ the pumping inversion and in particular $\sigma_{\text{thr}}=\kappa\gamma_\perp/g^2$, the threshold inversion, and g^2 the dipole transition strength. The quantum noise $\Gamma=\Gamma_1+i\,\Gamma_2$ is considered to be Gaussian with zero mean and second moment

$$\langle\Gamma(t)\,\Gamma^\dagger(t')\rangle=4q\,\delta(t-t')\ . \tag{3}$$

The noise strength is $q=Ng^2/4\gamma_\perp$, N being the number of active laser atoms.

The Langevin equation is statistically equivalent[3] to the following Fokker-Planck equation for the probability distribution $\rho(b,t)$ for a field strength b at time t:

$$\partial_t\rho(b,t)=[-\partial_i(\gamma-\beta|b|^2)b_i+q\partial_i\,\partial_i]\rho(b,t),\quad t\geq 0\ . \tag{4}$$

The stationary laser fluctuations are governed by the steady-state solution,[2] $\partial_t\rho_s(b)=0$:

$$\rho_s\propto\exp\left[-\frac{1}{q}\left(-\frac{1}{2}\gamma|b|^2+\frac{1}{4}\beta|b|^4\right)\right]\ . \tag{5}$$

This distribution only depends on the photon intensity $b^\dagger b$ and not on the field phase ϕ. It is nearly Gaussian in the field amplitude b far below threshold ($\gamma<0$, $|\gamma|\gg 1$), it exhibits large fluctuations near threshold ($\gamma\approx 0$), and it shows the typical Landau potential circular valley at $|b|_0^2=\gamma/\beta$ far above threshold ($\gamma>0$, $\gamma\gg 1$). This stationary laser distribution is experimentally well established.[4]

By solving the Fokker-Planck equation numerically for all (angular momentum zero) eigenvalues versus γ, Risken[2] calculated an effective Lorentzian linewidth λ_{eff} of the photon number fluctuations and also the linewidth factor $\lambda_{10}\times\langle|b|^2\rangle$ of the field fluctuations. A linewidth λ_{eff} has been measured by Arecchi *et al.*[5] and a linewidth factor by Gerhardt *et al.*[6] These experiments support the interpretation of the single-mode laser in terms of the Fokker-Planck equation (4).

This paper is aimed at understanding the details of the dynamical correlations near a laser's threshold not by numerical methods but as far as possible by an analytical thought approximate treatment which is applicable for two-mode lasers and other lasers as well. The Fokker-Planck equation (4) is taken for granted. Its steady-state solution is explicitly known and will be used. The idea is to use the Zwanzig-Mori[7,8] memory formalism to explain the dynamics of a system, once its statics (steady state) are known.

This memory (projector) representation has been very successful in equilibrium statistical physics of N-particle systems.[9] But, while there the equilibrium properties [like the pair distribution $g(r)$, etc.] have to be taken from experiments or have to be approximated more or less reasonably (like the triple density correlations), the laser provides an example where the stationary moments are explicitly and exactly known. This paper therefore is also aimed at testing the general method. It will turn out that all typical features met in dynamical problems are present in the laser dynamics as well. The only difference (which, of course, is an essential simplification) is the missing wave-number dependence: Because of the large coherence length of the light field, the single-mode laser can be treated as spatially homogeneous (dimensionality zero).

There is another aspect which the laser dynamics are able to exemplify: It is possible to extend the memory formalism and its essential idea to avoid perturbation expansion, namely, the mode-coupling approximation,[10] to nonequilibrium statistical physics. Some commonly used concepts have to be given up (like time-reversal invariance, Hermiticity of the Liouvillian, etc.) but essential features (like projecting onto the slow modes, the resolvent identity, continued-fraction representation, spectral function and spectral representation, etc.) can well be introduced to understand physically and calculate explicitly dynamical statistical fluctuations of open nonequilibrium systems in steady states.

The power of the continued-fraction representation[8] in a nonequilibrium system has been shown already by Bixon and Zwanzig,[11] who treated the damped anharmonic Duffing oscillator. Even the lowest (memory-free) approximation agrees quite well with corresponding computer experiments and much more elaborate perturbational (direct interaction) equations.[12] Bixon and Zwanzig were able to work out for this one-real-variable system the continued fraction for the memory kernel up to eighth order, with only moderate improvement from order to order. I had a similar experience with the two-real-variable system "laser": mode coupling seems to give better results then the continued fraction of some manageable order.

A general outline of the memory method in nonequilibrium systems as it will be used in this paper has been given by Zwanzig.[13] Lücke[14] applied it to explain the dynamical correlations of the Lorenz model, an open system with three real dynamical variables creating its own statistics by the nonlinearities.

So far these are the methodological aspects of the laser example. On the other hand explicit results may be of interest in themselves, since they shed some light on the photon dynamics of a laser, in particular on the importance of memory effects to reduce the field linewidth. Since the theory is very easy it is possible to get not only the Lorentzian linewidths versus pumping far beyond quasilinear theory[15] but also the laser line shapes themselves, showing appreciable deviations from the Lorentz profile in the proper threshold regime only. It is furthermore possible to treat similarly the multimode laser, which so far could be studied in quasilinear approximation only.[16]

The paper is organized as follows: First, dynamical variables and correlations are introduced. Then the general ideas of the memory-projector formalism are formulated for the laser system. The equal-time correlations are calculated by relations which apparently have not been realized before. Using these one can quantitatively determine all dynamical properties solely from the knowledge of the laser intensity. The high-frequency restoring matrix is discussed, which already describes the intensity fluctuations pretty well. The field correlations are essentially influenced by memory effects. The most important process is the decay of the field into another field plus a photon mode. Mode coupling and continued-fraction results are derived and compared. Details of the line shape are determined by solving the nonlinear coupled equations for the field and intensity spectral functions. Finally some modified approximations are discussed, which have been used in equilibrium many-body systems, where their effects or defects could not be checked.

II. EQUATIONS OF MOTION

Let us introduce reduced variables by

$$u=(\beta/q)^{1/4}b,\quad I=|u|^2=(\beta/q)^{1/2}|b|^2,\quad t=(\beta q)^{1/2}\times\text{time}. \tag{6}$$

The basic equation of motion then reads ($t\geq 0$):

$$\partial_t\rho(u,t)=[-\partial_i(p-I)u_i+\partial_i\partial_i]\rho(u,t)\equiv\mathfrak{D}\rho\ . \tag{7}$$

There is only one parameter left, representing the external pump rate

$$p=\gamma/(\beta q)^{1/2}=(\gamma_{\parallel}/\kappa N)^{1/2}(\sigma-\sigma_{\text{thr}})\ . \tag{6'}$$

The Fokker-Planck (FP) operator $\mathfrak{D}$ defines the stationary state by $\mathfrak{D}\rho_s(u)=0$,

$$\rho_s(u)\propto\exp(\tfrac{1}{2}p|u|^2-\tfrac{1}{4}|u|^4)\ . \tag{8}$$

The FP equation provides a Schrödinger-like description of the physical process, i.e., the state ρ is time dependent but the physical variables $A(u)$ are not. It is useful to introduce a Heisenberg-like description with time-dependent variables $A(u,t)$ but time-independent state

$$a(t)=\int d^2u\,A(u)\rho(u,t)=\int d^2u\,A(u,t)\rho(u,o)\ .$$

This is achieved with the adjoint Fokker-Planck operator $\mathfrak{D}^\dagger$,

$$\int d^2u\,A(u)\,[e^{\mathfrak{D}t}\rho(u,0)]=\int d^2u\,[e^{\mathfrak{D}^\dagger t}A(u)]\rho(u,0)\ .$$

Since $\mathfrak{D}^+$ determines the time dependence of the physical variables $A(u)$ as in classical mechanics, this adjoint operator is henceforth called the Liouvillian and denoted by $\mathfrak{L}$. Thus

$$\int d^2u\,A\mathfrak{D}\rho=\int d^2u\,(\mathfrak{L}A)\rho \tag{9}$$

defines $\mathfrak{L}$ and

$$A(t)=e^{\mathfrak{L}t}A,\quad \partial_t A(t)=\mathfrak{L}A(t) \tag{10}$$

describes its use and justifies its name "Liouvillian." $\mathcal{L}$ is different from $\mathfrak{D}$.

$$\mathcal{L}=(p-I)u_i\,\partial_i+\partial_i\,\partial_i\ . \tag{11}$$

Note that because of the last term $\mathcal{L}A^2 \neq 2A\mathcal{L}A$ in general. Therefore the product rule does *not* hold for ∂_t acting on dynamical variables.

The dynamical operators $\mathfrak{D}$ and $\mathcal{L}$ as well as the steady-state distribution ρ_s show the symmetries under the following operations: (i) permutation of components, $u_1 \rightleftharpoons u_2$; (ii) reflection of either component independently, $u_1 \rightleftharpoons -u_1$, u_2 fixed; (iii) $u_2 \rightleftharpoons -u_2$, u_1 fixed; and (iv) gauge (rotational) symmetry, $u \rightleftharpoons ue^{i\alpha}$. Both $\mathfrak{D}$ and $\mathcal{L}$ do not explicitly depend on time, expressing the stationarity property of the laser-light fluctuations.

III. DYNAMICAL CORRELATION FUNCTIONS

The macroscopically relevant dynamical variables are the field u_1,u_2 or $u,u^\dagger$ as well as the photon number $I=|u|^2$. Because of the symmetries there is only one independent field correlation function in the steady state, since $\langle u_1u_1(t)\rangle=\langle u_2u_2(t)\rangle$ and $\langle u_1u_2(t)\rangle=0$. Thus the correlation matrix is introduced as

$$C_{ij}(t)=(\delta A_i\,|\,\delta A_j(t))$$

with $\delta A_1=u$ and $\delta A_2=\delta I=I-\langle I\rangle$. We use

$$(A\,|\,B)=\int d^2u\,\rho_s(u)A^\dagger(u)B(u) \tag{12}$$

as an inner product. It is u orthogonal to δI as a result of the reflection symmetries [(ii), (iii)] of ρ_s. This is true also for $t\neq 0$. Hence $C_{ij}(t)$ is diagonal:

$$C_{ij}(t)=\begin{pmatrix}(u\,|\,u(t)) & 0\\ 0 & (\delta I\,|\,\delta I(t))\end{pmatrix}\equiv\begin{pmatrix}C_u(t) & 0\\ 0 & C_I(t)\end{pmatrix}. \tag{13}$$

The correlation functions $C(t)$ do not contain the entire time-development information but only its projection on the field or the intensity. Let us therefore introduce the projection operator $P=P^\dagger=P^2$,

$$P=|u)\frac{1}{\langle I\rangle}(u\,|+|\,\delta I)\frac{1}{\langle(\delta I)^2\rangle}(\delta I\,|\,. \tag{14}$$

The Laplace transform of the correlation matrix is the resolvent in the P subspace.

$$C_{ij}(w)=\int_0^\infty dt\,e^{-wt}\,C_{ij}(t)\,,\quad \text{holomorph in } w\geq 0\,,$$

$$=\left(\delta A_i\,\middle|\,\frac{1}{w-\mathcal{L}}\,\delta A_j\right). \tag{15}$$

If the equation of motion (7) described a conservative N-particle system (i.e., $\mathfrak{D}$ and $\mathcal{L}$ would be the Poisson bracket), $i\mathcal{L}$ would be Hermitian in the inner product (12). In open systems far from equilibrium this is *not* the case in general. In the particular case considered here, we even find $\mathcal{L}$ itself to be Hermitian:

$$(A\,|\,\mathcal{L}B)=(\mathcal{L}A\,|\,B)\,. \tag{16}$$

To prove this, the properties of ρ_s are important:

$$(A\,|\,\mathcal{L}B)=\int d^2u\,\rho_s\,A^\dagger\mathcal{L}B=\int d^2u\,(\mathfrak{D}\rho_sA^\dagger)B\,.$$

It is

$$\mathfrak{D}\rho_sA^\dagger=A^\dagger\mathfrak{D}\rho_s+\rho_s[-(p-I)u_iA^\dagger_{|i}+2(\ln\rho_s)_{|i}A^\dagger_{|i}+A^\dagger_{|i|i}]$$

$$=0+\rho_s\mathcal{L}A^\dagger=\rho_s(\mathcal{L}A)^\dagger.$$

Consequently, the poles of the resolvent are on the negative real axis. Hence the single-mode laser's correlations decay monotonically and cannot oscillate.

IV. RESOLVENT IDENTITY AND MEMORY-KERNEL REPRESENTATION

In the P subspace the resolvent $(w-\mathcal{L})^{-1}$ satisfies the resolvent identity

$$P(w-\mathcal{L})^{-1}P[w-\mathcal{L}_{PP}-\mathcal{L}_{PQ}(w-\mathcal{L}_{QQ})^{-1}\mathcal{L}_{QP}]=P\,, \tag{17}$$

where $Q=1-P$, the space of dynamical variables orthogonal to u, δI and $\mathcal{L}_{PP}\equiv P\mathcal{L}P$, etc. The $(\delta A_i|\cdots|\delta A_j)$ matrix element of (17) gives

$$C_{ij}(w)=N_{im}[wN-i\Omega-K(w)]^{-1}_{mn}N_{nj}\,. \tag{18}$$

This is the memory-kernel representation with

$$N_{ij}=(\delta A_i\,|\,\delta A_j)=C_{ij}(t=0) \tag{19}$$

the matrix of the stationary equal-time correlations (corresponding to the susceptibility);

$$i\Omega_{ij}=(\delta A_i\,|\,\mathcal{L}\delta A_j)=\dot{C}_{ij}(t=0)\,, \tag{20}$$

the high-frequency restoring matrix, also a stationary equal-time mean. Because of Eq. (16), $i\Omega_{ij}$ is a Hermitian matrix in the single-mode laser's dynamics. Finally, the memory matrix is

$$K_{ij}(w)=\left(Q\mathcal{L}\delta A_i\,\middle|\,\frac{1}{w-\mathcal{L}_{QQ}}\,Q\mathcal{L}\delta A_j\right). \tag{21}$$

It has the same resolvent structure in the Q subspace as $C_{ij}(w)$ had itself. Iterating the same procedure with $Q\mathcal{L}\delta A_i$, $(w-\mathcal{L}_{QQ})^{-1}$, etc., $K_{ij}(w)$ has a representation analogous to (18). This generates a continued-fraction representation of the correlation functions of interest.

Since a continued-fraction representation of a holomorphic function mathematically converges uniformly[17] with respect to w, any approximate calculation of the memory matrix will provide a uniform approximation of the correlation functions. Similarly, a continued-fraction of finite order for $K_{ij}(w)$ itself uniformly approximates K_{ij} as well as C_{ij}. This is the mathematical basis of the quality of the approximate calculations of the memory effects, given in the following sections.

It is useful to introduce the normalized correlation matrix $C^0_{ij}(t) = C_{im}(t)N^{-1}_{mj}$, being $C^0_{ij}(t=0) = \delta_{ij}$. This normalized matrix satisfies (from Eq. (18)

$$C^0_{ij}(w) = \left(\frac{1}{w - i\Omega^0 - K^0(w)}\right)_{ij}, \tag{22}$$

where

$$\Omega^0 = \Omega N^{-1}, K^0(w) = K(w)N^{-1}.$$

Note that the equation of motion of the matrix $C(t)$ [or $C^0(t)$] corresponding to Eq. (22) is

$$\partial_t C(t) = i\Omega^0 C(t) + \int_0^t dt' K^0(t')C(t-t'). \tag{23}$$

This is a linear equation but with memory $K^0(t')$.

V. EQUAL-TIME CORRELATIONS AND THE HIGH-FREQUENCY RESTORING MATRIX OF THE SINGLE-MODE LASER

Both matrices N_{ij} and $i\Omega_{ij}$ are diagonal because of reflection symmetry [(ii), (iii)]:

$$N_{ij} = \begin{pmatrix} \langle I\rangle & 0 \\ 0 & \langle(\delta I)^2\rangle \end{pmatrix}, \quad i\Omega_{ij} = \begin{pmatrix} (u\,|\,\mathcal{L}u) & 0 \\ 0 & (\delta I\,|\,\mathcal{L}\delta I) \end{pmatrix}. \tag{24}$$

From

$$\mathcal{L}u = (p - I)u, \tag{25a}$$

$$\mathcal{L}\delta I = 2(p - I)I + 4, \tag{25b}$$

it is found that

$$(u\,|\,\mathcal{L}u) = p\langle I\rangle - \langle I^2\rangle,$$

$$(\delta I\,|\,\mathcal{L}\delta I) = 2p\langle(\delta I)^2\rangle - \langle I^2\delta I\rangle.$$

These static moments can be calculated simply by using the steady-state distribution ρ_s from Eq. (8). Thus

$$K_0 = \ln \int_0^\infty dI \exp(\tfrac{1}{2}pI - \tfrac{1}{4}I^2)$$

$$= \tfrac{1}{4}p^2 + \ln(1 + \operatorname{erf}\tfrac{1}{2}p) + \ln\sqrt{\pi}, \tag{26}$$

with

$$\operatorname{erf} x = (2/\sqrt{\pi}) \int_0^x \exp(-t^2)\, dt,$$

is the partition function, generating the cumulants

$$K_1 = 2\partial_p K_0 = \langle I\rangle, \tag{27a}$$

$$K_2 = 2\partial_p K_1 = \langle(\delta I)^2\rangle, \tag{27b}$$

$$K_3 = 2\partial_p K_2 = \langle(\delta I)^3\rangle, \tag{27c}$$

$$K_4 = 2\partial_p K_3 = \langle(\delta I)^4\rangle - 3K_2^2, \tag{27d}$$

$$K_5 = 2\partial_p K_4 = \langle(\delta I)^5\rangle - 10K_2K_3, \quad \text{etc.} \tag{27e}$$

In particular,

$$K_1(p) = \langle I\rangle = p + \frac{2\exp(-\frac{1}{4}p^2)}{\sqrt{\pi}\,(1 + \operatorname{erf}\frac{1}{2}p)}. \tag{28}$$

Recent measurements[18] of $\langle I\rangle$ vs p have confirmed Eq. (28) for the single-mode laser. Far below threshold $K_1 \approx 2/|p|$ and far above threshold $K_1 \approx p$.

It may be useful to point out an easy method to find the higher cumulants, once $K_1(p)$ is known. It is applicable in other physical problems too, in which ρ_s might not be known explicitly. Note that

$$(1\,|\,\mathcal{L}\delta I) = 0,$$

whence $\langle 2(p - I)I + 4\rangle = 0$, implying the *exact* relation

$$K_2 = 2 + K_1(p - K_1). \tag{29}$$

Either from $(1\,|\,\mathcal{L}(\delta I)^2) = 0$ or from $K_3 = 2\partial_p K_2$ one finds the exact relations

$$K_3 = 2K_1(1 - K_2) + pK_2, \tag{30}$$

$$K_4 = 2K_2(2 - K_2) - K_3(2K_1 - p), \tag{31}$$

etc. These relations express the time independence of the steady state moments, e.g. $\partial_t\langle I\rangle = 0$. They allow the elimination of all higher-order equal-time correlations in favor of K_1 (and p) and apparently have not been realized before. Hence all steady-state equal-time means are available once $K_1(p)$ is known.

The high-frequency restoring force can be simplified with the relations (29)–(31):

$$i\Omega_{ij} = \begin{pmatrix} -2 & 0 \\ 0 & -4K_1 \end{pmatrix}.$$

The corresponding normalized matrix is

$$i\Omega^0_{ij} \equiv \begin{pmatrix} -\lambda_{0u} & 0 \\ 0 & -\lambda_{0I} \end{pmatrix}$$

with

$$\lambda_{0u} = 2/K_1, \quad \lambda_{0I} = 4K_1/K_2. \tag{32}$$

If one neglects memory effects, the λ_0 are the bare widths of a Lorentzian-shaped time correlation for the field u and photon intensity δI. λ_{0I}, which can also be written as $4K_1 - 2p + 2K_3/K_2$, agrees with Risken's 1966 result[2] calculated by a variational procedure for the lowest eigenvalue of the Fokker-Planck equation and it also agrees exactly with our[15] earlier formula, found by Stratonovich's decoupling of higher-order correlations. λ_{0I} vs p is already rather close to the experimental curve measured by Arecchi *et al.*[5] Appreciable deviations are in the proper threshold region. Here the memory effects are non-negligible.

λ_{0u} also agrees qualitatively with the experimental findings,[6] being $\approx |p|$ far below threshold and $\sim\langle I\rangle^{-1}$ above, thus showing the decrease of the laser linewidth with increasing intensity. However, the linewidth factor $\alpha = \lambda_u \langle I\rangle$ corresponding to λ_{0u} is $\alpha_0 = 2$ below, at, and above threshold. Its monotonic decrease from 2 below threshold to 1 above threshold due to the reduction of the effective degrees of freedom (which above threshold is the phase motion only) is missing. This therefore must be a memory effect.

VI. SPECTRAL FUNCTION AND SPECTRAL REPRESENTATION

In conservative N-particle systems the singularities of $C_{ij}(w)$ are on the imaginary axis, since $i\mathcal{L}$ is real. In the laser considered here $\mathcal{L}$ itself is real. Thus $C_{ij}(w)$ has its singularities on the negative real axis. In other open systems far from equilibrium the singularities may be somewhere in the left half-plane. Nevertheless a spectral density may be introduced, although not as a discontinuity between a $C^{>}_{ij}$ and a $C^{<}_{ij}$. $C_{ij}(t)$ is defined only in forward direction, $t \geq 0$.

Consider a real correlation matrix $C_{ij}(t)$ with $C_{ij}(t\to\infty)\to 0$. Define

$$C''_{ij}(\omega) = 2\int_0^\infty dt\, C_{ij}(t)\cos\omega t\,, \quad \text{real, even in } \omega\,, \tag{33}$$

$$C'_{ij}(\omega) = 2\int_0^\infty dt\, C_{ij}(t)\sin\omega t\,, \quad \text{real, odd in } \omega. \tag{34}$$

Then along the imaginary axis

$$C_{ij}(w = i\omega + \eta) = \frac{1}{2}C''_{ij}(\omega) + \frac{1}{2i}C'_{ij}(\omega)\,. \tag{35}$$

The function

$$C''_{ij}(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega t} C''_{ij}(\omega)$$

is equal to $C_{ij}(t)$ for $t \geq 0$ and defines an even continuation for $t \leq 0$. Using C''_{ij} one gets a spectral representation for the complex correlation function:

$$C_{ij}(w) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i}\frac{C''(\omega)}{\omega - iw}\,, \quad \mathrm{Re}\, w \geq 0\,. \tag{36}$$

If $w \to i\omega + \eta$ one finds the dispersion relation

$$C'_{ij}(\omega) = -\int_{-\infty}^{+\infty} \frac{d\omega'}{\pi}\frac{C''_{ij}(\omega')}{\omega' - \omega}\,. \tag{37}$$

Because of its analogous structure the memory matrix can be described by similar formulas. Thus

$$K_{ij}(w = i\omega + \eta) = \frac{1}{2}K''_{ij}(\omega) + \frac{1}{2i}K'_{ij}(\omega), \text{ etc.} \tag{38}$$

VII. THE MEMORY MATRIX

Using the reflection symmetry (ii), (iii) one finds the $(u, \delta I)$ elements of $K_{ij}(w)$ to be zero, i.e.,

$$K_{ij}(w) = \begin{pmatrix} K_u(w) & 0 \\ 0 & K_I(w) \end{pmatrix}\,. \tag{39}$$

Thus there are two separate memory representations for the field and for the intensity:

$$C^0_u(w) = [w + \lambda_{0u} - K^0_u(w)]^{-1}, \tag{40}$$

$$C^0_I(w) = [w + \lambda_{0I} - K^0_I(w)]^{-1}. \tag{41}$$

The bare widths $\lambda_{0u}, \lambda_{0I}$ are given by (32), with

$$K^0_u(w) = \left(Q\mathcal{L}u\left|\frac{1}{w - \mathcal{L}_{QQ}}Q\mathcal{L}u\right)\right/(u|u), \tag{42}$$

$$K^0_I(w) = \left(Q\mathcal{L}\delta I\left|\frac{1}{w - \mathcal{L}_{QQ}}Q\mathcal{L}\delta I\right)\right/(\delta I|\delta I). \tag{43}$$

The u and δI dynamics are coupled, nevertheless, since $Q\mathcal{L}u$ has a component in the two-mode subspace, $\sim u\delta I$; see Eq. (25a). From Eq. (25b) on the other hand one concludes that the most important contribution to $Q\mathcal{L}\delta I$ is $(\delta I)^2$, i.e., the state with two δI modes.

VIII. TWO-MODE COUPLING

As we have just learned, the most important contributions to the $Q\mathcal{L}\delta A_i$ dynamics are two-mode states. This suggests the following approximation of the memory kernels K_u, K_I: Instead of the whole Q space only the two-mode subspace is considered, and within this the dynamics of the two modes is taken as being independent because of the Q-projected Liouvillian [$\mathcal{L}_{QQ}$ instead of $\mathcal{L}$ in (42), (43)].

There are the following six possible two-mode states: $uu, u^\dagger u^\dagger, uu^\dagger; u\delta I, u^\dagger\delta I; \delta I\delta I$. But uu and $u^\dagger u^\dagger$ do not contribute to K_u or K_I (use reflection or gauge symmetry). $uu^\dagger$ is $\delta I + K_1$ which is projected out by Q. $u\delta I$ violates reflection symmetry in K_I but contributes to K_u. $u^\dagger\delta I$ is not gauge invariant. $(\delta I)^2$ only overlaps with $Q\mathcal{L}\delta I$. So only two product states effectively contribute, one solely in K_u, the other one in K_I alone.

Since $u\delta I$ is orthogonal to $(\delta I)^2$, the two-mode projector can be written

$$P_2 = |u\delta I)\frac{1}{\langle I(\delta I)^2\rangle}(u\delta I| + |(\delta I)^2)\frac{1}{\langle(\delta I)^4\rangle}((\delta I)^2|. \tag{44}$$

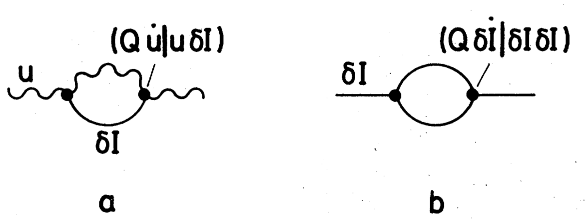

FIG. 1. (a) Decay of the amplitude u into an amplitude mode and an intensity mode. (b) The intensity δI decays into two intensity modes.

The two-mode contribution to $K_u^0(w)$ is

$$(u|u)K_u^0(w) = (Q\mathcal{L}u|u\delta I)\frac{1}{(u\delta I|u\delta I)}\left(u\delta I\left|\frac{1}{w-\mathcal{L}_{QQ}}\right|u\delta I\right) \times\frac{1}{(u\delta I|u\delta I)}(u\delta I|Q\mathcal{L}u).$$

It is represented by the graph Fig. 1(a). The dynamics are now factorized, but *not* the static vertices like $(Q\mathcal{L}u|u\delta I)$, etc., thus retaining the *exact* equal-time correlations,

$$\frac{\left(u\delta I\left|\frac{1}{w-\mathcal{L}_{QQ}}u\delta I\right.\right)}{(u\delta I|u\delta I)} = \int_0^\infty dt\,\exp(-wt)\frac{(u\delta I|\exp(\mathcal{L}_{QQ}t)u\delta I)}{(u\delta I|u\delta I)} \approx \int_0^\infty dt\,\exp(-wt)C_u^0(t)C_I^0(t). \quad (45)$$

The mode-coupling approximation now reads

$$K_u^0(w) = B_u\int\frac{dw_1}{2\pi i}C_u^0(w_1)C_I^0(w-w_1). \quad (46)$$

The memory strength B_u is a steady-state equal-time mean, which can be calculated exactly from ρ_s:

$$B_u = \frac{(Q\mathcal{L}u|u\delta I)(u\delta I|Q\mathcal{L}u)}{(u\delta I|u\delta I)(u|u)} = \frac{(K_1K_2+K_3-K_2^2/K_1)^2}{(K_1K_2+K_3)K_1} = \frac{4(K_1^2-K_2)^2}{K_1^3[2p+(p-K_1)^2K_1]}. \quad (47)$$

Analogously one finds a two-mode decay of the intensity mode [(see Fig. 1(b)] and

$$K_I^0(w) = \frac{B_I^2}{4}\int\frac{dw_1}{2\pi i}C_I^0(w_1)C_I^0(w-w_1) \quad (48)$$

with

$$\frac{B_I^2}{4} = \frac{(Q\mathcal{L}\delta I|\delta I^2)(\delta I^2|Q\mathcal{L}\delta I)}{(\delta I|\delta I)(\delta I^2|\delta I^2)}.$$

Again, this memory strength can be expressed in terms of the equal-time cumulants:

$$B_I = \frac{4(2K_2^2+K_4-K_3^2/K_2)}{[K_2(K_4+3K_2^2)]^{1/2}} = \frac{8(2K_2^2-K_1K_3)}{K_2^{3/2}[4K_2+K_2^2+K_3(p-2K_1)]^{1/2}}. \quad (49)$$

Far below and far above threshold $K_3=K_4=0$. Then $B_I=8(K_2/3)^{1/2}$, i.e., zero far below and $16/\sqrt{6}$ far above threshold. The field vertex is outside the proper threshold regime $B_u=K_2(1-K_2/K_1^2)^2$, i.e., $B_u=2$ far above threshold and zero below.

IX. LASER LINEWIDTHS IN THE LORENTZIAN APPROXIMATION

If the shape of the field as well as the intensity correlation function is considered to be nearly Lorentzian, the only parameters are the linewidths λ_u and λ_I. They have to be determined from Eqs. (40) and (41) together with (46) and (48):

$$\frac{1}{w+\lambda_u} \approx \frac{1}{w+\lambda_{0u}-B_u/(w+\lambda_u+\lambda_I)}, \quad (50a)$$

$$\frac{1}{w+\lambda_I} \approx \frac{1}{w+\lambda_{0I}-(B_I^2/4)/(w+2\lambda_I)}. \quad (50b)$$

The poles on the left-hand side and on the right-hand side shall coincide, $w=-\lambda_u$ and $w=-\lambda_I$, respectively:

$$\lambda_u = \lambda_{0u} - B_u/\lambda_I, \quad (51a)$$

$$\lambda_I = \lambda_{0I} - (\tfrac{1}{2}\lambda_{0I})\{1-[1-(B_I/\lambda_{0I})^2]^{1/2}\}. \quad (51b)$$

Thus λ_u can be found once λ_I has been calculated. λ_I is that solution of a quadratic equation which coincides with λ_{0I} far below threshold, where the nonlinearities are small and so there are no memory effects. (The existence of spurious solutions of the severely nonlinear mode coupling integral equations is a known effect[9]; in numerical procedures one may have to switch on the memory strength slowly from zero to its proper magnitude[9] in order to avoid spurious solutions.)

Using the static cumulants[19] $K_1,\ldots,K_4$, λ_{0u}, λ_{0I} were calculated [see Eq. (32)] as well as B_u, B_I

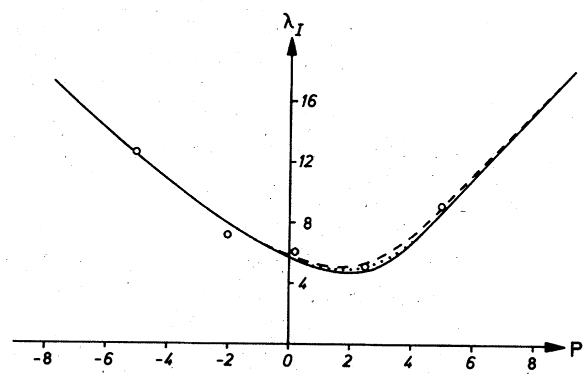

FIG. 2. Linewidth of photon number fluctuations. Full curve: mode-coupling approximation; dotted curve: continued-fraction results; dashed curve: Risken's λ_{eff}. ○, experimental values by Arecchi *et al.*

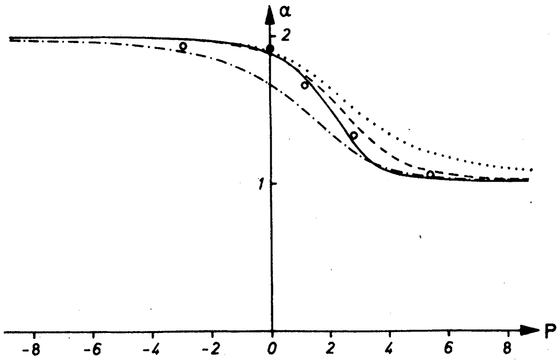

FIG. 3. Linewidth factor $\alpha = \lambda_u \langle I \rangle$ from mode-coupling theory (solid curve) and continued-fraction (dotted curve) compared with Risken's curve (dashed), Mandel and Nguyen Dinh's results, and experimental data (O) by Gerhardt *et al.*

[see Eqs. (47) and (49)] versus pumping p and from (51) the Lorentzian linewidth. The results (Figs. 2 and 3) agree rather well with Risken's[2] computer curves and with the experimental values.[5,6] For comparison the approximate linewidth factor given by Mandel and Nguyen Dinh[20] is indicated too.

X. LINE SHAPE

Inserting Eqs. (46) and (48) for the memory kernels into Eqs. (40) and (41) one gets two coupled equations for the normalized relaxation functions $C_u^0(w), C_I^0(w)$. The Lorentzian approximations to these are Eqs. (50), yielding the linewidths λ_u, λ_I. One can, of course, solve the coupled equations without further approximation and get the line shape within the two-mode coupling approximation of the memory.

Starting with some $C_I^{0\prime\prime}(\omega)$ (in fact the Lorentzian was chosen) first determine

$$K_I^{0\prime\prime}(\omega) = \frac{B_I^2}{4} \int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi}\, C_I^{0\prime\prime}(\omega_1) C_I^{0\prime\prime}(\omega - \omega_1). \quad (52)$$

The imaginary part of K_I^0 is the dispersion integral,

$$K_I^{0\prime}(\omega) = -\int_{-\infty}^{+\infty} \frac{d\omega_1}{2\pi}\, \frac{K_I^{0\prime\prime}(\omega_1)}{\omega_1 - \omega}. \quad (53)$$

From Eq. (41) then

$$C_I^{0\prime\prime}(\omega) = \frac{2\lambda_{0I} - K_I^{0\prime\prime}(\omega)}{[\omega + \frac{1}{2}K_I^{0\prime}(\omega)]^2 + [\lambda_{0I} - \frac{1}{2}K_I^{0\prime\prime}(\omega)]^2}. \quad (54)$$

Iterating this scheme yields the spectral function for the intensity correlations. The input is the known static vertices B_I, λ_{0I} and the Lorentzian width λ_I for different pumping rates p. Some numerical results are shown in Fig. 4.

An analogous iteration gives the spectral func-

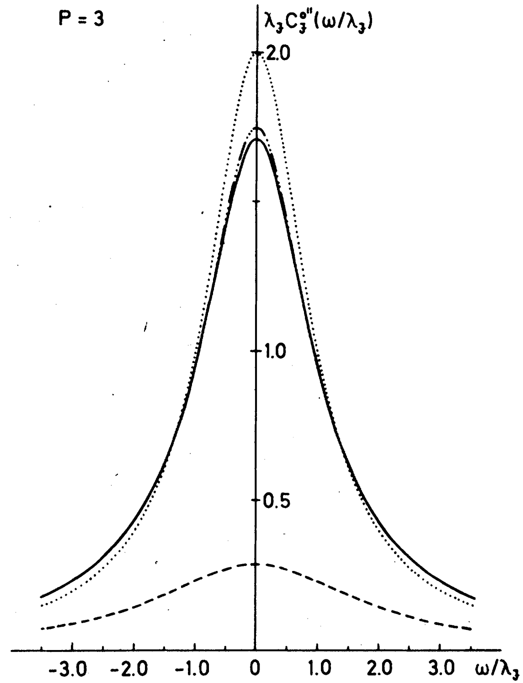

FIG. 4. Intensity fluctuation line shape for $p = 3$. Dotted line: Lorentzian; dashed-dotted line: first iteration; full line: second iteration; and dashed line: the memory spectral function.

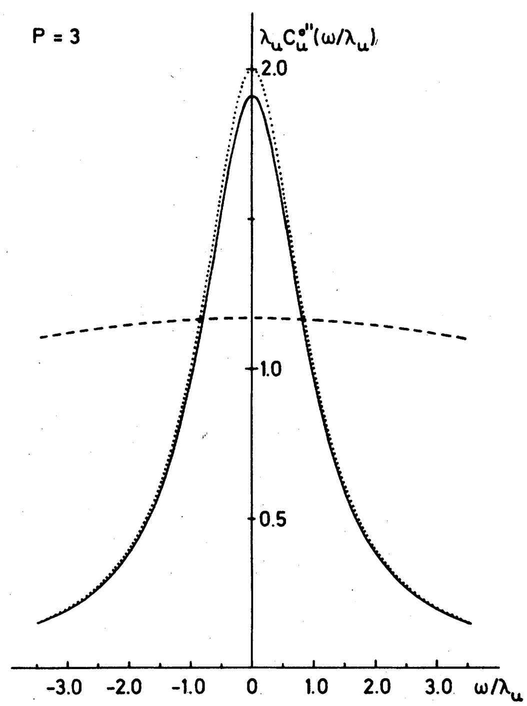

FIG. 5. Spectral line shape of field fluctuations for $p = 3$. Dotted line: Lorentz profile; full line: first iteration; dashed line: memory spectral function.

tion of the field itself:

$$K_u^{0\prime\prime}(\omega)=B_u\int_{-\infty}^{+\infty}\frac{d\omega_1}{2\pi}C_u^{0\prime\prime}(\omega_1)C_I^{0\prime\prime}(\omega-\omega_1) \quad (55)$$

is the memory kernel spectral function. $K_u^{0\prime}(\omega)$ is the dispersion integral of it:

$$C_u^{0\prime\prime}(\omega)=\frac{2\lambda_{0u}-K_u^{0\prime\prime}(\omega)}{[\omega+\frac{1}{2}K_u^{0\prime}(\omega)]^2+[\lambda_{0u}-\frac{1}{2}K_u^{0\prime\prime}(\omega)]^2}\,. \quad (56)$$

The input now is $B_u, \lambda_{0u}, \lambda_u$ Lorentzian, and $C_I^{0\prime\prime}(\omega)$ as calculated before. Figure 5 shows that in spite of the large memory contribution the deviations of the spectral lineshape are $\approx 10\%$ only.

It should be remarked that $C_u^{0\prime\prime}(\omega)$ and $C_I^{0\prime\prime}(\omega)$ just found from mode coupling agree very well with the spectral functions according to the continued fraction of first order for $K_{u,I}$ (c.f. Sec. XII).

XI. COMMENTS ON MODIFIED APPROXIMATIONS

There are some *ad hoc* assumptions in the two-mode coupling approximation which cannot be checked systematically. It seems useful to consider reasonable modifications.

(i) One might factorize the two-mode projector's normalization

$$((\delta I)^2\,|\,(\delta I)^2)\approx\langle(\delta I)^2\rangle\langle(\delta I)^2\rangle$$

in Eq. (44). This amounts to using K_2^2 instead of $3K_2^2+K_4$ in the denominator of (49). This reduction increases the memory strength B_I so much that λ_I from Eq. (51b) becomes complex in the proper threshold region, disqualifying this simple factorization of the equal-time correlations.

One might prefer $((\delta I)^2\,|\,(\delta I)^2)\approx 2K_2^2$. The argument is more consistent with the natural *dynamical* factorization

$$(\delta I\delta I\,|\,\delta I(t)\delta I(t))\approx 2C_I(t)C_I(t)\,.$$

This also increases the memory strength B_I, yielding too small a linewidth (e.g., $\lambda_I=3.84$ if $p=2$; 4.03 if $p=3$; $\approx 30\%$ too small).

There is, of course, an ambiguity in any factorization. (Statically even $3K_2^2$ is reasonable, while dynamically $2K_2^2$ is to be preferred.) I have circumvented this by factorizing the *normalized* dynamical correlation but taking into account all remaining equal time correlations *exactly*, although this amounts to a different treatment of the two-mode state's normalization denominators [e.g., in Eq. (45) compared with Eq. (47)].

(ii) There is some ambiguity in the definition of the two-mode states, too. I used the simple product state $(\delta I)^2$. Another reasonable state would be $\delta[(\delta I)^2]=(\delta I)^2-\langle(\delta I)^2\rangle$. It has the same overlap (vertex) with $Q\mathfrak{L}\delta I$ as $(\delta I)^2$ itself. [By the way, $\delta(u\delta I)=u\delta I$.] Therefore only the denominator of P_2 [c.f. Eq. (44)] changes and consequently that of B_I: substitute $K_4+2K_2^2$ instead of $K_4+3K_2^2$ in formula (49). This again leads to an increase of the memory strength, resulting in linewidths that are too small. [Even asymptotically: one finds $B_I=8$, so a reduction of λ_I of 2.1% ($p=10$) or 3.5% ($p=8$).]

(iii) An argument in favor of $\delta[(\delta I)^2]$ is that it is orthogonal to 1; an argument against it is that it already contains two-mode correlations (K_2), so that its dynamics hardly should factorize. Thus one expects the failure of another reasonable two-mode state, $Q\delta[(\delta I)^2]$. It is orthogonal to $1, u, \delta I$. From (25b) and (29) one concludes that

$$Q\delta[(\delta I)^2]=-\tfrac{1}{2}Q\mathfrak{L}\delta I\,. \quad (57)$$

Hence this state has full overlap with $Q\dot{\delta I}$, yielding the maximum possible memory strength

$$B_I\to\overline{B}_I=2\left[\frac{(Q\mathfrak{L}\delta I\,|\,Q\mathfrak{L}\delta I)}{(\delta I\,|\,\delta I)}\right]^{1/2}. \quad (58)$$

Note that it represents the exact high-frequency limit of the memory kernel.

$\overline{B}_I$ is systematically larger than B_I,

$$\left(\frac{\overline{B}_I}{B_I}\right)^2=\frac{3K_2^2+K_4}{2K_2^2+K_4-K_3^2/K_2}\,; \quad (59)$$

thus λ_I is systematically too small. In the proper threshold region λ_I even gets complex (e.g., at $p=3$) far above threshold, $\overline{B}_I=8$, so the linewidth coincides with that of (ii). Analogously $Qu\delta I$ ($=-Q\mathfrak{L}u$) yields an increase in the field memory strength B_u corrresponding to a linewidth factor which is systematically too small:

$$\frac{\overline{B}_u}{B_u}=\frac{K_1K_2+K_3}{K_1K_2+K_3-K_2^2/K_1}>1\,. \quad (60)$$

XII. CONTINUED FRACTION OF THE MEMORY KERNELS

A systematic way to calculate the memory effects is to repeat the procedure of Sec. IV and to deduce a representation of $K_{u,I}(w)$ with the same structure as Eq. (18) provides for the correlation itself. This yields a continued-fraction representation of $K_{u,I}(w)$. It is free of the ambiguities of the mode-coupling approximation, but soon gets rather elaborate.

Let us consider the first step only. It qualitatively gives correct linewidths which, of course, is not surprising for the intensity fluctuations, since for these the memory effects altogether are moderate only. But also the lowest-order continued fraction of the field memory kernel already reduces the linewidth factor α from 2 below threshold to 1 far above; α vs p quantitatively differs only slightly from the mode-coupling results.

The continued fraction of the field memory kernel $K_u(w)$ and the intensity kernel $K_I(w)$ is found by comparing

$$K(w)=(1/w)(Q\mathcal{L}\delta A\,|\,1+\mathcal{L}_{QQ}/w+(\mathcal{L}_{QQ}/w)^2+\cdots\,|\,Q\mathcal{L}\delta A) \qquad (61)$$

with

$$K(w)=\frac{a}{w-b-[c/(w-\cdots)]} \qquad (62)$$

asymptotically for $w\to\infty$. One gets

$$a=(Q\mathcal{L}\delta A\,|\,Q\mathcal{L}\delta A),\ \text{real, positive,} \qquad (63a)$$

$$ab=(Q\mathcal{L}\delta A\,|\,\mathcal{L}Q\mathcal{L}\delta A),\ \text{etc.} \qquad (63b)$$

The Lorentzian linewidth is found as in the formulas (50) by determining the poles from

$$-\lambda+\lambda_0-\frac{a/N}{-\lambda-b-\cdots}=0\,.$$

In lowest order, $c=0$, one gets

$$\lambda=\lambda_0+\frac{b+\lambda_0}{2}\left[\left(1+\frac{4a/N}{(b+\lambda_0)^2}\right)^{1/2}-1\right]. \qquad (64)$$

λ_0 is diminished by the memory effects $\sim a$, since $b<0$ (as $\mathcal{L}$ is real with negative spectrum). a is equal to zero in the linear regime below threshold. λ is always real (as it should because $\mathcal{L}$ is Hermitian), for the radicand is always larger than 1.

a. Field amplitude memory

Putting $\delta A=u$ one finds [with relations (30), (29)]

$$a_u=(Q\mathcal{L}u\,|\,Q\mathcal{L}u)=2K_1-2K_2/K_1\,. \qquad (65)$$

Far below threshold $K_2\approx K_1^2$, so $a_u=0$. Far above threshold $a_u\to 2p$, indicating an increasing memory of the field decay with increased pumping.

While a_u originally depends on the triple cumulant K_3, the kernel's restoring force b_u even contains K_4. One finds a lengthy expression which by repeated application of the again useful relations (31), (30), and (29) can be reduced to

$$b_u=-4(pK_1+2+K_2)/a_u+K_2/K_1\,. \qquad (66)$$

Far above threshold one gets $b_u\to -2p$. Together with $N_u=K_1\to p$ and $\lambda_{0u}=2/K_1\to 2/p$ the spectral linewidth for $p\gg 1$ is

$$\lambda_u=2/p-p\,[(1+2/p^2)^{1/2}-1]\approx 1/p\,. \qquad (67)$$

Indeed, the memory contribution reduces the bare linewidth factor from 2 to 1. The precise form of α vs p is given in Fig. 3.

b. Intensity memory

Putting $\delta A=\delta I$ from (63a) one gets

$$\begin{aligned}a_I&=4(2K_2^2+K_4-K_3^2/K_2)\\&=8(2K_2^2-K_1K_3)/K_2\\&=8(pK_1+4)-16K_1^2/K_2\,.\end{aligned} \qquad (68)$$

One has to use repeatedly the relations (29)–(31) to reduce the higher cumulants to the lower ones, until finally $K_1(p)$ is needed only. The asymptotic memory strength is $a_I\to 0$ far below threshold and $a_I\to 32$ far above it.

a_Ib_I according to (63b) is a lengthy expression, containing cumulants up to K_5. From Eq. (31) one finds

$$K_5=K_4(p-2K_1)+6K_3(1-K_2)\,. \qquad (69)$$

A tedious but straightforward calculation yields

$$\begin{aligned}b_I&=-32(2K_1K_2+K_3)/a_I+2K_3/K_2\\&=-32(2K_1+pK_2)/a_I+2p+4K_1(1-K_2)/K_2\,.\end{aligned} \qquad (70)$$

Asymptotically $b_I=-4p$ far above threshold. Together with $\lambda_{0I}=2p$ and $N_I=K_2=2$, expression (64) yields $\lambda_I=19.230$ for $p=10$, 15.056 for $p=8$, etc., being a bit smaller than the mode-coupling linewidth (about 1%); within the proper threshold regime it is slightly above (see Fig. 2).

We conclude that already the lowest-order continued fraction for the memory, which is free of any ambiguities, yields quite satisfactory results throughout the laser threshold, where the nonlinearities are quite essential. It has already been noted that the line-shape details of the continued-fraction approximation agree very closely with those of the mode-coupling approximation.

In conclusion it may be stated that the laser dynamics are quite well understood, if only the intensity $\langle I\rangle$ vs pumping p is known (which implies the exact knowledge of all steady-state cumulants). The statics has to be used exactly to get a dynamics which already contains renormalized vertices and propagators.[21]

ACKNOWLEDGMENTS

I would like to thank Dr. E. Schnedler, who checked the calculations using the reduction relations and who prepared the figures, and Frau B. Sonneborn-Schmick, who made the line-shape calculations. A preliminary report has been presented at STATPHYS 13 in Haifa.

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