Solitons under perturbations

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A direct perturbation method is developed in order to study the interactions of N solitons with imperfections and with each other in the presence of imperfections. The leading-order effects are obtained directly from the N-soliton waveform without invoking methods from inverse-scattering theory. The method is based upon a Green's function and a "two time" procedure from classical perturbation theory. An example of a single soliton is developed in detail. Finally, this perturbation method is compared with other approaches in the literature.

I. INTRODUCTION AND OUTLINE OF THE ABSTRACT PERTURBATION SCHEME

The interaction of solitons, both with imperfections in the medium and with each other in the presence of imperfections, is an important aspect of nonlinear physics.[1,2] This importance becomes particularly apparent when solitons are used to model real physical situations in areas such as plasma,[3,4] solid state,[5,6] or high-energy physics.[7-10] The main purpose of this paper is to introduce a direct perturbation method, which is simple, and yet sufficiently general to apply to "breather" and "N-soliton" wave forms.

In order to calculate the leading-order effects with this perturbation scheme, we need only the general N-soliton wave form; no knowledge of inverse scattering is required to obtain the most important modulations. Our method uses a Green's function together with a rather standard "two-timing" procedure. Since it consists of natural extensions of concepts from classical perturbation theory, we believe our scheme is easy to follow and to use.

The paper is organized in the following manner: An abstract outline of the perturbation scheme is summarized in the introduction. In Sec. II the general method is applied to a nonlinear Schrödinger equation. An example of simple dissipation is presented in Sec. III for purposes of illustration. Throughout these first three sections no inverse-scattering is used. However, to represent a Green's function which is needed for the first-order correction, we do use calculations from inverse scattering theory. Both this Green's function and the first-order correction are described in Sec. IV. In the conclusion, we summarize the merits of our perturbation method, and discuss the equivalence of our method with an alternative perturbation scheme[11,12] whose application relies heavily upon inverse-scattering techniques.

Our approach can be described in very general, although somewhat abstract, terms. We study solutions of a perturbed nonlinear equation

$$\partial_t r + H(r) = \epsilon f(r), \quad 0 \le \epsilon \ll 1, \tag{1.1}$$

where r denotes a function of (one-dimensional) space and time, and where $H(r)$ and $f(r)$ are nonlinear differential (in space) operators on r. We assume that the reduced equation ($\epsilon = 0$) supports N-soliton wave forms, and we seek a perturbation expansion of r in the form

$$r \simeq r_0 + \epsilon r_1 + \cdots, \tag{1.2}$$

where r_0 denotes the pure N-soliton wave form. If r_0 exactly satisfies the reduced equation,

$$\partial_t r_0 + H(r_0) = 0, \tag{1.3}$$

then the first-order correction r_1 is defined by

$$[L(r_0)]r_1 \equiv \partial_t r_1 + [\delta H(r_0)]r_1 = f(r_0) \tag{1.4a}$$

$$r_1\big|_{t=0} = 0. \tag{1.4b}$$

Here $L(r_0)$ is the linearization of the operator $\partial_t + H$ about r_0, and initially $r = r_0$.

To solve the linearized Eq. (1.4), we represent the operator $[L(r_0)]^{-1}$ by means of a "Green's function." We define a linear operator $\hat{G}(t, t')$, which maps a Hilbert space $\mathcal{H}$ into itself, by the differential equation

$$[L(r_0)]\hat{G} \equiv \partial_t \hat{G} + [\delta H(r_0)]\hat{G} = 0, \quad 0 < t' < t, \tag{1.5a}$$

$$\lim_{t \to t'} \hat{G}(t, t') = I \tag{1.5b}$$

Here $\mathcal{H}$ is the Hilbert space which is natural for the linear operator $\delta H(r_0)$, and I is the identity operator on $\mathcal{H}$. The inner product for $\mathcal{H}$, denoted by (,), includes an integral over all space. In terms of this Green's operator $\hat{G}$, the first-order correction r_1 is represented by

$$r_1(x,t) = \int_0^t \{\hat{G}(t,t')f[r_0(t')]\}(x)\,dt'$$

$$= \int_0^t \big(G(x,t\,|\,\cdot\,,t'), f[r_0(\cdot\,,t')]\big)\,dt', \tag{1.5c}$$

where $G(x,t|x',t')$ denotes the kernel of the linear operator $\hat{G}(t,t')$. For notational purposes we treat the kernel $G(x,t|x',t')$, as a function of x', as if it belongs to the Hilbert space $\mathcal{H}$, and denote this convention by $G(x,t|\cdot,t')$.

The function $r_0+\epsilon r_1$ is considered a valid asymptotic expansion of $r(\cdot,t)$ for times t of order $1/\epsilon$ if the function $r_1(\cdot,t)$ has the property

$$\lim_{\epsilon\to 0}\epsilon r_1(\cdot,\tau/\epsilon)=0 \tag{1.6}$$

for each fixed τ. This requirement demands that $\epsilon r_1(t)$ remain small for a long time. It is generally called a "secularity condition." If Eq. (1.6) is valid, one has an indication that, for fixed finite t,

$$r(\cdot,t)=r_0(\cdot,t)+\epsilon r_1(\cdot,t)+O(\epsilon^2),$$

and also that

$$r(\cdot,t)=r_0(\cdot,t)+O(\epsilon),$$

is valid uniformly over the long-time interval $0\le t\le \tau/\epsilon$.

Frequently the secularity condition Eq. (1.6) cannot be satisfied for an r_0 which exactly satisfies Eq. (1.3), and one must allow additional flexibility in the function r_0. In particular, one allows free parameters in r_0 to modulate on the (ϵt) time scale. When r_0 is an N-soliton wave form, the free parameters fix the speeds and the locations of the solitons in r_0, and one expects a perturbation to alter these characteristics of the soliton. With such modulations r_0 no longer satisfies Eq. (1.3) exactly; instead

$$\partial_t r_0+H(r_0)=O(\epsilon). \tag{1.7}$$

Furthermore, because of this slow modulation, it may no longer be practical to find an exact inverse for $L(r_0)$, but only an approximate inverse $[\tilde{L}(r_0)]^{-1}$ for which

$$[\tilde{L}(r_0)]^{-1}L(r_0)=I+O(\epsilon).$$

The function r_1 now satisfies

$$[L(r_0)]r_1=\partial_t r_1+[H(r_0)]r_1=F(r_0), \tag{1.8}$$

where an effective source $F(r_0)$ arises because of the modulations. Using Eq. (1.7), we compute this effective source,

$$F(r_0)=f(r_0)-[\partial_t r_0+H(r_0)]/\epsilon. \tag{1.9}$$

The first-order correction r_1 is then given by

$$r_1(t)=\int_0^t \tilde{G}(t,t')F[r_0(t')]dt'+O(\epsilon), \tag{1.10}$$

where we have represented the approximate inverse through a Green's operator $\tilde{G}(t,t')$. If the modulation in the parameters of r_0 can be picked so that the secularity condition (1.6) is satisfied, then the method has produced an asymptotic expansion of $r(t)$ valid for times of order $O(1/\epsilon)$.

This classical approach to finding the asymptotic behavior of $r(t)$ relies heavily on the ability to find the inverse operator for a given zeroth-order solution r_0. For most partial differential equations, this task is not practical. However, for those evolution equations which can be integrated by inverse scattering techniques, the appropriate Green's function can always be found.[13] In order to solve (1.5a) and (1.5b) for the Green's operator $\hat{G}$, we study the null space of the linear operator $L(r_0)\equiv\partial_t+H(r_0)$. Since $L(r_0)$ arises in the linearization of $\partial_t r+H(r)=0$ about r_0, members of its null space can be found by differentiating r_0 with respect to free parameters. Using scattering methods, sufficient free parameters can be identified to generate a complete set of functions which spans this null space. With this complete set, an explicit representation of $\hat{G}$ can be constructed.[13]

Fortunately, to obtain the modulations in r_0 we do not need the technical details of this construction. We only need the structure of the null space of $L(r_0)$ from which we obtain the structure of $\hat{G}$. It turns out[13] that this space consists of two very distinct parts: the "discrete" and "continuous" subspaces. Physically, the discrete subspace is associated with the solitons in r_0, while the continuous subspace is associated with dispersive wave trains. Speaking rather generically, the N-soliton component r_0 is described by $2N$ free parameters. Half of these locate the solitons and the other half fix their speeds. Any choice of parameters $\{p_j\}$ which carries these $2N$ pieces of independent information yields a family of functions,

$$\left\{\frac{\partial r_0}{\partial p_j},\quad j=1,\ 2,\dots 2N\right\},$$

which spans the discrete subspace. In this manner, the Green's operator $\hat{G}$ takes the generic form

$$\hat{G}(t,t')=\hat{G}_d(t,t')+\hat{G}_c(t,t'),$$

where the kernel of the discrete component $\hat{G}_d$ admits a representation of the type[14]

$$G_d(x,t|x',t')=\sum_{j=1}^{2N}A_j(x,t)\frac{\partial r_0}{\partial p_j}(x',t'), \tag{1.11}$$

and the continuous component $\hat{G}_c$ consists of continuous wave trains. We shall not need the explicit form of the expansion coefficients A_j.

Formula (1.11) is central to our perturbation scheme. Recall that as t' becomes large, an N-soliton wave $r_0(x',t')$ decomposes into a sum of N solitons, each of which travels at its own speed c_j. Thus, for sufficiently large t', $r_0(x',t')$ depends upon x' and t' only through the combinations $\{(x'-c_jt')\}$. Formula (1.11) shows that the same space and time dependence appears in the discrete Green's function. Since both the wave r_0 and the discrete Green's function G_d consist of components which translate at speeds $\{c_j\}$, secularities can be introduced into r_1. In particular, since the inner product (,) involves an integral over all space, the first-order correction (1.10) contains integrals which behave for large t like

$$\int^t \int_{-\infty}^{\infty} \mathfrak{F}(x' - c_j t')\, dx'\, dt' .$$

The spatial (x') integral eliminates all t' dependence leaving a constant integrand for the temporal (t') integral. This integral over t' then introduces linear growth in t into r_1 which violates the secularity condition (1.6).

Stated more concisely, formula (1.11) shows that the Green's function has discrete components which translate with the solitons in r_0. Any part of the perturbation $F(r_0)$ which is "parallel" to one of these discrete components resonates with the Green's function and introduces secularities into the perturbation scheme. In order to eliminate such leading secularities, we select the modulations of the free parameters $\{p_j\}$ so that the *full* source

$$F(r_0) = f(r_0) - [\partial_t r_0 + H(r_0)]/\epsilon$$

is orthogonal to the entire discrete subspace. Explicitly, we demand

$$\left(\frac{\partial r_0}{\partial p_k}, f(r_0) - \sum_{j=1}^{2N} \frac{\partial p_j}{\partial \tau}\, \frac{\partial r_0}{\partial p_j}\right) = 0, \quad \text{for each } k = 1, 2, \ldots, 2N,$$

where the slow time $\tau \equiv \epsilon t$. This orthogonality condition yields a system of $2N$ differential equations which govern the modulation of the parameters $\{p_j(\tau)\}$:

$$\sum_{j=1}^{2N} \frac{dp_j}{d\tau}\left(\frac{\partial r_0}{\partial p_k}, \frac{\partial r_0}{\partial p_j}\right) = \left(\frac{\partial r_0}{\partial p_k}, f(r_0)\right), \quad k = 1, 2, \ldots, 2N. \qquad (1.12)$$

The solutions $\{p_j(\tau)\}$ of Eq. (1.12) describe changes in the speeds and locations of the solitons in r_0, changes which are induced by the perturbation. We emphasize that in order to find Eqs. (1.12), we use only the general N-soliton wave form r_0 and the appropriate inner product (,), both of which can be obtained without inverse-scattering theory.

Once the proper modulation of the parameters $\{p_j\}$ has been selected, the first-order correction r_1 is given by

$$r_1(x,t) = \int_0^t \left(\tilde{G}_c(x, t \,|\, \cdot\,, t'), F[r_0(\cdot\,, t')]\right) dt' , \qquad (1.13)$$

where inverse scattering is used to find the continuous component of the Green's function. In Sec. II we discuss the physical structure of the first-order correction r_1.

In later sections of this paper, we use formula (1.13) to identify some lesser secularities which may still remain. These secondary secularities arise because the full source F, which has been made orthogonal to the discrete subspace, may still contain components in the continuous subspace. Although such components can resonate with G_c, enough oscillation is present to lessen the effects of these resonances. The effects do increase with time, but their growth is slower than $O(t)$. Although we can use the integral representation (1.13) to identify the strength and the physical origin of these secularities, a practical scheme to remove these terms from the perturbation scheme is not currently available.

In summary, our perturbation scheme consists of seven computational steps: (1) Take a general N-soliton wave form r_0 and allow its parameters $\{p_j\}$ to modulate on a slow time scale. (2) Compute an effective source F, together with the equation which defines the first-order correction r_1. (3) Consider solving this equation for r_1 by means of a Green's function, and use these considerations to identify the appropriate inner product. (4) Using this inner product, find the equations which govern the modulations of the parameters $\{p_j\}$. These equations follow from the orthogonality conditions $(\partial r_0/\partial p_j, F) = 0$. (5) Solve these ordinary differential equations for the modulations of the parameters $\{p_j(\tau)\}$. (6) Using scattering theory, find a representation of the continuous Green's function G_c, and use this representation to compute the first-order correction r_1 through formula (1.13). (7) Study this representation in order to identify the structure of the first-order correction. In particular, identify and interpret any secondary secularities which remain.

In this paper, we specialize these abstract steps to the concrete case of a nonlinear Schrödinger equation with simple dissipation. More substantial examples for the sine-Gordon equation will be published elsewhere.[15,16] Of course, these methods extend to all nonlinear wave equations which support N-soliton solutions and which can be solved by the inverse-scattering transform.

II. EXPLICIT PERTURBATION SCHEME FOR A NONLINEAR SCHRÖDINGER EQUATION

A. Background information

Consider a nonlinear Schrödinger equation in the form

$$-i\partial_t r+\partial_{xx}r+2|r|^2 r=\epsilon f(r,r^*),$$
$$0\leq\epsilon\ll 1,\quad t>0,\quad -\infty<x<\infty\,. \tag{2.1}$$

This equation arises in a wide variety of physical situations ranging from nonlinear optics,[17,18] through waves in water,[19] to plasma waves.[20] In fact, a nonlinear Schrödinger equation governs the amplitude modulations of any weakly nonlinear wave which consists of a rapidly oscillating carrier with a slowly varying envelope, and which is dispersive to leading order.[12,19,21,22]

When the perturbation is missing ($\epsilon=0$), the general solution of the nonlinear Schrödinger Eq. (2.1) consists of two very distinct types of waves, solitons, and dispersive wave trains. A soliton is a localized pulse[23] which is described analytically by the four parameter formula

$$r(x,t)=2i\eta\,\mathrm{sech}\{2\eta[(x-x_0)+4\xi t]\}$$
$$\times\exp\{i[2\xi x+4(\xi^2-\eta^2)t+\phi]\}\,, \tag{2.2}$$

where ξ, η, x_0, ϕ are constants. Notice the physical information which is carried by these parameters. A single soliton has both an amplitude envelope and a phase. The envelope translates at *constant* speed 4ξ; its width and height are determined by the constant η; initially the pulse is centered at x_0. The phase of the soliton has wave number 2ξ; the constant ϕ centers this phase. There also exist analytical solutions containing $4N$ parameters which are called "pure N-soliton states." One representation of these waves takes the form

$$r(x,t)=2\sum_{j=1}^{N}[k_1^{(j)}(x,t)]e^{i\zeta_j x}\,. \tag{2.3a}$$

Here $k_1^{(j)}$ is a solution of the linear algebraic equation

$$k_1^{(j)}(x,t)+\sum_{l=1}^{N}M_{jl}(x,t)k_1^{(l)}(x,t)=i\gamma_j^* e^{-i\zeta_j^* x}\,,$$
$$j=1,\,2,\,\ldots N, \tag{2.3b}$$

where the matrix M is specified by

$$M_{jl}(x,t)\equiv\sum_{n=1}^{N}\gamma_j^*\gamma_n\frac{\exp[-i(\zeta_j^*-2\zeta_n+\zeta_l^*)x]}{(\zeta_j^*-\zeta_n)(\zeta_n-\zeta_l^*)}\,. \tag{2.3c}$$

The entire t dependence enters through the formula

$$\gamma_j(t)=\Gamma_j\exp(4i\zeta_j^2 t)\,. \tag{2.3d}$$

The $4N$ parameters are $\{\zeta_j=\xi_j+i\eta_j;\ \Gamma_j=|\Gamma_j|\exp(i\phi_j);\ j=1,\,2,\,\ldots N\}$. In the case $N=1$, the identifications

$$\zeta_1=\xi+i\eta\,,\qquad \Gamma_1=|\Gamma_1|e^{i\phi}\,,\qquad |\Gamma_1|=2\eta e^{+2\eta x_0}$$

reduce representation (2.3) to the single soliton (2.2). Although initially complicated, the N-soliton formula simplifies to a sum of solitons as t becomes large. The sequence of solitons which results is ordered according to the size (speed) of the individual solitons.

In addition to soliton components, the general solution r contains dispersive wave trains. These waves are quite different from solitons. Although nonlinear, they resemble periodic solutions of linear, dispersive wave equations in that they spread out because of dispersion and also travel in group packets. There exists a continuum of such waves labeled by their wave number k. For each real wave number k, the wave possesses two additional degrees of freedom, amplitude, and phase. These nearly linear waves are referred to as "radiation," as "phonons," and as "packets of phonons."[24]

In the absence of the perturbation ($\epsilon=0$), a general wave can consist of a (nonlinear) superposition of N solitons together with dispersive wave trains at all wave numbers. All of these degrees of freedom are available to the wave motion. This general structure of the wave r is apparent in numerical calculations and it can be established analytically by the inverse-scattering transform.[25-27]

Returning to the full nonlinear Schrödinger Eq. (2.1), we ask in what manner the weak perturbation affects a wave. It seems clear that a perturbation alters the speeds and shifts the locations of any solitons present in the wave. It can also modulate the phonon packets which are present. Since the perturbation is weak, these effects take place on slow time and long space scales. In addition, the perturbation could open additional degrees of freedom which are not excited in the initial wave. In other words, it could create (or destroy) solitons, as well as generate phonons.

Both numerical experiments and the theory of inverse scattering indicate that it takes a more violent perturbation to create or to destroy solitons than it takes to modulate their speeds. At present, a practical scheme which describes the creation or destruction of solitons is not available. All existing schemes fix the number of soliton components in the wave. In the next section, we outline the steps for computing modulations for the speeds and locations of a fixed number of solitons, give a formula which describes the (first order) phonons which are created by the perturbation, and give an indication of the validity of these calculations.

B. Computational steps

Consider a wave which initially is a pure N-soliton state. We wish to compute the modification of this wave due to the perturbation ϵf. We begin with the ansatz that the wave is of the form

$$r(x,t)\simeq r_0(x,t)+\epsilon r_1(x,t) \ . \tag{2.4a}$$

Here r_0 is the N-soliton state, such as represented by Eq. (2.3), except that all parameters are allowed to modulate on the slow time scale as follows:

$$\zeta_j=\zeta_j(\epsilon t) \ , \qquad \gamma_j=\Gamma_j(\epsilon t)\exp\left(\left[4i\int_0^t \zeta_j^2(\epsilon t')\,dt'\right]\right). \tag{2.4b}$$

We next compute the effective source $F(r_0)$ which is given by

$$F(r_0)\equiv f(r_0)-\frac{1}{\epsilon}\,[-i\partial_t r_0+\partial_{xx}r_0+2|r_0|^2 r_0] = f(r_0)+i\sum_{j=1}^{N}\left(\frac{\partial\zeta_j}{\partial\tau}\frac{\partial}{\partial\zeta_j}+\frac{\partial\Gamma_j}{\partial\tau}\frac{\partial}{\partial\Gamma_j}+\text{c.c.}\right)r_0 \ , \tag{2.5a}$$

where $\tau=\epsilon t$ and c.c. denotes the complex-conjugate operation. In terms of this effective source, the first-order correction r_1 is defined by

$$[-i\sigma_3\partial_t+H(\vec{r}_0)]\vec{r}_1=\vec{F} \ , \qquad \vec{r}_1(x,t=0)=\vec{0} \ , \tag{2.5b}$$

where we have written the first-order correction as a column vector $\vec{r}_1$ with components r_1 and r_1^*; similarly, the vector $\vec{F}$ has components F and F^*. The matrix σ_3 denotes the third Pauli spin matrix, and the matrix operator H is given by

$$H\equiv\begin{pmatrix}\partial_{xx}+4|r_0|^2 & 2r_0^2\\ 2(r_0^*)^2 & \partial_{xx}+4|r_0|^2\end{pmatrix} . \tag{2.6}$$

In this framework, the natural inner product is given by

$$(\vec{u},\vec{v})\equiv\int_{-\infty}^{\infty}[\vec{u}^\dagger(x)\cdot\vec{v}(x)]\,dx \ , \tag{2.7a}$$

where † denotes the Hermitian conjugate. The linear Eq. (2.5b) may be inverted to yield a representation of the first-order correction $\vec{r}_1$ in terms of a (matrix) Green's function G,

$$\vec{r}_1(x,t)=\int_0^t (G(x,t|\cdot,t'),\vec{F}(\cdot,t'))\,dt' \ , \tag{2.7b}$$

where, because H is self-adjoint with respect to the inner product (2.7a), the matrix kernel $G(x,t|x',t')$ is defined by the final value problem in (x',t')

$$[-i\sigma_3\partial_{t'}+H']G(\cdot,\cdot\,|x',t')=0, \qquad \lim_{t'\uparrow t} G(x,t|x',t')=-i\sigma_3\delta(x-x') \ . \tag{2.7c}$$

In this representation of $\vec{r}_1$, we have adopted the convention for the (matrix) operator G that

$$(G(x,t|\cdot,t'),\vec{F}(\cdot,t'))\equiv\int_{-\infty}^{\infty}[G(x,t|x',t')]^\dagger\vec{F}(x',t')\,dx' \ . \tag{2.8}$$

Since G satisfies Eq. (2.7c), it may be constructed by considering the null space of the operator $L\equiv i\sigma_3\partial_t+H$. The discrete component of this null space is $4N$ dimensional [13] and is spanned by the set

$$\left\{\frac{\partial\vec{r}_0}{\partial\zeta_j},\ \frac{\partial\vec{r}_0}{\partial\zeta_j^*},\ \frac{\partial\vec{r}_0}{\partial\Gamma_j},\ \frac{\partial\vec{r}}{\partial\Gamma_j^*},\ j=1,2,\ldots,N\right\} .$$

The next step is to insure that leading secularities are absent in the first-order correction $\vec{r}_1$. One demands that the full source $\vec{F}$ be orthogonal to the discrete subspace,

$$\left(\frac{\partial\vec{r}_0}{\partial\zeta_k},\vec{F}\right)=0\ ,\quad \left(\frac{\partial\vec{r}_0}{\partial\Gamma_k},\vec{F}\right)=0\ ,\quad \left(\frac{\partial\vec{r}_0}{\partial\zeta_k^*},\vec{F}\right)=0\ ,\quad \left(\frac{\partial\vec{r}_0}{\partial\Gamma_k^*},\vec{F}\right)=0\ ,\quad k=1,2,\ldots,N \ . \tag{2.9a}$$

In view of the definition of the full source F (2.5b), these orthogonality conditions provide $4N$ equations which determine the slow modulations of the parameters $\{\zeta_j\}$ and $\{\Gamma_j\}$.

In order to obtain these equations explicitly, we must calculate inner products of the type

$$\left(\frac{\partial\vec{r}_0}{\partial\zeta_k},\ \sigma_3\frac{\partial\vec{r}_0}{\partial\zeta_j}\right),\ \left(\frac{\partial\vec{r}_0}{\partial\zeta_k},\ \sigma_3\frac{\partial\vec{r}_0}{\partial\Gamma_j}\right),\ \text{etc}\ .$$

In the Appendix, we evaluate these integrals using orthogonality relations from inverse-scattering theory; however, since the integrands depend only upon the N-soliton wave form r_0, these integrals could be evaluated directly without reference to inverse scattering. In any event, the equations which result can be placed in the form

$$\frac{d\zeta_j}{d\tau}=-\frac{1}{2}\,\Gamma_j^- e^{-i\theta_j}\left(\vec{f}(\cdot,t),\ \frac{\partial\vec{r}_0(\cdot,t)}{\partial\gamma_j^-(t)}\right) \ , \qquad \frac{d}{d\tau}\ln\Gamma_j=-\frac{1}{2}\left(\vec{f}(\cdot,t),\ \frac{\partial\vec{r}_0}{\partial\zeta_j^-(t)}\right)-\frac{a_j''}{a_j'}\frac{d\zeta_j}{d\tau} \ , \tag{2.9b}$$

Here

$$\gamma_j^-(t)\equiv\Gamma_j^-\exp(-i\theta_j),$$

$$\theta_j(t)\equiv\int_0^t 4\zeta_j^2(\epsilon t')\,dt' \ ,$$

$$\Gamma_j^-\equiv[(a_j')^2\Gamma_j]^{-1} \ ,$$

and

$$a_j'\equiv\frac{d}{d\zeta}\,a(\zeta)\big|_{\zeta_j} \ ,$$

where

$$a(\zeta)\equiv\prod_{j=1}^{N}\left(\frac{\zeta-\zeta_j}{\zeta-\zeta_j^*}\right) .$$

In (2.9b) the independent variables in the partial derivatives consist of the set $\{\gamma_j^-, \zeta_j, (\gamma_j^-)^*, \zeta_j^*, j=1, 2, \ldots, N\}$.

Next, it is necessary to compute the inner products

$$\left(\frac{\partial \vec{r}_0}{\partial \Gamma_j}, \vec{f}\right) \text{ and } \left(\frac{\partial \vec{r}_0}{\partial \zeta_j}, \vec{f}\right), \quad j=1, 2, \ldots, N,$$

for the particular perturbation at hand, solve the differential equations for $\{\Gamma_j(\tau), \zeta_j(\tau)\}$, and insert the result into the zeroth-order wave form $\delta\vec{r}$. At the completion of this step, we have computed the perturbation's effect on the speeds, widths, and locations of the N solitons. To obtain this information, we have used scattering theory only to identify the structure of the perturbation scheme. In the actual calculations, no scattering theory is necessary.

We do employ methods from scattering theory to represent the complete Green's function[13]

$$G(x,t|x',t') \simeq \tilde{G}_d(x,t|x',t') + \tilde{G}_c(x,t|x',t'), \tag{2.10a}$$

where $\tilde{G}_c$ is given by

$$\tilde{G}_c(x,t|x',t') \equiv i\pi\int_{-\infty}^{\infty} e^{4ik^2(t-t')}\left[\prod_{j=1}^{N}\left(\frac{k-\zeta_j^*}{k-\zeta_j}\right)^2\right]\left(\frac{\delta\vec{r}_0(x',t')}{\delta\rho_0(R,t')}\right)\left(\frac{\delta\vec{r}_0(x,t)}{\delta\rho_+^*(k,t)}\right)^\dagger dk$$
$$+ i\pi\int_{-\infty}^{\infty} e^{-4ik^2(t-t')}\left[\prod_{j=1}^{N}\left(\frac{k-\zeta_j}{k-\zeta_j^*}\right)\right]\left(\frac{\delta\vec{r}_0(x',t')}{\delta\rho_-^*(k,t')}\right)\left(\frac{\delta\vec{r}_0(x,t)}{\delta\rho_+(k,t)}\right)^\dagger dk, \tag{2.10b}$$

where ρ_+ and ρ_- denote reflection coefficients. (The reflection coefficient ρ_+ is defined at $x=+\infty$, while ρ_- is defined at $x=-\infty$. Throughout this paper, the $\pm$ subscripts refer to this convention.)

A few remarks are in order here. First, the precise definitions of the integrands in this representation of $\tilde{G}_c$ are given in Sec. IV. Secondly, we do not need the formula for the discrete component $\tilde{G}_d$ because the full source $\vec{F}$ has been made orthogonal to the discrete subspace. Finally, representation (2.10a) and (2.10b) of G is exact provided the parameters $\{\zeta_j, \Gamma_j\}$ are constants; it becomes an approximation when these parameters are allowed to modulate on the slow (ϵt) time scale.

With formulas (2.10a) and (2.10b) for $\tilde{G}_c$, we represent the first-order correction r_1 as a triple integral,

$$r_1(x,t) = \int_0^t (\tilde{G}_c(x,t|\cdot,t'), \vec{F}(\cdot,t'))\,dt' + 0(\epsilon), \tag{2.11}$$

a representation which admits an interesting physical interpretation. After interchanging the orders of integration, we obtain

$$r_1(x,t) = \int_{-\infty}^{\infty} \hat{r}_1(x,t;k)\,dk + O(\epsilon),$$

where

$$\hat{r}_1(x,t;k) \equiv -i\pi\int_0^t dt'\int_{-\infty}^{\infty} dx'\left\{e^{-4ik^2(t-t')}\left[\prod_{j=1}^{N}\left(\frac{k-\zeta_j}{k-\zeta_j^*}\right)^2\right]\left(\frac{\delta\vec{r}_0(x,t)}{\delta\rho_+^*(k,t)}\right)\left(\frac{\delta\vec{r}_0(x',t')}{\delta\rho_-(k,t')}\right)^\dagger\right.$$
$$\left.+ e^{4ik^2(t-t')}\left[\prod_{j=1}^{N}\left(\frac{k-\zeta_j^*}{k-\zeta_j}\right)^2\right]\left(\frac{\delta\vec{r}_0(x,t)}{\delta\rho_+(k,t)}\right)\left(\frac{\delta\vec{r}_0(x',t')}{\delta\rho_-(k,t')}\right)^\dagger\right\}\vec{F}[\vec{r}_0(x',t')].$$

The function $\hat{r}_1(x,t;k)$ can be interpreted as the density of phonons near wave number k which have been created by the perturbation $\vec{f}$ acting on the N-soliton wave form $\vec{r}_0$. The first-order correction $r_1(x,t)$ consists of a superposition of these "nonlinear phonons."

It remains to introduce enough scattering theory to compute $\tilde{G}_c$ and then to estimate the asymptotic behavior of r_1 for large t. First, however, we discuss a single-soliton example in more detail.

III. A SINGLE SOLITON

Suppose that r_0 is a single-soliton wave form which is allowed to modulate on the $\tau=\epsilon t$ time scale. Using Eqs. (2.3) we represent $r_0(x,t)$ in the form

$$r_0(x,t) = \frac{-2i(\zeta-\zeta^*)^2\gamma^{-*}e^{2i\zeta x}}{\gamma^-\gamma^{-*} - (\zeta-\zeta^*)^2 e^{2i(\zeta-\zeta^*)x}}; \tag{3.1a}$$

where the time dependence enters through

$$\gamma^- = \Gamma^-(\epsilon t)\exp[-i\theta(t,0)],$$
$$\zeta = \zeta(\epsilon t) = \xi(\epsilon t) + i\eta(\epsilon t),$$
$$\theta(t,0) = \int_0^t 4\zeta^2(\epsilon t)\,dt = \int_0^t 4(\xi^2-\eta^2)\,dt' + i\int_0^t 8\xi\eta\,dt'$$
$$= \theta_r(t,0) + i\theta_I(t,0). \tag{3.1b}$$

Alternately, the identifications

$$\zeta = \xi + i\eta, \quad \Gamma = |\Gamma|e^{i\phi}, \quad |\Gamma| = 2\eta e^{2\eta x_0},$$
$$\phi = \phi(\epsilon t), \quad x_0 = x_0(\epsilon t),$$

reduce (2.3) to a representation of $r_0(x,t)$ in the form

$$r_0(x,t) = -2i\eta(\epsilon t)\,\mathrm{sech}\Big(2\eta(\epsilon t)\Big[x - x_0(\epsilon t) + 4\int_0^t \xi(\epsilon t')\,dt'\Big]\Big)\exp\Big(i\Big[2\xi(\epsilon t)x + 4\int_0^t [\xi^2(\epsilon t') - \eta^2(\epsilon t')]\,dt' + \phi(\epsilon t)\Big]\Big).$$

For this example, we assume that the soliton is propagating in the presence of dissipation, and that the inhomogeneous perturbation is given by

$$\vec{f} = ig\sigma_3\vec{r},\; g \text{ constant}. \qquad (3.2)$$

This perturbation will modulate the constants in the wave form. To obtain these modulations, we use (3.1a) to calculate explicitly that

$$\frac{\partial\vec{r}_0}{\partial\gamma^-} = \frac{-1}{\gamma^-\left[1 - \dfrac{(\zeta-\zeta^*)^2}{\gamma^-\gamma^{-*}}e^{2i(\zeta-\zeta^*)x}\right]} \times \begin{pmatrix} r_0 \\ \dfrac{(\zeta-\zeta^*)^2}{\gamma^-\gamma^{-*}}e^{2i(\zeta-\zeta^*)x}r_0^* \end{pmatrix}, \qquad (3.3a)$$

and

$$\frac{\partial\vec{r}_0}{\partial\zeta} = \frac{-2}{\left(1 - \dfrac{(\zeta-\zeta^*)^2}{\gamma^-\gamma^{-*}}e^{2i(\zeta-\zeta^*)x}\right)} \times \begin{pmatrix} \left(\dfrac{1}{(\zeta-\zeta^*)} + ix\right)r_0 \\ \left(\dfrac{1}{(\zeta-\zeta^*)} + ix\dfrac{(\zeta-\zeta^*)}{\gamma^-\gamma^{-*}}e^{2i(\zeta-\zeta^*)x}\right)r_0 \end{pmatrix} \qquad (3.3b)$$

We now compute the inner products

$$(\delta\vec{r}_0/\delta\gamma^-, \vec{f}) \text{ and } (\delta\vec{r}_0/\delta\zeta, \vec{f})$$

by elementary integration. For example,

$$\left(\vec{f}, \frac{\partial\vec{r}_0}{\partial\gamma^-}\right) = \int_{-\infty}^{\infty} \vec{f}^\dagger \frac{\partial\vec{r}_0}{\partial\gamma^-}\,dx = i\frac{g}{\gamma^-}\int_{-\infty}^{\infty} r_0 r_0^*\,dx$$
$$= \frac{4ig}{\gamma^-}\eta^2\int_{-\infty}^{\infty}\mathrm{sech}^2[2\eta(x+x_0) + \theta_I(t,0)]\,dx$$
$$= \frac{4ig\eta}{\gamma^-} = \frac{4ig\eta}{\Gamma^-}e^{i\theta(t,0)} \qquad (3.4a)$$

Similarly, the inner product $(\vec{f}, \delta\vec{r}_0/\delta\zeta)$ is a simple integral involving hyperbolic functions with the result that

$$\left(\vec{f}, \frac{\delta\vec{r}_0}{\delta\zeta}\right) = -4g[\theta_I + \ln(|\Gamma^-/2\eta|)] \qquad (3.4b)$$

Using these inner products in (2.4b), we find that the slow evolution of the parameters ζ and Γ^- must satisfy the differential equations

$$\frac{d\zeta}{d\tau} = -2ig\eta, \quad \zeta = \xi + i\eta$$
$$\frac{d}{d\tau}\ln\left|\frac{\Gamma^-}{2\eta}\right| = -2g\left(\theta_I + \ln\left|\frac{\Gamma^-}{2\eta}\right|\right), \qquad (3.5)$$

Integrating these equations shows that the modulating parameters take the form

$$\xi(\epsilon t) = \xi(0), \quad \eta(\epsilon t) = \eta(0)\exp(-2g\epsilon t)$$
$$\ln\left|\frac{\Gamma^-}{2\eta}\right| = \ln\left|\frac{\Gamma^-(0)}{2\eta(0)}\right|e^{-2g\epsilon t} - \frac{4\xi\eta(0)}{\epsilon g}\{1 - e^{-2g\epsilon t}(1 + 2g\epsilon t)\}. \qquad (3.6a)$$

Expressed in terms of x_0 and ϕ, this last equation is

$$2\eta(\epsilon t)x_0(\epsilon t) = 2\eta(0)x_0(0)e^{-2g\epsilon t} - \frac{4\xi\eta(0)}{\epsilon g}\{1 - e^{-2g\epsilon t}(1+2g\epsilon t)\}, \qquad (3.6b)$$
$$\phi(\epsilon t) = \phi(0).$$

Of course, the modulations of the parameters ξ, η, x_0, ϕ can also be derived directly from Eq. (1.12) using the representation Eq. (3.1c) of r_0. Then one needs to calculate the set of functions

$$\left\{\frac{\delta\vec{r}_0}{\delta\eta}, \frac{\delta\vec{r}_0}{\delta\xi}, \frac{\delta\vec{r}_0}{\delta x_0}, \frac{\delta\vec{r}_0}{\delta\phi}\right\}$$

and require that these functions be orthogonal to the full source $\vec{F}(\vec{r}_0)$. These orthogonality calculations again involve integrals of hyperbolic functions and no scattering theory.

We can get an idea of the validity of this solution by examining more closely the wave (3.1c) with the prescribed modulations (3.6a) and (3.6b). In particular, after a long time, the amplitude of the wave $\eta(\epsilon t)$ becomes very small, the envelope of the wave is spread out (since η is small), so that the wave $r_0(x,t)$ resembles a linear dispersive wave at wave number 2ξ. Since this solution $r_0(x,t)$ is used as a first-order approximation to the solution $r(x,t)$, it is quite likely that r_0 begins to resonate with the continuous spectrum near wave number 2ξ as t becomes large. To understand when this resonance becomes important and how it actually shows itself, it is necessary to calculate the first-order correction $r_1(x,t)$. To do this we need to study scattering theory and calculate the continuous part of the Green's function $G_c(t,t')$.

IV. RADIATION (PHONON) COMPONENTS

In this section we obtain the continuous component of the Green's function G_c, and thereby complete the representation of the first-order correction r_1. Much of the background material in this section can be found in various forms elsewhere.[12,25-27] We have specifically included it here to clarify the computation of G_c, and hence to assure that all the ingredients of our perturbation scheme are self-contained.

A. Notation from scattering theory

We begin with a brief summary of the direct-scattering theory of Zakharov and Shabat.[25] For fixed t, consider the eigenvalue problem

$$(\partial_x + i\zeta)v_1 = -R^* v_2\,,$$
$$(\partial_x - i\zeta)v_2 = Rv_1, \quad \text{for every } x\in(-\infty,\infty),\quad R\equiv r_0, \tag{4.1}$$

and define two solutions ϕ and ψ by the asymptotic boundary conditions

$$\phi(x,\zeta)\simeq\begin{pmatrix}1\\0\end{pmatrix}e^{-i\zeta x}\ \text{as } x\to-\infty\,, \tag{4.2a}$$

$$\psi(x,\zeta)\simeq\begin{pmatrix}0\\1\end{pmatrix}e^{i\zeta x}\ \text{as } x\to+\infty,\quad \mathrm{Im}(\zeta)\geq 0. \tag{4.2b}$$

Symmetries on the eigenvalue problem immediately yield two other solutions, $\overline{\phi}$ and $\overline{\psi}$, which are defined by

$$\overline{\phi}(x,\zeta)\equiv -i\sigma_2[\phi(x,\zeta^*)]^*\,, \tag{4.3}$$
$$\overline{\psi}(x,\zeta)\equiv -i\sigma_2[\psi(x,\zeta^*)]^*,\quad \mathrm{Im}(\zeta)\leq 0.$$

The pair of solutions $(\psi,\overline{\psi})$ is linearly independent as is the pair $(\phi,\overline{\phi})$, $\mathrm{Im}(\zeta)=0$. These solutions are related by

$$\phi(x,\zeta)=a(\zeta)\overline{\psi}(x,\zeta)+b(\zeta)\psi(x,\zeta),$$
$$\overline{\phi}(x,\zeta)=-\overline{a}(\zeta)\psi(x,\zeta)+\overline{b}(\zeta)\overline{\psi}(x,\zeta),\quad \mathrm{Im}(\zeta)=0. \tag{4.4}$$

The scattering coefficients $(a,b,\overline{a},\overline{b})$ can be obtained from the asymptotic behavior

$$\phi(x,\zeta)\simeq\begin{pmatrix}a(\zeta)e^{-i\zeta x}\\ b(\zeta)e^{i\zeta x}\end{pmatrix}\quad \text{as } x\to+\infty, \tag{4.5a}$$

$$\overline{\phi}(x,\zeta)\simeq\begin{pmatrix}\overline{b}(\zeta)e^{-i\zeta x}\\ -\overline{a}(\zeta)e^{i\zeta x}\end{pmatrix}\quad \text{as } x\to+\infty,\ \mathrm{Im}(\zeta)=0. \tag{4.5b}$$

The coefficient $a(\zeta)$ admits an analytic continuation into the upper-half ζ plane where its only zeros occur at the bound-state eigenvalues ζ_j. At these eigenvalues ϕ and ψ are linearly dependent,

$$\phi(x,\zeta_j)=b_j\psi(x,\zeta_j), \tag{4.6}$$

where the "normalization constant" b_j is given, in the case r_0 has compact support, by $b_j=b(\zeta_j)$. The coefficient $\overline{a}(\zeta)$ admits a continuation into the lower-half ζ plane where its only zeros occur at $\overline{\zeta}_j$. In this case, symmetries yield the relationships

$$\overline{a}(\zeta)=[a(\zeta^*)]^*,\quad \mathrm{Im}(\zeta)\leq 0,$$
$$\overline{b}(\zeta)=[b(\zeta)]^*,\quad \mathrm{Im}(\zeta)=0, \tag{4.3$'$}$$
$$\overline{\zeta}_j=[\zeta_j]^*,\quad \overline{b}_j=[b_j]^*.$$

This notation enables us to define the following set of *scattering data* $\mathcal{S}_+$

$$\mathcal{S}_+=S_+ U \overline{S}_+, \tag{4.7}$$

where

$$S_+\equiv\{\rho_+(\xi)\ \text{for every}\ \xi\in(-\infty,\infty)\,|\,\zeta_j,\gamma_j^+\ \text{for each}\ j\in(1,\ 2,\dots,N)\}.$$

Here the reflection coefficient

$$\rho_+(\xi)\equiv b(\xi)/a(\xi),\quad \gamma_j^+\equiv b_j/a_j',\quad a_j'\equiv\frac{d}{d\zeta}a\Big|_{\zeta=\zeta_j},\quad \overline{S}_+=S_+^*$$

[because of Eq. (3.3b)], and the notation { | } separates the "continuous" scattering data from the "discrete". (In the preceding sections, we have used γ_j to mean γ_j^+.)

At the foundation of the inverse method is the map between $\vec{r}_0(\cdot,t)$ and the scattering data at time t, $\mathcal{S}_+(t)$. At any fixed time t this map is one-to-one and invertible. The evolution of $\vec{r}_0$ in t induces an equivalent evolution of $\mathcal{S}_+$ in t. For example, when this evolution of $\vec{r}_0$ is dictated by the (unperturbed) nonlinear Schrödinger equation, the temporal behavior of the scattering data is given explicitly by

$$S_+(t)=\{\rho_+(\xi,t)=e^{4i\xi^2 t}\rho_+(\xi,0)\ \text{for every}\ \xi\in(-\infty,\infty)\,|\,\zeta_j(t)=\zeta_j(0),\quad \gamma_j^+(t)=e^{4i\zeta_j^2 t}\gamma_j^+(0)\ \text{for each}\ j\in(1,\ 2,\dots,N)\}. \tag{4.8}$$

Since the map from $\vec{r}_0$ to S_+ is invertible, knowledge of $\mathcal{S}_+(t)$ is equivalent to knowledge of $\vec{r}_1(\cdot,t)$.

The main point of the preceding paragraph is that any solution of the nonlinear Schrödinger equation is parametrized by its scattering data at $t=0$. The variation of $\vec{r}_0(x,t)$ with respect to each one of these parameters will provide a member of the null space of $L=-i\sigma_3\partial_t+H(r_0)$. That is, the set

$$\frac{\delta\vec{r}_0}{\delta\mathcal{S}_+(t=0)}\equiv\left\{\frac{\delta\vec{r}_0(x,t)}{\delta\rho_+(\xi,0)},\ \frac{\delta\vec{r}_0(x,t)}{\delta\overline{\rho}_+(\xi,0)},\quad \xi\in(-\infty,\infty)\,\Big|\ \frac{\partial\vec{r}_0(x,t)}{\partial\zeta_j},\ \frac{\partial\vec{r}_0(x,t)}{\partial\overline{\zeta}_j},\ \frac{\partial\vec{r}_0(x,t)}{\partial\gamma_j^+(0)},\ \frac{\partial\vec{r}_0(x,t)}{\partial\overline{\gamma}_j^+(0)},\quad j=1,2,\dots,N\right\} \tag{4.9}$$

provides an infinite number of solutions of the linearized equation. This set, which certainly consists of two types of components (discrete and continuous), actually spans the null space, and can be used to construct G.[13]

In this construction it is convenient to introduce an additional set of scattering data $\mathcal{S}_-$ which is equivalent to $\mathcal{S}_+$. The scattering data $\mathcal{S}_-$ is defined by

$$\mathcal{S}_- = S_- U \bar{S}_-, \tag{4.10a}$$

$$S_- \equiv \{\rho_-(\xi) \text{ for every } \xi \in (-\infty, \infty) \,|\, \zeta_j, \gamma_j^- \text{ for each } j \in (1, 2, \ldots, N)\}.$$

where $\rho_-(\xi) \equiv \bar{b}(\xi)/a(\xi)$, $\gamma_j^- \equiv [(a_j')^2 \gamma_j^+]^{-1}$, and $\bar{S}_- = (S_-)^*$. The time evolution of $\mathcal{S}_-$ under the unperturbed flow is given by

$$S_-(t) = \{\rho_-(\xi, t) = e^{-4i\xi^2 t}\rho_-(\xi, 0) \text{ for every } \xi \in (-\infty, \infty) \,|\, \zeta_j(t) = \zeta_j(0),\ \gamma_j^-(t) = e^{-4i\zeta_j^2 t}\gamma_j^-(0) \text{ for each } j \in (1, 2, \ldots, N)\}. \tag{4.10b}$$

With this notation from scattering theory, we can write the Green's function[13] G as

$$G(x, t \,|\, x', t') = i\pi \int_{\mathcal{C}_a} \frac{1}{[a(\zeta)]^2} \left(\frac{\delta \vec{r}_0(x', t')}{\delta S_-(\zeta, 0)}\right) \left(\frac{\delta \vec{r}_0(x, t)}{\delta \bar{S}_+(\zeta, 0)}\right)^\dagger d\zeta + i\pi \int_{\mathcal{C}_b} \frac{1}{[\bar{a}(\zeta)]^2} \left(\frac{\delta \vec{r}_0(x', t')}{\delta \bar{S}_-(\zeta, 0)}\right) \left(\frac{\delta \vec{r}_0(x, t)}{\delta S_+(\zeta, 0)}\right)^\dagger d\zeta. \tag{4.11}$$

Here the contours of integration run along the real axis from $\zeta = -\infty$ to $\zeta = +\infty$, the first ($\mathcal{C}_a$) being indented above all zeros of $a(\zeta)$ while the second ($\mathcal{C}_b$) runs below all zeros of $\bar{a}(\zeta)$. Deforming these contours to the real axis yields

$$G(x, t \,|\, x', t') = G_d(x, t \,|\, x', t') + G_c(x, t \,|\, x', t'),$$

where the discrete component G_d arises from the residues at the poles of $[a(\zeta)]^{-2}$ which occur at the bound-state eigenvalues. We will not need the exact form of G_d. On the other hand, the continuous component is given by

$$G_c(x, t \,|\, x', t') = i\pi \int_{-\infty}^{\infty} \frac{e^{4ik^2(t-t')}}{[a(k)]^2} \left(\frac{\delta \vec{r}_0(x', t')}{\delta \rho_-(k, t')}\right) \left(\frac{\delta \vec{r}_0(x, t)}{\delta \bar{\rho}_+(k, t)}\right)^\dagger dk + i\pi \int_{-\infty}^{\infty} \frac{e^{-4ik^2(t-t')}}{[\bar{a}(k)]^2} \left(\frac{\delta \vec{r}_0(x', t')}{\delta \bar{\rho}_-(k, t')}\right) \left(\frac{\delta \vec{r}_0(x, t)}{\delta \rho_+(k, t)}\right)^\dagger dk, \tag{4.12}$$

B. Formula for the continuous eigenfunctions

Without an explicit formula for $\bar{\Psi}$, representation (4.14) of G_c will not be very useful; moreover, when $\vec{r}_0$ denotes some arbitrary solution of the nonlinear Schrödinger equation, such explicit expressions for $\bar{\Psi}$ are not available. Fortunately, when $\vec{r}_0$ denotes an exact N-soliton solution, $\bar{\Psi}$ can be computed analytically. In this section, we show how to obtain these squared eigenfunctions. Our presentation begins with an arbitrary exact solution r_0 of the unperturbed Schrödinger equation. We specialize to the N-soliton wave from only at the necessary stage in the derivation.

The construction is based upon inverse-scattering theory. It begins with the solutions (ψ and $\bar{\psi}$) of the linear scattering problem (4.1). These solutions are specified by boundary conditions at $x = +\infty$, such as (4.2b). These boundary conditions may themselves be considered as solutions (ψ^0, $\bar{\psi}^0$) of the free-scattering problem with $r_0 = 0$,

$$\psi^0(x, \zeta) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta x}, \tag{4.13}$$

$$\bar{\psi}^0(x, \zeta) \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\zeta x}. \tag{4.14}$$

The foundation of the inverse-scattering method is an operator $\hat{\mathcal{K}}$ which transforms these free-scattering functions (ψ^0, $\bar{\psi}^0$) into the full scattering functions (ψ, $\bar{\psi}$),

$$\psi = (I + \hat{\mathcal{K}})\psi^0, \qquad \bar{\psi} = (I + \hat{\mathcal{K}})\bar{\psi}^0. \tag{4.15}$$

This transformation operator $\hat{\mathcal{K}}$ can be represented as a Volterra integral operator,

$$[(I + \hat{\mathcal{K}})\vec{f}\,](x) \equiv \vec{f}(x) + \int_x^\infty \mathcal{K}(x, y)\vec{f}(y)\,dy,$$

with matrix kernel

$$\mathcal{K}(x, y) = \begin{bmatrix} [K_2(x, y)]^* & K_1(x, y) \\ -[K_1(x, y)]^* & K_2(x, y) \end{bmatrix}.$$

The wave form r_0 is given in terms of the transformation kernel K_1 by

$$r_0(x) = 2[K_1(x, x)]^*. \tag{4.16}$$

Thus, an explicit representation of $\mathcal{K}$ will yield a representation of the eigenfunctions ψ and $\bar{\psi}$ and a representation of the wave r_0.

To compute a representation of $\mathcal{K}$, we use the Marchenko equations which define $\mathcal{K}$ in terms of the scattering data S_+. These integral equations take the form

$$K_1(x,y)-\int_x^\infty [K_2(x,s)F(s+y)]^*\,ds=[F(x+y)]^*,$$
$$[K_2(x,y)]^*+\int_x^\infty K_1(x,s)F(s+y)\,ds=0, \qquad (4.17)$$

where F is a transform of the scattering data S_+,

$$F(x)\equiv\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho_+(\xi)e^{i\xi x}\,d\xi - i\sum_{j=1}^{N}\gamma_j^+ e^{i\xi_j x}.$$

In general, analytical formulas for the solutions K_j of these integral equations are not known. However, when r_0 is an N-soliton wave form, one can solve these integral equations analytically. In this case, the reflection coefficient $\rho_+(\xi)=0$ for all $\xi\in(-\infty,\infty)$, and the kernels $K_j(x,y)$ take the form

$$K_1(x,y)=\sum_{j=1}^{N}k_1^{(j)}(x)e^{-i\xi_j^* y},$$
$$K_2(x,y)=\sum_{j=1}^{N}k_2^{(j)}(x)e^{-i\xi_j^* y}. \qquad (4.18)$$

Upon setting $\rho_+(\xi)=0$ and inserting this ansatz for $K_j(x,y)$ into the Marchenko equations (4.17), one finds the function $k_1(x)$ is defined by the linear algebraic system

$$k_1^{(j)}(x)+\sum_{l=1}^{N}M_{jl}(x)k_1^{(l)}(x)=i[\gamma_j^+e^{i\xi_j x}]^*,$$
$$j=1,\,2,\,\dots,\,N, \qquad (4.19)$$

where the matrix M is specified in terms of the scattering data S_+ by

$$M_{jl}\equiv\sum_{n=1}^{N}(\gamma_j^+)^*\gamma_n^+\frac{\exp[-i(\xi_j^*-2\xi_n+\xi_l^*)x]}{(\xi_j^*-\xi_n)(\xi_n-\xi_l^*)}\,. \qquad (4.20)$$

Moreover, k_2 is given in terms of k_1 by

$$[k_2^{(j)}(x)]^*=-\gamma_j^+\sum_{l=1}^{N}\frac{e^{i(\xi_j-\xi_l^*)x}}{(\xi_j-\xi_l^*)}k_1^{(l)}(x). \qquad (4.21)$$

In this N-soliton case, the kernel $\mathcal{K}(x,y)$ can be computed explicitly. Moreover, the (x,y) dependence in $\mathcal{K}(x,y)$ separates, as can be seen from (4.18). This separation permits explicit integration in Eq. (4.17). Thus, in the N-soliton case, both the eigenfunction ψ (4.15) and the wave form r_0 (4.16) can be represented explicitly in terms of the discrete scattering data $\{\xi_j,\gamma_j^+\}$. This construction is the origin of the particular N-soliton wave form given in Eq. (2.3) although for Secs. II and III any general expression for the N-soliton state is sufficient. Here we emphasize that the explicit formula for $\mathcal{K}(x,y)$ yields the continuous eigenfunction ψ,

$$\psi(x,k)=\begin{pmatrix}0\\1\end{pmatrix}e^{ikx}+\int_x^\infty\mathcal{K}(x,y)\begin{pmatrix}0\\1\end{pmatrix}e^{iky}\,dy \qquad (4.22a)$$
$$=\begin{pmatrix}0\\1\end{pmatrix}e^{ikx}+\sum_{j=1}^{N}\frac{i}{(k-\xi_j)}\begin{pmatrix}k_1^{(j)}(x)\\k_2^{(j)}(x)\end{pmatrix}e^{i(k-\xi_j^*)x}, \qquad (4.22b)$$

where $k(x)$ can be obtained by inverting the linear algebraic system (4.19).

This representation of ψ, together with the formula

$$a(k)=\prod_{j=1}^{N}\left(\frac{k-\xi_j}{k-\xi_j^*}\right),\quad \mathrm{Im}(k)\ge 0, \qquad (4.23)$$

which is valid anytime $\rho(k)\equiv 0$, provides all the ingredients of the continuous component of the Green's function, G_c.

At this point, we have derived (for a pure N soliton) representations of the wave from $\vec{r}_0$ and of the continuous component of the Green's function G_c. The dependence of these representations upon the discrete scattering data $\{\xi_j$ and $\gamma_j^+\}$ has been displayed explicitly. If this scattering data evolves in time as

$$\xi_j(t)=\xi_j(0),\quad \gamma_j^+(t)=\gamma_j^+(0)e^{+4i\xi_j^2 t},$$

the wave form r_0 is an exact solution of the nonlinear Schrödinger equation, and the formula for G_c is exact. On the other hand, when modulations are introduced,

$$\xi_j=\xi(\epsilon t),$$
$$\gamma_j^+=\Gamma_j^+(\epsilon t)\exp\left[4i\int_0^t\xi_j^2(\epsilon t')\,dt'\right],$$

r_0 solves the nonlinear Schrödinger equation to leading order,

$$-i\partial_t r_0+\partial_{xx}r_0+2|r_0|^2r_0=O(\epsilon),$$

and our formula for the continuous component of the Green's function is also an approximation. With this approximation denoted by $\tilde{G}_c$, the first order correction $\vec{r}_1$ is given in formula (2.11) by

$$\vec{r}_1(x,t)=\int_0^t(\tilde{G}_c(x,t|x',t'),\vec{F}(\cdot,t'))dt'+O(\epsilon).$$

C. Single soliton case

With these ingredients at hand, we return to the example of a single soliton. Recall that the structure of the modulated soliton as described in Sec. III indicates the existence of a resonance in the continuous spectrum near wave number (2ξ).

For this single soliton case, $k(x)$ is given by

$$k_1(x)=\frac{i(\xi-\xi^*)^2(\gamma^+)^*\exp[-i\xi^*x]}{(\xi-\xi^*)^2-\gamma^+(\gamma^+)^*\exp[2i(\xi-\xi^*)x]}\,,$$

$$k_2(x)=\frac{i\gamma^+(\gamma^+)^*(\zeta^*-\zeta)\exp[-i(\zeta^*-2\zeta)x]}{(\zeta-\zeta^*)^2-\gamma^+(\gamma^+)^*\exp[2i(\zeta-\zeta^*)x]},$$

in terms of which $\tilde{G}_c$ can be constructed and used in our representation of $\vec{r}_1(x,t)$. A rather cursory stationary phase analysis of the triple integral representation of $\vec{r}_1$, shows that r_1 grows like $\ln t$ for large t. Only the leading secularity has been removed. Moreover, as is apparent from this representation, the source of the $\ln t$ secularity is indeed a resonant coupling between the continuous spectrum and the phase $\exp[i(2\xi x+\theta_r(t,0)+\phi)]$ of the soliton $\vec{r}_0$. As $\vec{r}_0$ decays in (ϵt), this phase generates continuous components with wave number in the neighborhood of $(2\xi_1)$. The secularities which result presumably could be removed by introducing some continuous spectrum in $\vec{r}_0$ which would modulate on the (ϵx) scale as well as (ϵt).

These results on the modulation of a single soliton due to damping are not new. They were obtained by Kaup[11] and have recently been analyzed by Kaup and Newell[28] in an attempt to eliminate the secularities which remain. This problem does provide a good example to illustrate our perturbation scheme, and it permits a direct comparison of our method with the approach of Kaup and Newell. This comparison is discussed in the final section.

V. CONCLUSION

In this paper we have introduced a perturbation scheme for N-soliton waves which is a natural extension of classical perturbation methods. The approach is based upon a Green's function and a rather standard "two-timing" procedure. In the scheme, the modulations in the speeds and in the locations of the solitons are computed directly from the N-soliton wave form. Inverse-scattering techniques are not needed at this leading order. Since the entire approximation is given in space time, it admits direct physical interpretation. The actual construction of the Green's function and, therefore, the first-order correction does require inverse scattering. The first-order correction is represented as a triple integral from which its physical structure and its asymptotic behavior can be seen.

In these final paragraphs we compare our scheme with another due to Kaup[11] and Newell.[12] Their calculations use a very simple idea. Since systems which support solitons are conservative, the energy integral is a constant of the motion. Weak perturbations cause this energy to change slowly, a change which is easy to compute and to approximate. Most wave equations which can be integrated by inverse-scattering methods possess an infinite number of constants of the motion. In fact, half the degrees of freedom are constant in time, while the other half satisfy trivial equations in t. A perturbation causes these "constants" to change slowly. Kaup and Newell approximate these modulations in the unperturbed "constants of motion." In this manner they obtain the identical modulation equations that result from our scheme. Both approaches are equivalent to leading order in ϵ.

Unfortunately, this beautiful idea of computing changes in the invariants relies heavily upon scattering theory, which is used even to identify the constants of motion for the unperturbed flow. Kaup and Newell begin by transforming the full dynamics (with perturbations) into scattering space. That is, they use scattering methods to find a differential equation for the scattering data $\mathcal{S}(t)$. This equation is equivalent to the *full* nonlinear Schrödinger equation. The map which defines this transformation is implicit in the sense that it depends upon the unknown $\vec{r}(\cdot,t)$; or equivalently, the map depends nonlinearly and implicitly upon the scattering data itself. However, to leading order, it depends only upon the zeroth-order solution $\vec{r}_0$ and is explicit to this order. Using the leading order of the map, they obtain an approximate differential equation for $\mathcal{S}_+(t)$. For example, $d\zeta_j/d\tau$ is given by (2.9b) of Sec. II, with a similar equation for $d\gamma_j/d\tau$. One then solves these equations in scattering space using techniques of singular perturbation theory to eliminate leading secularities. These discrete components of the approximate dynamics define the modulations in the N-soliton wave forms. In order to check the accuracy of the scheme, they estimate the size of the continuous components by studying the approximate equation for the reflection coefficient $\rho(\xi,t)$. If ρ remains small for all time, the scheme has eliminated all secularities; if not, only leading secularities have been removed without a modulation of the continuous components.

Although these two schemes are equivalent to leading order, their derivations differ substantially. We believe that our scheme, which minimizes the use of scattering methods, should be easier to follow and to use, particularly for those who are not experts in inverse scattering theory. Furthermore, we believe that the Green's function will provide estimates of accuracy in a more direct manner. In any case, it is helpful to have an alternative derivation available as an aid to understanding the nature of the perturbation theory.

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TABLE I. Variations of $\vec{r}$ with respect to members of the sets of scattering data.

$$\frac{\delta\vec{r}(x,t)}{\delta\rho_+(\xi,t)}=-\frac{i}{\pi}\sigma_2\Psi(x,t,\xi)\quad\text{for every }\xi\in(-\infty,\infty),$$

$$\frac{\delta\vec{r}(x,t)}{\delta\zeta_j(t)}=-2\gamma_j^+\sigma_2\left(\frac{d}{d\zeta}\Psi(x,t;\zeta)\right)_{\zeta_j},$$

$$\frac{\delta\vec{r}(x,t)}{\delta\gamma_j^+(t)}=-2\sigma_2\Psi(x,t;\zeta_j).$$

$$\frac{\delta\vec{r}(x,t)}{\delta\bar{\rho}_+(\xi,t)}=\frac{i}{\pi}\sigma_2\bar{\Psi}(x,t;\xi)\quad\text{for every }\xi\in(-\infty,\infty),$$

$$\frac{\delta\vec{r}(x,t)}{\delta\bar{\zeta}_j(t)}=-2\bar{\gamma}_j^+\sigma_2\left(\frac{d}{d\zeta}\bar{\Psi}(x,t;\zeta)\right)_{\bar{\zeta}_j},$$

$$\frac{\delta\vec{r}(x,t)}{\delta\bar{\gamma}_j^+(t)}=-2\sigma_2\bar{\Psi}(x,t;\bar{\zeta}_j).$$

$$\frac{\delta\vec{r}(x,t)}{\delta\rho_-(\xi,t)}=-\frac{1}{\pi}\Psi^A(x,t;\xi)\quad\text{for every }\xi\in(-\infty,\infty),$$

$$\frac{\delta\vec{r}(x,t)}{\delta\zeta_j(t)}=2i\gamma_j^-\left(\frac{d}{d\zeta}\Psi^A(x,t;\zeta)\right)_{\zeta_j},$$

$$\frac{\delta\vec{r}(x,t)}{\delta\gamma_j^-(t)}=2i\Psi^A(x,t;\zeta_j).$$

$$\frac{\delta\vec{r}(x,t)}{\delta\bar{\rho}_-(\xi,t)}=\frac{1}{\pi}\bar{\Psi}^A(x,t;\xi)\quad\text{for every }\xi\in(-\infty,\infty),$$

$$\frac{\delta\vec{r}(x,t)}{\delta\bar{\zeta}_j(t)}=2i\bar{\gamma}_j^-\left(\frac{d}{d\zeta}\bar{\Psi}^A(x,t;\zeta)\right)_{\bar{\zeta}_j},$$

$$\frac{\delta\vec{r}(x,t)}{\delta\bar{\gamma}_j^-(t)}=2i\bar{\Psi}^A(x,t;\bar{\zeta}_j).$$

during the development of this work. Also, we would like to take this opportunity to thank D. Kaup and A. Newell for keeping us informed of their work. In particular, preprints of Refs. 11 and 12 were extremely useful.

APPENDIX

In this appendix we summarize one derivation of the modulational equations (2.9b). Because of our own background, we use some orthogonality relations from scattering theory in order to evaluate certain integrals; however, as we have emphasized in the text, this use of scattering is not necessary.

TABLE II. Completeness and orthogonality of the squared eigenfunctions. *Note*: All inner products of barred with unbarred squared eigenfunctions are zero.

Completeness

$$\begin{pmatrix}1&0\\0&1\end{pmatrix}\delta(x-x')=-\frac{1}{\pi}\int_{\mathfrak{C}_a}\frac{1}{[a(\zeta)]^2}\Psi^A(x',t;\zeta)[\Psi(x,t;\zeta)]^T d\zeta+\frac{1}{\pi}\int_{\mathfrak{C}_b}\frac{1}{[\bar{a}(\zeta)]^2}\bar{\Psi}^A(x',t;\zeta)[\bar{\Psi}(x,t;\zeta)]^T d\zeta$$

Orthogonality

$$(\Psi^*(\cdot,t;\zeta),\Psi^A(\cdot,t;\zeta'))=-\pi a^2(\zeta)\delta(\zeta-\zeta'),\quad \mathrm{Im}(\zeta)=\mathrm{Im}(\zeta')=0,$$
$$=0,\quad\text{all other cases}.$$

$$\left(\Psi^*(\cdot,t;\zeta_k),\frac{\partial}{\partial\zeta}\Psi^A(\cdot,t;\zeta)\Big|_{\zeta_j}\right)=\left(\left[\frac{\partial}{\partial\zeta}\Psi(x,t;\zeta)\Big|_{\zeta_k}\right]^*,\Psi^A(\cdot,t;\zeta_j)\right)=-\tfrac{1}{2}i(a_k')^2\delta_{kj}.$$

$$\left(\left[\frac{\partial}{\partial\zeta}\Psi(\cdot,t;\zeta)\Big|_{\zeta_k}\right]^*,\frac{\partial}{\partial\zeta}\Psi^A(\cdot,t;\zeta)\Big|_{\zeta_j}\right)=-\tfrac{1}{2}ia_k'a_k''\delta_{kj}.$$

$$(\bar{\Psi}^*(\cdot,t;\zeta),\bar{\Psi}^A(\cdot,t;\zeta'))=\pi\bar{a}^2(\zeta)\delta(\zeta-\zeta'),\quad \mathrm{Im}(\zeta)=\mathrm{Im}(\zeta')=0,$$
$$=0,\quad\text{all other cases}.$$

$$\left(\bar{\Psi}^*(\cdot,t;\bar{\zeta}_k),\frac{\partial}{\partial\zeta}\bar{\Psi}^A(\cdot,t;\zeta)\Big|_{\bar{\zeta}_j}\right)=\left(\left[\frac{\partial}{\partial\zeta}\bar{\Psi}(\cdot,t;\zeta)\Big|_{\bar{\zeta}_k}\right]^*,\bar{\Psi}^A(\cdot,t;\bar{\zeta}_j)\right)=-\tfrac{1}{2}i(\bar{a}_k')^2\delta_{kj}.$$

$$\left(\left[\frac{\partial}{\partial\zeta}\bar{\Psi}(\cdot,t;\zeta)\Big|_{\bar{\zeta}_k}\right]^*,\frac{\partial}{\partial\zeta}\bar{\Psi}^A(\cdot,t;\zeta)\Big|_{\bar{\zeta}_j}\right)=-\tfrac{1}{2}i\bar{a}_k'\bar{a}_k''\delta_{kj}.$$

The null space of $L=-i\sigma_3\partial_t+H$ can be parameterized by the initial values of the scattering data $\mathcal{S}_-(t=0)$ which is defined by equation (4.10a). In particular, the discrete component of this null space is spanned by

$$\left\{\frac{\partial\vec{r}_0}{\partial\zeta_j^-},\ \frac{\partial\vec{r}_0}{\partial(\zeta_j^-)^*},\ \frac{\partial\vec{r}_0}{\partial\Gamma_j^-},\ \frac{\partial\vec{r}_0}{\partial(\Gamma_j^-)^*},\quad j=1,2,\ldots,N\right\}.$$

To derive the modulational equations, we demand that the full source $\vec{F}$ be orthogonal to this discrete subspace. For example,

$$0=\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \vec{F}X\right)=\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \vec{f}+i\sigma_3\sum_{j=1}^{N}\left[\frac{\partial\zeta_j}{\partial\tau}\frac{\partial}{\partial\zeta_j}+\frac{\partial\Gamma_j}{\partial\tau}\frac{\partial}{\partial\Gamma_j}+\text{c.c.}\right]\vec{r}_0\right).$$

In this manner we obtain the modulational equations in the form

$$\sum_{j=1}^{N}\left[\frac{\partial\zeta_j}{\partial\tau}\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \sigma_3\frac{\partial\vec{r}_0}{\partial\zeta_j}\right)+\frac{\partial\Gamma_j}{\partial\tau}\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \sigma_3\frac{\partial\vec{r}_0}{\partial\Gamma_j}\right)+\frac{\partial\zeta_j^*}{\partial\tau}\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \sigma_3\frac{\partial\vec{r}_0}{\partial\zeta_j^*}\right)+\frac{\partial\Gamma_j^*}{\partial\tau}\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \sigma_3\frac{\partial\vec{r}_0}{\partial\Gamma_j^*}\right)\right]=i\left(\frac{\partial\vec{r}_0}{\partial(\zeta_k^-)^*},\ \vec{f}\right),\tag{A1}$$

together with three similar equations in which $\partial \tilde{\mathbf{r}}_0/\partial(\zeta_k^-)^*$ is replaced by $\partial \tilde{\mathbf{r}}_0/\partial \zeta_k^-$, $\partial \tilde{\mathbf{r}}_0/\partial(\Gamma_k^-)^*$, and $\partial \tilde{\mathbf{r}}_0/\partial \Gamma_k^-$. These four equations define the modulations of the discrete part of the scattering data $\mathcal{S}_+$; that is, of $\zeta_j \equiv \zeta_j^+$, $\zeta_j^* \equiv (\zeta_j^+)^*$, $\Gamma_j \equiv \Gamma_j^+$, and $\Gamma_j^* = (\Gamma_j^+)^*$.

To evaluate the inner products which appear on the left-hand side of the modulational equations, we find it convenient to change variables from Γ_j to γ_j,

$$\gamma_j^{\pm} \equiv \Gamma_j^{\pm} \exp[\pm i\theta_j], \quad \theta_j = 4\int_0^t \zeta_j^2(\epsilon t')\,dt'.$$

These variables and the identities

$$\frac{\partial}{\partial \zeta_j} \to \frac{\partial}{\partial \zeta_j} + \left(\frac{\partial \gamma_j^{\pm}}{\partial \zeta_j}\right)\frac{\partial}{\partial \gamma_j^{\pm}}, \qquad \frac{\partial}{\partial \Gamma_j^{\pm}} \to [\exp(\pm i\theta_j)]\frac{\partial}{\partial \gamma_j^{\pm}}, \tag{A2}$$

permit the inner products which we must compute to be expressed in terms of

$$\left(\frac{\partial \tilde{\mathbf{r}}_0}{\partial(\zeta_k^-)^*}, \sigma_3 \frac{\partial \tilde{\mathbf{r}}_0}{\partial \zeta_j}\right), \left(\frac{\partial \tilde{\mathbf{r}}_0}{\partial(\gamma_k^-)^*}, \sigma_3 \frac{\partial \tilde{\mathbf{r}}_0}{\partial \zeta_j}\right), \left(\frac{\partial \tilde{\mathbf{r}}_0}{\partial(\zeta_k^-)^*}, \sigma_3 \frac{\partial \tilde{\mathbf{r}}_0}{\partial \gamma_j}\right), \left(\frac{\partial \tilde{\mathbf{r}}_0}{\partial(\gamma_k^-)^*}, \sigma_3 \frac{\partial \tilde{\mathbf{r}}_0}{\partial \gamma_j}\right), \tag{A3}$$

together with their "conjugate expressions."

Next we use Table I to express the inner products of (A3) in terms of squared eigenfunctions. For example,

$$\left(\frac{\partial \tilde{\mathbf{r}}_0}{\partial(\zeta_k^-)^*}, \sigma_3 \frac{\partial \tilde{\mathbf{r}}_0}{\partial \zeta_j}\right) = 4i\gamma_k^-\gamma_j(\Psi^{A\prime}(\cdot, t; \zeta_k^*), \sigma_3\sigma_2\Psi'(\cdot, t; \zeta_j)) = -4\gamma_k^-\gamma_j(\Psi^{A\prime}(\cdot, t; \zeta_k^*), [\Psi'(\cdot, t; \zeta_j^*)]^*).$$

The orthogonality relations for the squared eigenfunctions are known[12] and are summarized in Table II. With these orthogonality relations, we compute

$$\left(\frac{\partial \tilde{\mathbf{r}}_0}{\partial(\zeta_k^-)^*}, \sigma_3 \frac{\partial \tilde{\mathbf{r}}_0}{\partial \zeta_j}\right) = -4\gamma_k^-\gamma_j([\Psi'(\cdot, t; \zeta_j^*)]^*, \Psi^{A\prime}(\cdot, t; \zeta_k^*))^* = -2i\frac{a_k''}{a_k'}\delta_{kj}.$$

In this manner, Table II can be used to compute all inner products of type (A3). These, together with (A2), can be used to place the modulational equation (A1) in the form (2.9b).

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$$\int_0^t (f, \partial_{t'} + H(r_0)g)\,dt' = -\int_0^t ([\partial_{t'} + H(r_0)]f, g)\,dt'.$$

When this symmetry does not hold, one must consider both $L(r_0)$ and its formal adjoint. However, in these cases very similar formulas apply. One example is the Korteweg–de Vries equation.

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