Asymptotic form of the third Born amplitude for forward electron capture by a bare ion incident on a hydrogenlike ion

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The nonrelativistic asymptotic behavior of the third Born amplitude is determined for electron capture from a hydrogenlike ion or atom by a bare ion incident with a very high velocity. It is assumed that the incident ion is scattered through only a narrow range of angles, of the order of the electron/proton mass ratio. The result for the asymptotic form of the amplitude differs from two previous results. However, for the asymptotic contibution of the amplitude to the forward cross section (i.e., to the cross section integrated over the narrow forward cone) I obtain the same result as obtained previously.

I. INTRODUCTION

In this note we examine, within the nonrelativistic framework, the asymptotic form of the third Born amplitude for the capture of an electron from a hydrogenlike ion or atom by a bare ion that is incident with a very high velocity. We assume that the incident ion is scattered through only a narrow range of angles, of the order of the electron/proton mass ratio. This restriction to scattering into a narrow forward cone is not unduly severe since the differential cross section for scattering outside the cone is porportional to the square of the electron/proton mass ratio, and is therefore small.[1] In effect, we are neglecting corrections of the order of the electron/proton mass ratio. For consistency, we therefore neglect the internuclear potential, since when corrections of the above order are neglected, the internuclear potential depends only on the coordinate connecting the incident ion to the center of mass of the target "atom," and it cannot then effect the internal state of the "atom."

The asymptotic behavior of the first and second Born amplitudes has been examined in detail by Dettmann and Leibfried.[2,3] It is now well established that for sufficiently high impact velocities the contribution from the second Born amplitude to the forward capture cross section (i.e. to the cross section integrated over the narrow forward cone) dominates over the contribution from the first Born amplitude, a result which was first proved in a remarkable thesis by Drisko.[4] For example, in the case of ground-state to ground-state capture the first Born contribution decreases as $1/v^{12}$ with increasing impact velocity v, whereas the second Born contribution decreases only as $1/v^{11}$. The asymptotic behavior of the *third* Born amplitude, and its contribution to the forward capture cross section, is therefore of considerable interest. The third Born amplitude was, in fact, examined for the case of ground-state to ground-state capture by both Drisko[4] and Dettmann.[3] However, their expressions for the asymptotic form of this amplitude differ significantly. Nevertheless, their resulting estimates of the asymptotic contribution of the third Born amplitude to the forward capture cross section are the same! Now Drisko's analysis, while it is reasonably convincing, lacks proper justification and, as Drisko himself states, his result for the asymptotic contribution of the third Born amplitude "cannot be completely credited." Dettmann's analysis also involves an unjustified approximation, one which we discuss in detail below. In fact, the outcome of the more rigorous analysis presented here is that both Drisko's and Dettmann's estimates of the asymptotic form of the third Born amplitude are incorrect. However, their estimates of the asymptotic contributions of the third Born amplitude to the forward capture cross section are correct!

We restrict the discussion of this paper to ground-state to ground-state capture. However, the methods used here can be readily generalized to arbitrary initial and final states.

II. NOTATION

Let m, M_A, and M_B denote the masses of the electron, target nucleus, and incident nucleus, respectively. We refer to the particles by their masses. We define the mass ratios

$$\alpha = M_A/(M_A+m), \quad \beta = M_B/(M_B+m),$$

and the reduced masses

$$\mu_A = mM_A/(M_A+m), \quad \mu_B = mM_B/(M_B+m),$$
$$\nu_A = M_B(M_A+m)/(M_A+M_B+m),$$
$$\nu_B = M_A(M_B+m)/(M_A+M_B+m).$$

We denote the interactions of m with M_A and m with M_B by $W_A(r_A)$ and $W_B(r_B)$, respectively; the coordinate system is defined in Fig. 1. Let e, $-Z_Ae$, and $-Z_Be$ denote the charges of m, M_A, and M_B, respectively. We have

$$W_A(r_A) = -Z_A e^2/r_A, \quad W_B(r_B) = -Z_B e^2/r_B.$$

Initially, m is bound to M_A in the state i, characterized by a wave function $\phi_i(\vec{r}_A)$, and M_B is incident with a velocity $\vec{v}_i$ relative to $(m+M_A)$; finally m is bound to M_B in the state f, characterized by a wave function $\phi_f(\vec{r}_B)$, and $(m+M_B)$ has a velocity $\vec{v}_f$ relative to M_A. Let ϵ_i and ϵ_f denote the initial and final bound-state energies. Assuming i and f are both ground states, we have, omitting subscripts,

$$\epsilon = \frac{-Z^2e^2}{2a_0}, \quad \phi = \left(\frac{Z^3}{\pi a_0^3}\right)^{1/2} \exp\left(\frac{-Zr}{a_0}\right), \tag{2.1}$$

where $a_0 = \hbar^2/me^2$ is the Bohr radius of the hydrogen atom. We work in the center-of-mass frame of all three particles. In this frame the initial and final wave functions are

$$\begin{aligned} \psi_i &= \exp(i\vec{K}_i \cdot \vec{R}_A)\phi_i(\vec{r}_A), \\ \psi_f &= \exp(i\vec{K}_f \cdot \vec{R}_B)\phi_f(\vec{r}_B), \end{aligned} \tag{2.2}$$

where $\hbar\vec{K}_i = \nu_A\vec{v}_i$ and $\hbar\vec{K}_f = \nu_B\vec{v}_f$. If E denotes the total energy of the system in the center-of-mass frame, we have

$$E = (\hbar^2/2\nu_A)K_i^2 + \epsilon_i = (\hbar^2/2\nu_B)K_f^2 + \epsilon_f. \tag{2.3}$$

We define the momentum-transfer vectors

$$\vec{K} = \beta\vec{K}_f - \vec{K}_i, \quad \vec{J} = \alpha\vec{K}_i - \vec{K}_f; \tag{2.4}$$

conservation of energy can also be written as

$$\hbar^2K^2/\beta - \hbar^2J^2/\alpha = 2m(\epsilon_f - \epsilon_i), \tag{2.5}$$

where $K = |\vec{K}|$ and $J = |\vec{J}|$. Note that

$$\hbar\vec{J} = -m\vec{v}_i - \hbar\vec{K}/\beta, \quad \hbar\vec{K} = -m\vec{v}_f - \hbar\vec{J}/\alpha. \tag{2.6}$$

The cross section for the process under consideration is[5]

$$\sigma = \frac{1}{2\pi\beta\hbar^2 v_i^2}\frac{\nu_B}{\nu_A}\int_{K_{\min}}^{K_{\max}} |\mathcal{T}|^2 K\,dK, \tag{2.7}$$

where $K_{\min} = |\beta K_f - K_i|$ and $K_{\max} = \beta K_f + K_i$ and where, omitting the interaction between the nuclei and keeping only the first three Born terms,

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3,$$

with

$$\begin{aligned} \mathcal{T}_1 &= \langle\psi_f | W_A | \psi_i\rangle, \\ \mathcal{T}_2 &= \langle\psi_f | W_A G_0^+(E) W_B | \psi_i\rangle, \\ \mathcal{T}_3 &= \mathcal{T}_{3A} + \mathcal{T}_{3B}, \\ \mathcal{T}_{3A} &= \langle\psi_f | W_A G_0^+(E) W_A G_0^+(E) W_B | \psi_i\rangle; \end{aligned} \tag{2.8}$$

the definition of $\mathcal{T}_{3B}$ should be obvious. Here, $G_0^+(E)$ is the Green's function for three noninteracting particles with total energy $E + i\eta$, where η is positive and infinitesimal.

The Fourier transform of any function $f(\vec{r})$ is denoted by $\tilde{f}(\vec{k})$ where

$$\tilde{f}(\vec{k}) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3r \exp(-i\vec{k}\cdot\vec{r}) f(\vec{r}). \tag{2.9}$$

When we write $a \sim b$, we mean that the relative difference of a and b is of order $1/v$, where v is equal to $v_i \equiv |\vec{v}_i|$ or $v_f \equiv |\vec{v}_f|$. When we write $a \approx b$, we mean that the relative difference of a and b is of order unity.

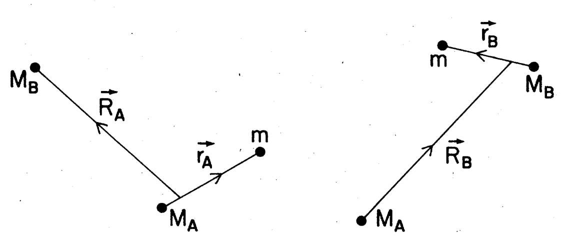

FIG. 1. Initially m is bound to M_A. The coordinate of m relative to M_A is $\vec{r}_A$ and the coordinate of the incident nucleus M_B relative to center of mass of the target "atom" $(m+M_A)$ is $\vec{R}_A$. Finally m is bound to M_B. The coordinate of m relative to M_B is $\vec{r}_B$ and the coordinate of the center of mass of the outgoing "atom" $(m+M_B)$ relative to the stripped nucleus M_A is $\vec{R}_B$.

III. PRELIMINARY ANALYSIS

The first two Born terms can be expressed in the form[3]

$$\mathcal{T}_1 = -(2\pi)^3[(\hbar^2/2\mu_A)J^2 - \epsilon_i]\tilde{\phi}_f^*(\vec{K})\tilde{\phi}_i(-\vec{J}), \tag{3.1}$$

$$\begin{aligned} \mathcal{T}_2 = \int d^3t\, d^3T\, &\tilde{\phi}_f^*(\beta\vec{K}_f - \vec{T})\tilde{W}_A(\vec{K}_f - \alpha\vec{T} - \vec{t})\tilde{G}_0^+(E; t, T) \\ &\times \tilde{W}_B(\vec{K}_i - \vec{T})\tilde{\phi}_i(\vec{t} - \alpha\vec{K}_i + \alpha\vec{T}), \end{aligned} \tag{3.2}$$

where, for arbitrary E, T, and t,

$$\tilde{G}_0^+(E; t, T) \equiv \frac{1}{E + i\eta - \hbar^2T^2/2\nu_A - \hbar^2t^2/2\mu_A}, \tag{3.3}$$

and where, omitting the subscripts,

$$\tilde{W}(\vec{k}) = \frac{-2^{1/2}Ze^2}{\pi^{1/2}k^2}, \quad \tilde{\phi}(\vec{k}) = \frac{2^{3/2}(Z/a_0)^{5/2}}{\pi[(Z/a_0)^2 + k^2]^2}. \tag{3.4}$$

The particular third Born term $\mathcal{T}_{3A}$ can be expressed in the form[3]

$$\begin{aligned} \mathcal{T}_{3A} = (2\pi)^{-3/2}\int d^3t\, d^3T\, &\tilde{\phi}_f^*(\beta\vec{K}_f - \vec{T}) \\ &\times V_A^+(E, T; \vec{K}_f - \alpha\vec{T}, \vec{t})\tilde{G}_0^+(E; t, T) \\ &\times \tilde{W}_B(\vec{K}_i - \vec{T})\tilde{\phi}_i(\vec{t} - \alpha\vec{K}_i + \alpha\vec{T}), \end{aligned} \tag{3.5}$$

where, for arbitrary E, T, $\vec{q}$, and $\vec{t}$,

$$V_A^+(E,T;\vec{q},\vec{t}) = \int d^3p\, \tilde{W}_A(\vec{q}-\vec{p})\tilde{G}_0^+(E;p,T)\tilde{W}_A(\vec{p}-\vec{t}). \tag{3.6}$$

In order to obtain the asymptotic form of $\mathcal{T}_3$ for high v, we first obtain the asymptotic form of $\mathcal{T}_2$, briefly repeating the analysis of Dettmann and Leibfried.[2,3] We change the variables of integration to $\vec{t}\,'$ and $\vec{T}\,'$ where

$$\vec{T}\,' = \vec{T} - \beta\vec{K}_f, \quad \vec{t}\,' = \vec{t} - \alpha\vec{K}_i + \alpha\vec{T} = \vec{t} + \alpha\vec{T}\,' + \alpha\vec{K}. \tag{3.7}$$

In terms of these new variables we have

$$\tilde{G}_0^+(E;t,T) = 1/(D+D_1+D_2), \tag{3.8a}$$

where D is constant, D_1 is linear, and D_2 is quadratic in $\vec{t}\,'$ and $\vec{T}\,'$:

$$D = \tfrac{1}{2}\alpha m v_f^2 - \alpha\frac{\hbar^2K^2}{2m} + \epsilon_f + i\eta\ , \tag{3.8b}$$

$$D_1 = -\vec{T}\,'\cdot\vec{T}_0' - \vec{t}\,'\cdot\vec{t}_0'\ , \tag{3.8c}$$

$$D_2 = \frac{-\hbar^2}{2\nu_A}T'^2 - \frac{\hbar^2}{2\mu_A}(\vec{t}\,' - \alpha\vec{T}\,')^2\ , \tag{3.8d}$$

$$\vec{T}_0' \equiv \frac{\hbar^2\beta}{2\nu_A}\vec{K}_f + \frac{\hbar^2\alpha}{m}\vec{K}, \quad \vec{t}_0' \equiv -\frac{\hbar^2}{m}\vec{K}. \tag{3.8e}$$

Using Eq. (2.6) to solve Eq. (2.5) for either J or K, we see that if $\frac{1}{2}mv^2 \gg |\epsilon_f - \epsilon_i|$ then $\hbar K > \frac{1}{2}mv_i$ and $\hbar J > \frac{1}{2}mv_f$, and therefore $\vec{T}_0'$ and $\vec{t}_0'$, defined in Eq. (3.8e), are each of order v. If follows that unless $\vec{T}\,'$ and $\vec{t}\,'$ are simultaneously perpendicular, or almost perpendicular, to $\vec{T}_0'$ and $\vec{t}_0'$, respectively, D_1 is a factor of order v larger than D_2, and therefore, D_2 can be neglected in Eq. (3.8a) for $\tilde{G}_0^+$. (The region of integration where D_2 cannot be neglected in comparison with D_1 is of order $1/v^2$ and is insignificant to the leading term in the asymptotic expansion of $\mathcal{T}_2$.) For "most" values of K, D is a factor of order v larger than D_1. However for $K \approx K_c$, where $\hbar K_c \sim mv_f$, we have $D \approx i\eta$ and D_1 cannot be neglected in comparison with D; we therefore retain D_1. Note that $\mathcal{T}_2$ is largest when K equals the "critical" value K_c since then $\tilde{G}_0^+$ is of order $1/v$, rather than $1/v^2$. Physically this corresponds to the fact that when $K \approx K_c$ the electron and incoming nucleus have roughly the same final velocity and hence they can become bound.[5]

Changing variables in Eq. (3.2) we have

$$\mathcal{T}_2 = \int d^3t'\, d^3T'\, \tilde{\phi}_f^*(-\vec{T}\,')\tilde{W}_A(\vec{t}\,'+\vec{J}) \times \tilde{G}_0^+(E;t,T)\tilde{W}_B(\vec{T}\,'+\vec{K})\tilde{\phi}_i(\vec{t}\,'). \tag{3.9}$$

The main contribution to this integral comes from the region $\vec{t}\,' \approx \vec{T}\,' \approx 0$ since $\tilde{\phi}_i(\vec{t}\,')$ and $\tilde{\phi}_f(-\vec{T}\,')$ decrease rapidly outside this region. Now $\tilde{W}_A(\vec{t}\,'+\vec{J})$ and $\tilde{W}_B(\vec{T}\,'+\vec{K})$ vary very slowly over this region since K and J each exceed $mv/2\hbar$. Therefore to obtain the asymptotic form of $\mathcal{T}_2$ it is legitimate to set $\vec{t}\,' = \vec{T}\,' = 0$ in $\tilde{W}_A$ and $\tilde{W}_B$. We then obtain

$$\mathcal{T}_2 \sim \tilde{W}_A(\vec{J})\tilde{W}_B(\vec{K}) \int d^3t'\, d^3T'\, \tilde{\phi}_f^*(-\vec{T}\,') \times \tilde{G}_0^+(E;t,T)\tilde{\phi}_i(\vec{t}\,'). \tag{3.10}$$

If we approximate $\tilde{G}_0^+$ by $1/(D+D_1)$ and use

$$\frac{1}{D+D_1} = -i\int_0^\infty e^{i(D+D_1)s}\, ds, \tag{3.11}$$

the integral over $\vec{t}\,'$ and $\vec{T}\,'$ becomes a product of two Fourier transforms which can be evaluated immediately to give

$$\mathcal{T}_2 \sim -i(2\pi)^3\tilde{W}_A(\vec{J})\tilde{W}_B(\vec{K})\int_0^\infty ds\, e^{iDs}\phi_f^*(-\vec{T}_0's)\phi_i(-\vec{t}_0's). \tag{3.12}$$

With ϕ_i and ϕ_f defined by Eq. (2.1) the integration over s is trivial to perform; we obtain, with $t_0' = |\vec{t}_0'|$ and $T_0' = |\vec{T}_0'|$,

$$\mathcal{T}_2 \sim \frac{2^4\pi(Z_AZ_B)^{5/2}e^4}{a_0^3(JK)^2}\,\frac{1}{D + iZ_At_0'/a_0 + i\,Z_BT_0'/a_0}. \tag{3.13}$$

We now proceed to evaluate $\mathcal{T}_{3A}$ in the asymptotic limit. From Eqs. (3.5) and (3.7) we have

$$\mathcal{T}_{3A} = (2\pi)^{-3/2}\int d^3t'\, d^3T'\, \tilde{\phi}_f^*(-\vec{T}\,')V_A^+(E,T;\vec{K}_f - \alpha\vec{T},\vec{t}) \times \tilde{G}_0^+(E;t,T)\tilde{W}_B(\vec{T}\,'+\vec{K})\tilde{\phi}_i(\vec{t}\,')\ . \tag{3.14}$$

As before, we set $\vec{T}\,' = 0$ in $\tilde{W}_B(\vec{T}\,'+\vec{K})$, since this function varies very slowly in the region $\vec{T}\,' \approx 0$ where $\tilde{\phi}_f(-\vec{T}\,')$ is non-negligible. Dettmann[3] also assumed that V_A^+ is slowly varying in the region $t' \approx T' \approx 0$ and he set the arguments $\vec{t}$ and $\vec{T}$ equal to $-\alpha\vec{K}$ and $\beta\vec{K}_f$, respectively, which is equivalent to setting $\vec{t}\,' = \vec{T}\,' = 0$; it is this step which is unjustified, as we now show.

V_A^+ can be evaluated exactly; in fact, V_A^+ has already been evaluated by Dalitz[7] for certain ranges of its arguments. Extending Dalitz's expression to cover the entire range of arguments we obtain, with $\vec{q} \equiv \vec{K}_f - \alpha\vec{T}$,

$$V_A^+(E,T;\vec{q},\vec{t}) = \frac{4\pi Z_A^2e^4}{|\vec{q}-\vec{t}|}\,\frac{2\mu_A}{\hbar^2} \times \int_0^\infty du\,\frac{1}{k^2 - t^2 + (k^2-q^2)u^2 + 2ik|\vec{q}-\vec{t}|u}\ , \tag{3.15a}$$

where

$$k^2=(2\mu_A/\hbar^2)(E+i\eta-\hbar^2T^2/2\nu_A). \quad (3.15b)$$

The integral over u can be evaluated in closed form, but this is not necessary for our purpose. With regard to the denominator of the integrand in Eq. (3.15a), note that

$$\frac{\hbar^2(k^2-q^2)}{2\mu_A}=\epsilon_f-\frac{\hbar^2}{2}\left(\frac{1}{\nu_A}+\frac{\alpha^2}{\mu_A}\right)T'^2,$$
$$\equiv -(a+bT'^2). \quad (3.16)$$

Since a and b are of the same order of magnitude, the term in T'^2 cannot be neglected, and it is not legitimate to set $\vec{T}'$ equal to zero; a more serious objection to setting $\vec{t}'$ and $\vec{T}'$ (both) to zero will be given in a moment. Since $\vec{q}-\vec{t}=-\vec{J}-\vec{t}'$, we set $\vec{t}'=0$ in the term $|\vec{q}-\vec{t}|$ which occurs in the denominator both inside and outside the integral of Eq. (3.15a). We set $\vec{T}=\beta\vec{K}_f$ in Eq. (3.15b); after neglecting $\epsilon_f+i\eta$ in comparison with $\frac{1}{2}\mu_A v_f^2$ we obtain $k=\mu_A v_f/\hbar$. Note that k^2-t^2 is simply $(2\mu_A/\hbar^2)(D+D_1+D_2)$; as before we neglect D_2. With these approximations we have

$$V_A^+(E,T;\vec{q},\vec{t})\sim\frac{4\pi Z_A^2e^4}{J}\times\int_0^\infty du\,\frac{1}{D+D_1-(a+bT'^2)u^2+i\hbar Jv_fu}. \quad (3.17)$$

This integral has a logarithmic singularity when $D+D_1=0$. Now if $K=K_c$, we have $D=i\eta$; and if $\vec{t}'$ and $\vec{T}'$ are both near zero, D_1 is very small. In this case the integral has a near singularity, and its value depends strongly on the sign of D_1. Therefore if $K=K_c$, which is the value of K for which $\mathcal{T}_{3A}$ is largest, V_A^+ varies rapidly in the vicinity of $\vec{t}'=\vec{T}'=0$; therefore $\vec{t}'$ and $\vec{T}'$ cannot both be set to zero.

IV. ASYMPTOTIC FORM OF THE THIRD BORN TERM

A. Amplitude

In Eq. (3.14) we approximate $\tilde{G}_0^+(E;t,T)$ by $1/(D+D_1)$, we approximate V_A^+ by the right-hand side of Eq. (3.17), and we define

$$X\equiv D+D_1-(a+bT'^2)u^2+i\hbar Jv_fu,$$
$$Y\equiv D+D_1,$$

and use the Feynman identity

$$\frac{1}{XY}=\int_0^1\frac{dz}{[Y+(X-Y)z]^2} \quad (4.1)$$

followed by the identity

$$\frac{1}{[Y+(X-Y)z]^2}=-\int_0^\infty s\,ds\,e^{i[Y+(X-Y)z]s} \quad (4.2)$$

to combine the denominators of $\tilde{G}_0^+$ and V_A^+ in Eq. (3.14), yielding

$$\mathcal{T}_{3A}\sim\frac{2Z_A^2Z_Be^6}{\pi JK^2}\int d^3T'\int_0^\infty du\int_0^\infty s\,ds\int_0^1 dz\int d^3t'\tilde{\phi}_f^*(-\vec{T}')\exp(i\{D-\vec{t}_0'\cdot\vec{t}'-\vec{T}_0'\cdot\vec{T}'+[-(a+bT'^2)u^2+i\hbar Jv_fu]z\}s)\tilde{\phi}_i(\vec{t}'); \quad (4.3)$$

we have set $\vec{T}'=0$ in $\tilde{W}_B(\vec{T}'+\vec{K})$ in Eq. (3.14) and we have used Eq. (3.4) to substitute for $\tilde{W}_B(\vec{K})$. The integral over $\vec{t}'$ in Eq. (4.3) is just a Fourier transform, and can be evaluated immediately. With ϕ_i defined by Eq. (2.1) it is straightforward to perform the integration over z, s, and u (in that order) to give

$$\mathcal{T}_{3A}\sim\frac{2^{5/2}Z_A^{7/2}Z_Be^6}{a_0^{3/2}JK^2}\int d^3T'\,\frac{\tilde{\phi}_f^*(-\vec{T}')}{A}\,\frac{1}{(C^2-4AB)^{1/2}}\times\ln\left(\frac{-C+(C^2-4AB)^{1/2}}{-C-(C^2-4AB)^{1/2}}\right), \quad (4.4a)$$

where

$$A=D-\vec{T}_0'\cdot\vec{T}'+i(Z_A/a_0)t_0', \quad (4.4b)$$

$$B=-(a+bT'^2),\quad C=i\hbar Jv_f. \quad (4.4c)$$

Since the quantities C and AB are both of order v^2 for large v, we have $|C^2|\gg|4AB|$ and

$$\frac{1}{(C^2-4AB)^{1/2}}\ln\left(\frac{-C+(C^2-4AB)^{1/2}}{-C-(C^2-4AB)^{1/2}}\right)\sim\frac{1}{i\hbar Jv_f}\ln[A(P^2+Q^2T'^2)], \quad (4.5a)$$

$$P^2\equiv(a/\hbar^2J^2v_f^2),\quad Q^2\equiv(b/\hbar^2J^2v_f^2). \quad (4.5b)$$

Inserting the right-hand side of Eq. (4.5a) into Eq. (4.4a) and using Eq. (3.4) to substitute for $\tilde{\phi}_f$, we can perform the angular integration (the angular dependence is contained in A) without great difficulty to give

$$\mathcal{T}_{3A}\sim\frac{i2^4(Z_AZ_B)^{7/2}e^6}{a_0^4(JK)^2v_f\hbar T_0'}(I_1+I_2), \quad (4.6a)$$

where

$$I_1=\int_0^\infty \frac{T'dT'}{[(Z_B/a_0)^2+T'^2]^2} \times\{[\ln(F-T_0'T')]^2-[\ln(F+T_0'T')]^2\}, \tag{4.6b}$$

$$I_2=\int_0^\infty \frac{2T'dT'}{[(Z_B/a_0)^2+T'^2]^2} \times \ln(P^2+Q^2T'^2)\ln\left(\frac{F-T_0'T'}{F+T_0'T'}\right), \tag{4.6c}$$

$$F=D+i(Z_A/a_0)t_0'. \tag{4.6d}$$

To evaluate I_1 we integrate by parts once and extend the range of the integral to $(-\infty,\infty)$ to give

$$I_1=\frac{-T_0'}{2}\int_{-\infty}^\infty \frac{dT'}{(Z_B/a_0)^2+T'^2} \times\left(\frac{\ln(F-T_0'T')}{F-T_0'T'}+\frac{\ln(F+T_0'T')}{F+T_0'T'}\right). \tag{4.7}$$

This integral can be evaluated by contour integration, treating the term in $\ln(F-T_0'T')$ separately from the term in $\ln(F+T_0'T')$. The appropriate contour is a semicircle in the lower (or upper) half-plane; this contour does not enclose the branch-point singularity at $T'=F/T_0'$ (or $T'=-F/T_0'$). The integral I_2 can be evaluated similarly; we first integrate by parts once, and extend the range of integration, to obtain

$$I_2=\int_{-\infty}^\infty \frac{dT'}{(Z_B/a_0)^2+T'^2}\left[\left(\frac{Q^2T'}{P^2+Q^2T'^2}\right)\ln\left(\frac{F-T_0'T'}{F+T_0'T'}\right) -\left(\frac{FT_0'}{F^2-T_0'^2T'^2}\right)\ln(P^2+Q^2T'^2)\right]. \tag{4.8}$$

We decompose the logarithms so that each term in the integrand of Eq. (4.8) has only one branch-point singularity, and we perform the contour integration treating each term separately, as before. We obtain finally

$$\mathcal{T}_{3A}\sim\frac{i2^5\pi(Z_AZ_B)^{7/2}e^6}{a_0^4(JK)^2\hbar v_f}\left\{\frac{iQ^2}{T_0'[P^2-(QZ_B/a_0)^2]}\ln\left(\frac{F+iT_0'P/Q}{F+iT_0'Z_B/a_0}\right) +\frac{1}{F^2+(T_0'Z_B/a_0)^2}\left[iT_0'\ln\left(P-\frac{iQF}{T_0'}\right)-\frac{Fa_0}{Z_B}\ln\left(P+\frac{QZ_B}{a_0}\right)\right]-\frac{a_0}{2Z_B}\frac{\ln(F+iT_0'Z_B/a_0)}{F+iT_0'Z_B/a_0}\right\}. \tag{4.9}$$

Equation (4.9) is the asymptotic form of $\mathcal{T}_{3A}$. To obtain $\mathcal{T}_{3B}$ we simply interchange A and B in Eq. (4.9).

We now pass to the limit m/M_A, $m/M_B\to 0$. In this limit $\vec{v}_i=\vec{v}_f$, and therefore, we drop altogether the subscript i or f from v. We also have in this limit

$$\hbar\vec{v}\cdot\vec{K}\sim\hbar\vec{v}\cdot\vec{J}\sim-\tfrac{1}{2}mv^2,\quad J\sim K,$$

$$K_{\min}\sim mv/2\hbar,\quad K_{\max}=\infty,$$

$$t_0'\sim T_0'\sim(\hbar^2/m)K,\quad F\sim D+iZ_Ae^2K,$$

$$D\sim mv^2/2-\hbar^2K^2/2m,\quad P=QZ_B/a_0.$$

Letting $P\to QZ_B/a_0$ in Eq. (4.9), and using the above relations, we obtain

$$\mathcal{T}_{3A}\sim\frac{-i2^4\pi Z_A^{7/2}Z_B^{5/2}e^6}{a_0^3\hbar vK^4}\,\frac{S}{D+i(Z_A+Z_B)e^2K}, \tag{4.10a}$$

where

$$S=1-\frac{2iZ_Be^2K}{F-iZ_Be^2K}\ln\left(\frac{F+iZ_Be^2K}{2iZ_Be^2K}\right) +2\ln\left(\frac{2QZ_B}{a_0}\right)+\ln(F+iZ_Be^2K). \tag{4.10b}$$

We will need the imaginary part of S below. We have

$$\operatorname{Im}S=\frac{1}{D^2+(Z_A-Z_B)^2e^4K^2} \times\left[Z_B(Z_A-Z_B)e^4K^2\tan^{-1}\left(\frac{D}{(Z_A+Z_B)e^2K}\right) -Z_Be^2KD\ln\left(\frac{D^2+(Z_A+Z_B)^2e^4K^2}{4Z_B^2e^4K^2}\right)\right] -\tan^{-1}\left(\frac{D}{(Z_A+Z_B)e^2K}\right)+\frac{\pi}{2}. \tag{4.10c}$$

In writing down this last equation we have used the relation

$$\tan^{-1}(1/x)=\tfrac{1}{2}\pi-\tan^{-1}x$$

with it understood that

$$-\tfrac{1}{2}\pi\le\tan^{-1}x\le\tfrac{1}{2}\pi.$$

B. Cross section

In the limit m/M_A, $m/M_B\to 0$ we have, using Eqs. (2.7), (3.1), (3.3), (3.13), (4.10a), and the relations immediately above Eq. (4.10a),

$$\sigma=\frac{1}{2\pi\hbar^2v^2}\int_{mv/2\hbar}^\infty|\mathcal{T}|^2K\,dK, \tag{4.11}$$

$$\mathcal{T}_1\sim\frac{-2^5\pi\hbar^2(Z_AZ_B)^{5/2}}{ma_0^5K^6}, \tag{4.12}$$

$$\mathcal{T}_2 \sim \frac{2^4\pi(Z_AZ_B)^{5/2}e^4}{a_0^3K^4}\frac{1}{D+i(Z_A+Z_B)e^2K}, \tag{4.13}$$

$$\mathcal{T}_{3A} \sim -i(Z_Ae^2/\hbar v)S\mathcal{T}_2, \tag{4.14}$$

with S defined by Eq. (4.10b). Evidently, looking at Eq. (4.14), the leading contribution from $\mathcal{T}_{3A}$ to σ comes from the interference of $\mathcal{T}_{3A}$ with $\mathcal{T}_2$, that is, from the term

$$2\,\mathrm{Re}(\mathcal{T}_2\mathcal{T}_{3A}^*) \sim 2(Z_Ae^2/\hbar v)\,|\mathcal{T}_2|^2\,\mathrm{Im}S. \tag{4.15}$$

Let us change variables from K to y, where y is defined by

$$K=(mv/\hbar)(1+\hbar y/mva_0)^{1/2}, \tag{4.16}$$

and insert Eq. (4.15) into Eq. (4.11), using Eq. (4.10c) with $D=-\hbar yv/2a_0$. Neglecting corrections of order y/v, we obtain

$$\sigma_{2,3A} \equiv \frac{1}{2\pi\hbar^2v^2}\int_{mv/2\hbar}^{\infty} 2\,\mathrm{Re}(\mathcal{T}_2\mathcal{T}_{3A}^*)K\,dK$$

$$\sim 2^9Z_A^6Z_B^5\left(\frac{e^2}{\hbar v}\right)^{12}(\pi a_0^2)\int_{-\infty}^{\infty}dy\,\frac{1}{y^2+4(Z_A+Z_B)^2}\left\{\frac{4}{y^2+4(Z_A-Z_B)^2}\right.$$

$$\times\left[-Z_B(Z_A-Z_B)\tan^{-1}\left(\frac{y}{2(Z_A+Z_B)}\right)+\frac{Z_By}{2}\ln\left(\frac{y^2+4(Z_A+Z_B)^2}{16Z_B^2}\right)\right]$$

$$\left.+\tan^{-1}\left(\frac{y}{2(Z_A+Z_B)}\right)+\frac{\pi}{2}\right\}. \tag{4.17}$$

Only the last term in the integrand (that is, the term in $\frac{1}{2}\pi$) contributes to the integral over y since the remainder of the integrand is odd in y. We therefore obtain

$$\sigma_{2,3A} \sim \frac{2^7\pi^2Z_A^6Z_B^5}{Z_A+Z_B}\left(\frac{e^2}{\hbar v}\right)^{12}(\pi a_0^2). \tag{4.18}$$

We now briefly discuss the results of Dettmann and Drisko. Dettmann's expression for the asymptotic form of $\mathcal{T}_{3A}$ is[8]

$$-i(Z_Ae^2/\hbar v)[-\ln(2\hbar v/Z_Be^2)+i\pi/2]\mathcal{T}_2.$$

This expression differs significantly from the right-hand side of Eq. (4.14) and yet it yields the correct estimate of $\sigma_{2,3A}$. The reason for this coincidence is not hard to see. The real part of S does not contribute to $\sigma_{2,3A}$ to leading order, and the only term in the imaginary part of S which contributes is the last term of Eq. (4.10c), namely $\frac{1}{2}\pi$. Dettmann's expression corresponds to replacing ReS by $-\ln(2\hbar v/Z_Be^2)$ and ImS by $\frac{1}{2}\pi$, and so the correct result is obtained. Drisko's expression for the asymptotic form of $\mathcal{T}_{3A}$ is[9]

$$-i(Z_Ae^2/\hbar v)[\ln(F+iZ_Be^2K)+c]\mathcal{T}_2,$$

where c is *real* and independent of K. We need not specify c any further, other than to note that c does not diverge more rapidly than $\ln v$ as $v\to\infty$. Drisko's expression corresponds to replacing ImS by $\tan^{-1}[y/2(Z_A+Z_B)]+\frac{1}{2}\pi$; the arctangent term is odd in y and does not contribute to $\sigma_{2,3A}$ and the correct result is obtained.

We can evaluate the leading contribution to σ from $\mathcal{T}_{3B}$ in a fashion similar to the above. The result differs from the right-hand side of Eq. (4.18) only in that A and B are interchanged. The leading contribution to σ from $\mathcal{T}_3$ is therefore σ_{23}, where

$$\sigma_{23} \sim 2^7\pi^2Z_A^5Z_B^5(e^2/\hbar v)^{12}(\pi a_0^2)$$

$$\sim 0.0241\sigma_{\mathrm{BK}}, \tag{4.19}$$

where σ_{BK} is the Brinkman-Kramers cross section, which has the asymptotic form

$$\sigma_{\mathrm{BK}} \sim (2^{18}/5)Z_A^5Z_B^5(e^2/\hbar v)^{12}(\pi a_0^2). \tag{4.20}$$

The contributions to σ from $\mathcal{T}_1$ and $\mathcal{T}_2$ can be calculated to order $(e^2/\hbar v)^{12}$ without difficulty. The result can be found, for example, in Ref. 3. Adding this result to σ_{23} we find that to order $(e^2/\hbar v)^{12}$ the contribution from the first three Born terms to the forward capture cross section σ is

$$\sigma \sim [0.319+5\pi\hbar v2^{-11}/(Z_A+Z_B)e^2]\sigma_{\mathrm{BK}}. \tag{4.21}$$

However this result is probably not an accurate estimate of σ unless $\hbar v/e^2 > 20$. When $\hbar v/e^2=20$, relativistic corrections probably amount to 5% or less.[10] The relative contribution of the third Born term is seen to be very small.

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[1]Nevertheless, capture into the backward direction (in the center of mass frame) is significant at very high impact velocities when the incident and target nuclei have the same masses. See R. A. Mapleton, Proc. Phys. Soc. London 83, 895 (1964). See also Ref. 6 below.

[2]K. Dettmann and G. Leibfried, Z. Phys. 218, 1 (1969).

[3]K. Dettmann, in *Springer Tracts in Modern Physics*, edited by G. Hohler (Springer, Berlin, 1971), Vol. 58, p. 119.

[4]R. M. Drisko, thesis (Carnegie Institute of Technology, 1955) (unpublished).

[5]See, e.g., M. R. C. McDowell and J. P. Coleman, *Introduction to the Theory of Ion-Atom Collisions* (North-Holland, London, 1970).

[6]For a "physical" discussion of capture mechanisms at high impact velocities see R. Shakeshaft and L. Spruch (unpublished).

[7]R. H. Dalitz, Proc. R. Soc. London A 206, 509 (1951).

[8]We pass to the limit m/M_A, $m/M_B \to 0$ and we let the Yukawa potential become a pure Coulomb potential in Eq. (4.71b) of Ref. 3.

[9]Actually, Drisko considered only the case $Z_A = Z_B = 1$.

[10]M. H. Mittleman, Proc. Phys. Soc. London 84, 453 (1964).