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Coherent pulse propagation: A comparison of the complete solution with the McCall-Hahn theory and others

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The complete solution of the Maxwell-Bloch equations by the inverse scattering transform as given by Ablowitz, Kaup, and Newell is compared with the McCall-Hahn theory, and others. Not only it is shown that the earlier results are contained in the complete solution, but also several new results are obtained. Among these are an infinite set of "nonlinear moments" which evolve similar to the McCall-Hahn area, a closed-form solution for the nonlinear transmission, how one can determine absolute time delays, why the threshold area for lossless propagation is still exactly π (even when the initial profile is off resonance but unchirped), and first-order effects of relaxation on 2π hyperbolic-secant pulse propagation. For the last-mentioned results, we find that in general if a 2π hyperbolic-secant pulse is initially off resonance, it will move away from resonance. Also, we find that the McCall-Hahn result for first-order effects on the time delay must be modified for small-amplitude 2π hyperbolic-secant pulses.

I. INTRODUCTION

With the recent advent of the "inverse scattering transform" (IST),[1] it has become possible to obtain the complete solution of certain nonlinear systems, one of which is the "Maxwell-Bloch equation"[2,3] describing coherent pulse propagation in a two-level system and in particular, "self-induced transparency" (SIT). Although many of the theoretical results were originally obtained by McCall and Hahn[2,3] using more simple methods, it was recognized earlier[4-7] that certain solutions of this system (2π pulses) did indeed have soliton-like properties.[8] Mostly due to the work of Lamb,[5-7,9-13] the soliton part of the solution was developed and studied, which allowed one to understand the asymptotic form of the solution in the attenuator case. In 1974, Ablowitz, Kaup, and Newell[14] obtained the complete solution for the Maxwell-Bloch (MB) equations, including the non-soliton part (called "radiation") as well as the soliton part. Since that time, further results have shown a much closer connection between the McCall-Hahn theory and the complete solution[15,16] than was at first suspected.

The purpose of this paper is manyfold. First, SIT is one of the few examples of these completely solvable nonlinear systems for which we have experimental results to compare the theory against, the most accurate of which are the results of Slusher and Gibbs.[17] Second, almost all of the results of the McCall-Hahn theory are to be contrasted against those of the complete theory, pointing out necessary modifications (as in nonlinear transmission), and also extensions of their results (such as nonlinear moments) which so naturally arise from the complete solution. Third, we can show how other results, such as small area limits,[18,19] various numerical,[20-22] and experimental[3,17] phenomena, which cannot be explained by the McCall-Hahn theory, can be explained and understood with the complete IST solution. And fourth, this will illustrate how some of the more simple but powerful results can be obtained by the IST.

In Sec. II, we briefly review the complete solution,[14] this time retaining all the physical constants and using a notation[17] more familiar to the experimentalists. For a square unchirped pulse profile, one can obtain closed-form solutions for the scattering data, and this is done in Sec. III. Here we discuss the analogies between the IST and the linear Fourier transform, and pointing out how the asymptotic soliton state can be uniquely determined from the initial data. In Sec. IV, we show how the McCall-Hahn area theorem is contained in the complete solution, and then how a simple extension allows one to define a complete set of "nonlinear moments" and their equations of motion, which are discussed in more detail in the Appendix.

In Sec. V, we look at nonlinear transmission in terms of the scattering data, and discuss how the amount of transmission is affected by the initial pulse profile. This is expanded on in Sec. VI where we consider how the eigenvalue spectrum is affected by the initial pulse profile. In Sec. VII, we show how to obtain the *absolute* time delay from the initial data, and in Sec. VIII, we discuss off-resonance effects and show that even in this case, the threshold area is still π, contrary to the suggestion by Diels and Hahn.[21] In Sec. IX, we make a few brief remarks concerning the continuous spectrum. In Sec. X, we consider the first-order effects of various relaxations on 2π hyperbolic secant (h.s.) pulse propagation, using a perturbation theory for the IST.[23] Our result for the evolution of the 2π h.s. pulse height agrees with that of McCall and Hahn,[3] but for the time delay, we find that the McCall-Hahn value must be corrected for small amplitude pulses whose widths are close to relaxation times. In addition, we also find that in general a 2π h.s. pulse off resonance will move *away* from resonance due to homogeneous broadening.

II. COMPLETE SOLUTION OF THE MB EQUATIONS WITHOUT DAMPING

It will be convenient first to give a brief description of our notation, to define our quantities, and to point out any differences from the common usage. We start with the applied electric field, and define a real envelope $\mathcal{E}$ and phase ψ by

$$\vec{E}(z,t) = \mathcal{E}(z,t)\{\hat{i}\cos[\omega t - kz + \psi(z,t)] + \hat{j}\sin[\omega t - kz + \psi(z,t)]\}, \quad (2.1)$$

where by retaining a possible t dependence in ψ, we can consider also chirped pulses. We shall find it useful to define a *complex* envelope ϵ where

$$\epsilon(z,t) \equiv \mathcal{E}(z,t)e^{-i\psi(z,t)}. \quad (2.2)$$

For the atomic quantities of the two-level atom, we shall closely follow the notation of Slusher and Gibbs.[17] We define

$$W = \tfrac{1}{2}\hbar\omega_0 N(\rho_{aa} - \rho_{bb}), \quad (2.3a)$$

$$X = \tfrac{1}{2}\hbar\omega_0 N(\rho_{aa} + \rho_{bb}), \quad (2.3b)$$

where $\omega_0 = (E_a - E_b)/\hbar$, and we shall find it useful to define also a *complex* polarization Λ by

$$\Lambda = ipN\rho_{ba}e^{+i\psi} = (iu - v)e^{+i\psi}, \quad (2.3c)$$

where $u(v)$ is the electric dipole dispersion (absorption) component introduced by McCall and Hahn.[3] In (2.3), ρ_{ij} are the components of the atomic density matrix, N is the density of atoms, and p is the McCall-Hahn dipole moment.

For the atomic relaxations and lifetimes, we define γ_{ab} to be the inverse time constant for decay from the state a to b, and γ_{ac} (γ_{bc}) to be the inverse time constant for decay from the state a (b) to any and all other states c. Thus[17,20]

$$1/T_1 = \gamma_{ab} + \tfrac{1}{2}\gamma_{ac}, \quad (2.4a)$$

$$1/T_2 = \tfrac{1}{2}(\gamma_{ab} + \gamma_{ac} + \gamma_{bc}) + \Gamma_{\text{phase}}, \quad (2.4b)$$

where Γ_{phase} contains any additional effects on the relaxation of the polarization.[20] We then have

$$\partial_\chi \epsilon = \frac{2\pi\omega}{nc}\langle\Lambda\rangle - \frac{2\pi}{nc}\sigma\epsilon + \frac{i\pi}{n^2}\langle\partial_\chi\Lambda\rangle + \frac{2\pi i}{nc}\langle\partial_\tau\Lambda\rangle, \quad (2.5)$$

$$\partial_\tau\Lambda + i\Delta\omega\Lambda + \Lambda/T_2 = (\kappa^2/\omega_0)\epsilon W, \quad (2.6a)$$

$$\partial_\tau W = -\tfrac{1}{2}\omega_0(\epsilon^*\Lambda + \Lambda^*\epsilon) + \tfrac{1}{2}\gamma_{bc}(X - W) - (X + W)/T_1, \quad (2.6b)$$

$$\partial_\tau X = -\tfrac{1}{2}\gamma_{ac}(X + W) - \tfrac{1}{2}\gamma_{bc}(X - W). \quad (2.6c)$$

In (2.5), n is the index of refraction, σ is the effective conductivity, the last two terms give the corrections due to the second derivatives in Maxwell's equation, while

$$\langle\Lambda\rangle \equiv \int_{-\infty}^{\infty} g(\Delta\omega)\Lambda(\Delta\omega)\,d\Delta\omega, \quad (2.7)$$

where $g(\Delta\omega)$ is the inhomogeneous broadening factor, with $\Delta\omega = \omega_0 - \omega$. In (2.6a), $\kappa = 2p/\hbar$, and in these equations, we are using the coordinates χ and τ, where

$$\chi = z, \quad (2.8)$$

$$\tau = t - (n/c)z. \quad (2.9)$$

The last two terms in (2.5) are invariably smaller than all other effects, except for extremely short pulses. Finally, we note that when $\gamma_{ac} \neq 0 \neq \gamma_{bc}$, then our total population of atoms (in levels a or b) will decrease in time. This can be accounted for by allowing $g(\Delta\omega)$ in (2.7) to be time dependent, as given by

$$\partial_\tau \ln g = \partial_\tau \ln X, \quad (2.10)$$

since X is proportional to number density per unit frequency.

As shown by Lamb,[11] Eqs. (2.6) for $\gamma_{bc} = \gamma_{ac} = 0$, $T_1 = T_2 = \infty$, very naturally decompose into the Zakharov and Shabat (ZS) eigenvalue problem. (This decomposition is even more natural than one might at first suspect. See McLaughlin and Corones.[15]) What was even more significant was that the 2π h.s. pulses of SIT were exactly the "solitons"[4,8] of this inverse scattering theory.[11,12] This was enough to indicate that the complete general initial-value problem for coherent pulse propagation could be solved. After valuable conversations with Lamb, Ablowitz, Kaup, and Newell[14] were able to incorporate Eq. (2.5) (upon ignoring the last two terms and setting $\sigma = 0$) also into the inverse scattering theory, which then yielded the complete solution. The equations which they solved were

$$\partial_\chi \epsilon = (2\pi\omega/nc)\langle\Lambda\rangle, \quad (2.11a)$$

$$\partial_\tau\Lambda + i\Delta\omega\Lambda = (\kappa^2/\omega_0)\epsilon W, \quad (2.11b)$$

$$\partial_\tau W = -\tfrac{1}{2}\omega_0(\epsilon^*\Lambda + \Lambda^*\epsilon). \quad (2.11c)$$

In terms of the above notation, the complete solution was given as follows:[14] Consider the ZS eigenvalue problem[11,24]

$$\partial_\tau v_1 + i\zeta v_1 = \tfrac{1}{2}\kappa\epsilon v_2, \quad (2.12a)$$

$$\partial_\tau v_2 - i\zeta v_2 = -\tfrac{1}{2}\kappa\epsilon^* v_1 , \tag{2.12b}$$

where ζ is the eigenvalue. The continuous spectrum is described as follows. Define ϕ to be the solution of (2.12), for real ζ, where

$$\phi \to \begin{pmatrix}1\\0\end{pmatrix} e^{-i\zeta\tau} \quad \text{as } \tau \to -\infty . \tag{2.13a}$$

Then $a(\zeta,\chi)$ and $b(\zeta,\chi)$ are defined from

$$\phi \to \begin{pmatrix}a(\zeta,\chi)e^{-i\zeta\tau}\\ b(\zeta,\chi)e^{i\zeta\tau}\end{pmatrix} \quad \text{as } \tau \to +\infty , \tag{2.13b}$$

where we are assuming

$$\int_{-\infty}^{\infty} |\epsilon|\, d\tau < \infty . \tag{2.14}$$

It is this condition which guarantees the existence of a and b.[24] Similarly, define $\overline{\phi}$ to be the other linearly independent solution of (2.12) which satisfies

$$\overline{\phi} \to \begin{pmatrix}0\\-1\end{pmatrix} e^{i\zeta\tau} \quad \text{as } \tau \to -\infty , \tag{2.15a}$$

then $\overline{b}$ and $\overline{a}$ are defined from

$$\overline{\phi} \to \begin{pmatrix}\overline{b}(\zeta,\chi)e^{-i\zeta\tau}\\ -\overline{a}(\zeta,\chi)e^{i\zeta\tau}\end{pmatrix} \quad \text{as } \tau \to +\infty . \tag{2.15b}$$

It follows[1,24] for a general complex ζ that

$$\overline{\phi}(\zeta) = \begin{pmatrix}\phi_2^*(\zeta^*)\\ -\phi_1^*(\zeta^*)\end{pmatrix} , \tag{2.16}$$

so

$$\overline{a}(\zeta) = a^*(\zeta^*) , \tag{2.17a}$$

$$\overline{b}(\zeta) = b^*(\zeta^*) , \tag{2.17b}$$

where for real ζ

$$\overline{a}a + \overline{b}b = 1 . \tag{2.18}$$

From ϕ and $\overline{\phi}$, one can construct W and λ.[11,14]

$$W = -(\pm)\tfrac{1}{2}\hbar\omega_0 N(\phi_1\overline{\phi}_2 + \overline{\phi}_1\phi_2)\big|_{\zeta=(\Delta\omega)/2} , \tag{2.19}$$

$$\Lambda = -(\pm)\hbar\kappa N\phi_1\overline{\phi}_1\big|_{\zeta=(\Delta\omega)/2} , \tag{2.20}$$

where the upper (lower) sign is to be taken for the amplifier (attenuator) case.

In addition to the continuous spectrum, (2.12) may also have bound states which vanish exponentially as $\tau \to \pm\infty$. From (2.13), these occur at the zeros of $a(\zeta)$ in the upper-half ζ plane. Assuming these zeros to be simple (for handling the case of multiple zeros, see Refs. 1 and 24) and designating them by $\{\zeta_j\}_{j=1}^{J}$, we have at such a zero

$$\phi(\zeta_j) \to \begin{pmatrix}0\\ b_j(\chi)\end{pmatrix} e^{i\zeta_j\tau} \quad \text{as } \tau \to +\infty , \tag{2.21}$$

which defines $b_j(\chi)$. Each one of these bound states will correspond to a 2π h.s. pulse.

From the solution of the inverse scattering problem, one may reconstruct the solution for $\epsilon(\tau,\chi)$. First, one constructs the function[1]

$$G(\tau,\chi) = -\sum_{j=1}^{J} D_j(\chi)e^{-i\zeta_j\tau} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\overline{b}(\zeta,\chi)}{a(\zeta,\chi)} e^{-i\zeta\tau}\, d\zeta , \tag{2.22}$$

where

$$D_j^{-1} = -b_j \frac{\partial a}{\partial \zeta}\bigg|_{\zeta=\zeta_j} . \tag{2.23}$$

We note that at $a=0$, then due to (2.18), $b = 1/\overline{b}$ in (2.23). Alternately, one may also determine D_j as the normalization constant for $\phi(\zeta_j)$, since[1]

$$\int_{-\infty}^{\infty} \phi_1(\zeta_j,\tau)\phi_2(\zeta_j,\tau)\, d\tau = -\tfrac{1}{2}D_j^{-1} . \tag{2.24}$$

Next, we solve the linear integral equation

$$\overline{L}_1(\tau,\theta) + \int_{-\infty}^{\tau} d\beta \int_{-\infty}^{\tau} d\gamma\, \overline{L}_1(\tau,\gamma)G^*(\gamma+\beta)G(\beta+\theta) = -G(\tau+\theta) , \tag{2.25}$$

for $\overline{L}_1(\tau,\theta)$, which has a unique solution[1,24] for $\theta \le \tau$. Then $\epsilon(\tau,\chi)$ is given by

$$\kappa\epsilon(\tau,\chi) = 4\overline{L}_1(\tau,\tau;\chi) . \tag{2.26}$$

The method of solution is as follows. Take the initial data of $\epsilon(\tau,0)$ at $\chi = z = 0$, and determine $\overline{b}/a$ for ζ real, and the set of bound-state parameters $\{\zeta_j, D_j\}_{j=1}^{J}$, all at $\chi=0$. Then from the χ dependence of this scattering data, given by[14]

$$\partial_\chi\left(\frac{\overline{b}}{a}\right) = (\pm)\frac{1}{2\pi i}\,\frac{\overline{b}}{a}\int_{-\infty}^{\infty} \frac{\alpha(\Delta\omega)\, d\Delta\omega}{\Delta\omega - 2\zeta - i0^+} , \tag{2.27}$$

$$\partial_\chi D_j = \frac{(\pm)}{2\pi i} D_j \int_{-\infty}^{\infty} \frac{\alpha(\Delta\omega)\, d\Delta\omega}{\Delta\omega - 2\zeta_j} , \tag{2.28}$$

$$\partial_\chi \zeta_j = 0 , \tag{2.29}$$

where

$$\alpha(\Delta\omega) \equiv (2\pi^2\kappa^2/nc)\hbar\omega N g(\Delta\omega) . \tag{2.30}$$

[Note that $\alpha_0 \equiv \alpha(0)$ is simply the inverse Beer's length.] One can then determine the scattering data at any later χ. Now construct G by (2.22), and determine $\epsilon(\tau,\chi)$ from (2.25), (2.26).

It is well known that there is an infinity of local conservation laws[7,10] associated with (2.11). These are found from the asymptotic expansion of $\ln a(\zeta)$ for $|\zeta| \to \infty$ in the upper-half ζ plane. This gives[1,24]

$$\ln a(\zeta) \to \sum_{n=1}^{\infty} (-iC_n)\zeta^{-n} , \tag{2.31}$$

where for the first three C_n's, we have[1,7,10]

$$C_1 = \tfrac{1}{8}\kappa^2 \int_{-\infty}^{\infty} \epsilon^*\epsilon\, d\tau , \tag{2.32a}$$

$$C_2 = \tfrac{1}{32} i \kappa^2 \int_{-\infty}^{\infty} (\epsilon^* \partial_\tau \epsilon - \epsilon \partial_\tau \epsilon^*) \, d\tau \,, \tag{2.32b}$$

$$C_3 = \tfrac{1}{32} \kappa^2 \int_{-\infty}^{\infty} [\,|\partial_\tau \epsilon|^2 - \kappa^2 \tfrac{1}{4} (\epsilon^* \epsilon)^2] \, d\tau \,. \tag{2.32c}$$

One can also obtain the C_n's in terms of the scattering data. This is given by[1,24]

$$C_n = i \frac{(-)^n}{n} \left(\sum_{j=1}^{J} (\zeta_j^n - \zeta_j^{*n}) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \xi^{n-1} \ln[1 + \Gamma(\xi)] \, d\xi \right) , \tag{2.33}$$

where

$$\Gamma(\xi) \equiv |\bar{b}(\xi)/a(\xi)|^2 \,. \tag{2.34}$$

The χ dependence of Γ follows directly from (2.27) and is given by

$$\partial_\chi \Gamma(\xi) = (\pm)\alpha(2\xi)\Gamma(\xi) \,. \tag{2.35}$$

Note that C_1 is just proportional to the energy in the pulse, $\mathcal{T}$. Thus

$$\mathcal{T} = \frac{nc}{4\pi} \int_{-\infty}^{\infty} \epsilon^* \epsilon \, d\tau = \frac{4nc}{\pi \kappa^2} \left(\sum_{j=1}^{J} \eta_j + \frac{1}{4\pi} \int_{-\infty}^{\infty} d\xi \ln(1 + \Gamma) \right) , \tag{2.36}$$

where η_j is the imaginary part of $\zeta_j (\eta_j > 0)$, the bound-state eigenvalues.

Finally, we simply note that as long as the Maxwell-Bloch equations are valid, then this inverse scattering solution is valid. For example, if as the solution evolves, one finds that the dictates of the slowly varying envelope approximation are no longer valid, it then follows that the Maxwell-Bloch equations are no longer valid. The IST simply allows one to solve the undamped MB equations exactly.

III. EXACTLY SOLVABLE MODEL

Before entering into a comparison of the complete theory with others, we shall present an exactly solvable model which we shall refer to often. Furthermore, this example will illustrate what we mean when we refer to the inverse scattering transform as a "nonlinear" Fourier transform.

The model we take is a simple unchirped square pulse, where at $\chi = 0$,

$$\epsilon(0, \tau) = \begin{cases} E & \text{if } 0 < \tau < \tau_p \\ 0 & \text{otherwise} \end{cases} \,, \tag{3.1}$$

where E is the amplitude, and τ_p is the pulse width.[25] Solving (2.12) for the initial values of a and b gives

$$a(\zeta) = \tfrac{1}{2} e^{i\zeta\tau_p} [(1 - \zeta/\lambda) e^{i\lambda\tau_p} + (1 + \zeta/\lambda) e^{-i\lambda\tau_p}] \,, \tag{3.2}$$

$$b(\zeta) = -\tfrac{1}{2} \kappa E e^{-i\zeta\tau_p} (\sin\lambda\tau_p)/\lambda \,, \tag{3.3}$$

where

$$\lambda^2 = \zeta^2 + \tfrac{1}{4}\kappa^2 E^2 \,. \tag{3.4}$$

Thus the continuous part of the scattering data, $\bar{b}/a$, (for ζ real) is simply

$$\frac{\bar{b}}{a}(\zeta, 0) = -\tfrac{1}{2}\kappa E \frac{\sin\lambda\tau_p}{\lambda \cos\lambda\tau_p - i\zeta \sin\lambda\tau_p} \,, \tag{3.5}$$

and also

$$\Gamma(\zeta, 0) = \left| \frac{\bar{b}}{a}(\zeta, 0) \right|^2 = \tfrac{1}{4}\kappa^2 E^2 \frac{\sin^2\lambda\tau_p}{\zeta^2 + \frac{1}{4}\kappa^2 E^2 \cos^2\lambda\tau_p} \,. \tag{3.6}$$

For this model (3.1), the initial area θ_0 is simply

$$\theta_0 = \kappa E \tau_p \,. \tag{3.7}$$

Let us now consider (3.2) and (3.3) in the limit of small absolute areas ($\kappa \int |\epsilon| d\tau \ll 1$) and when $|\zeta|$ is much larger than the Rabi frequency κE. Then

$$a(\zeta) \simeq 1 \,, \tag{3.8a}$$

$$b(\zeta) \simeq -\tfrac{1}{2}\kappa E e^{-i\zeta\tau_p} (\sin\zeta\tau_p)/\zeta \,, \tag{3.8b}$$

or

$$b(\zeta) \simeq -\tfrac{1}{2}\kappa \hat{\epsilon}(2\zeta) \,, \tag{3.9}$$

where $\hat{\epsilon}(k)$ is the Fourier transform of $\epsilon(\tau)$. Note the relation between (3.3) and (3.8b). As we go to the fully nonlinear limit where $\theta_0 \simeq 1$, then the qualitative structure of $b(\zeta)$ does not change. For $|\zeta|$ larger than the Rabi frequency, (3.3) is essentially the Fourier transform, while for smaller values of $|\zeta|$, it is "stretched out." Similarly, $a(\zeta)$ only differs from unity when $|\zeta|$ is of the order of or less than the Rabi frequency. Thus we may consider $\bar{b}/a$ to be at least qualitatively, a "nonlinear Fourier transform" of $-\frac{1}{2}\kappa\epsilon$.

The bound states of this model are determined from the zeros of $a(\zeta)$ for ζ in the upper-half ζ plane. If we define

$$x \equiv \lambda\tau_p \,, \tag{3.10a}$$

$$y \equiv -i\zeta\tau_p \,, \tag{3.10b}$$

then requiring $a(\zeta)$ to be zero gives

$$x^2 + y^2 = \tfrac{1}{4}\theta_0^2 \,, \tag{3.11a}$$

$$y = -x \cot x \,. \tag{3.11b}$$

These equations, (3.11), are exactly the same as those for determining the bound states of a quantum-mechanical particle in an attractive, three-dimensional, spherical well potential.[26] Although, in general, the bound-state eigenvalues of (2.12) need not always lie on the imaginary ζ axis, in this case they must, since (3.11) also arises for the Schrödinger equation. From the solution given

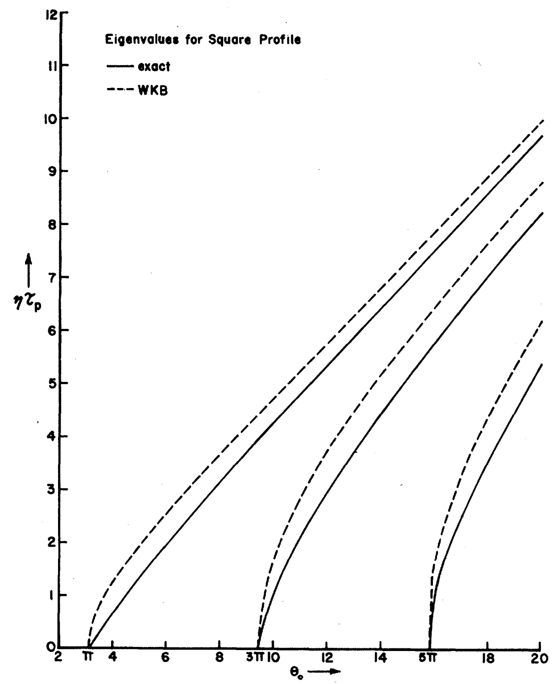

FIG. 1. Eigenvalues η vs the initial pulse area θ_0 for the square-pulse profile described in the text. The solid lines are the exact solutions while the dashed lines are the WKB solutions (see Sec. VI).

by Schiff,[26] we see that every time θ_0 crosses a value of $(2n+1)\pi$, a new bound state occurs, and the eigenvalues of these bound states may be determined either graphically or numerically. A plot of these eigenvalues versus the area of the initial pulse profile is shown in Fig. 1.

The last piece of scattering data required is the values of the D_j's. These may be calculated as the residue of $i\bar{b}/a$ at $\zeta=\zeta_j$. To obtain these when $\theta_0>\pi$, we define η_j by

$$\zeta_j = i\eta_j \quad (j=1,2,\ldots,J), \tag{3.12}$$

where J is the total number of bound states, and let

$$y_j=\eta_j\tau_p, \tag{3.13a}$$

$$x_j=(\tfrac{1}{4}\theta_0^2-y_j^2)^{1/2}, \tag{3.13b}$$

as given in (3.10). Then the residues of $i\bar{b}/a$ for each value of j are

$$D_j=\frac{2}{\theta_0\tau_p}\frac{x_j^2}{1+y_j}. \tag{3.14}$$

At this point, we have decomposed the initial pulse profile into the various independent components in "scattering space," like in any linear problem where one would decompose the initial profile into the independent components in "Fourier space." We know how these components in scattering space evolve in χ from (2.27)–(2.29), where the continuous spectrum, in general, decays exponentially,[14] while the bound-state spectrum remains invariant. From a plot such as Fig. 1, for a given pulse profile and initial area θ_0, we can determine the bound-state spectrum which then remains. For example, if $\pi<\theta_0<3\pi$, we have only one bound state and from (2.22)–(2.26), the solution will approach

$$\kappa\epsilon=\frac{4\eta_j}{\cosh[2\eta_j\tau+\ln(\frac{1}{2}D_j(\chi)/\eta_j)]}, \tag{3.15}$$

which is a 2π h.s. pulse. One should notice how the propagation of the continuous and the bound-state spectrum in scattering space has a one-to-one correspondence to the propagation of the actual pulse (which is a *nonlinear* combination of the continuous and bound-state spectrum). First, the continuous spectrum decays, which corresponds to the initial energy loss and pulse reshaping of the actual pulse. Meanwhile, the bound-state eigenvalues remain invariant, which corresponds to the 2π h.s. pulses propagating losslessly. Also, strictly speaking, a 2π h.s. pulse is never *created.* Rather, it is always present, even in the initial data, although hidden due to the nonlinear mixing of the continuous and bound-state spectrum. And it is only after the decay of the continuous spectrum that it emerges, allowing itself then to be obviously identified.

Returning to (3.15), we will define τ_{0j} to be the center of the pulse, propagated back to $\chi=0$. Thus by (3.14) and (3.15),

$$\tau_{0j}\equiv\tfrac{1}{2}\tau_p\frac{1}{y_j}\ln\left(\theta_0\frac{y_j}{x_j^2}(1+y_j)\right). \tag{3.16}$$

From (3.16), it is possible to determine the *absolute* time delay for each of those 2π h. s. pulses. Of course, when more than one soliton is present, there will be additional shifts arising from the differences of the various eigenvalues.[1,8] Equation (3.16) gives only the contribution to the time delay from D_j for this model. A plot of τ_{0j} for the first three branches is given in Fig. 2.

The χ dependence of D_j follows from (2.28), and if $\alpha(\Delta\omega)$ is an even function, then the spacial dependence in (3.15) is given by[3,11,13,14]

$$\kappa\epsilon(\tau,\chi)=4\eta_j\,\mathrm{sech}\left[2\eta_j\left(\tau-\tau_{0j}+(\pm)\chi\frac{1}{2\pi}\times\int_{-\infty}^{\infty}\frac{\alpha(\Delta\omega)\,d\Delta\omega}{4\eta_j^2+\Delta\omega^2}\right)\right]. \tag{3.17}$$

IV. AREA THEOREM AND EXTENSIONS

One of the most simple results of the McCall-Hahn theory[3] is the "area theorem." When ψ is

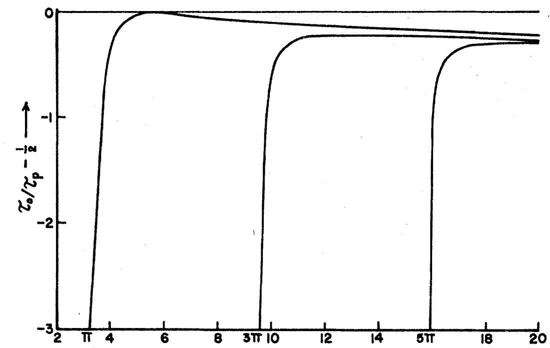

FIG. 2. $(\tau_{0j}/\tau_p - \frac{1}{2})$ vs θ_0 for the square profile. (The various curves actually go to minus infinity as each one approaches its threshold area.) The factor of $-\frac{1}{2}$ is entered since the center of the initial profile is at $\tau = \frac{1}{2}\tau_p$ (see text).

independent of t, McCall and Hahn showed that

$$\frac{d\theta}{d\chi} = (\pm)\tfrac{1}{2}\alpha_0 \sin\theta\,, \tag{4.1}$$

where the area θ is defined by

$$\theta(\chi) \equiv \kappa \int_{-\infty}^{\infty} \mathcal{E}(\chi, \tau)\, d\tau\,. \tag{4.2}$$

Of course, the area is just the linear Fourier transform at zero argument, so it is then not at all surprising that the zero argument of the "nonlinear" Fourier transform, $\bar{b}/a$, contains this area theorem also. From (3.5), we have $\bar{b}/a(\zeta=0) = -\tan(\frac{1}{2}\theta)$ for the model (3.1). This is also a general result whenever one is on resonance [$g(\Delta\omega)$ symmetric] and ψ vanishes, so that ϵ is real. To show this, one may readily verify that the general solution of (2.12) at $\zeta = 0$ and for $\epsilon = \mathcal{E}$ is[27]

$$v_1 = v_{10}\cos\tfrac{1}{2}\mathfrak{a} + v_{20}\sin\tfrac{1}{2}\mathfrak{a}\,, \tag{4.3a}$$

$$v_2 = -v_{10}\sin\tfrac{1}{2}\mathfrak{a} + v_{20}\cos\tfrac{1}{2}\mathfrak{a}\,, \tag{4.3b}$$

where v_{10} and v_{20} are arbitrary functions of χ, and

$$\mathfrak{a}(\chi, \tau) \equiv \kappa \int_{-\infty}^{\tau} \mathcal{E}(\chi, \tau)\, d\tau\,. \tag{4.4}$$

Thus from (2.13) and (2.17), upon setting $v_{10} = 1$ and $v_{20} = 0$, $\bar{b}/a$ at $\zeta = 0$ is therefore simply

$$\frac{\bar{b}}{a}(\zeta = 0, \chi) = -\tan\tfrac{1}{2}\theta(\chi)\,. \tag{4.5}$$

Turning now to (2.27), we find at $\zeta = 0$ that

$$\partial_\chi \ln(\bar{b}/a) = (\pm)\left(\tfrac{1}{2}\alpha_0 + \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \alpha(\Delta\omega)\frac{d\Delta\omega}{\Delta\omega}\right)\,, \tag{4.6}$$

where P indicates the Cauchy principle value integral. When $\alpha(\Delta\omega)$ is symmetric, the integral vanishes, then from (4.5) and (4.6) we obtain

$$\tan\tfrac{1}{2}\theta(\chi) = (\tan\tfrac{1}{2}\theta_0)e^{\pm\alpha_0\chi/2}\,, \tag{4.7}$$

which is the integrated form of the McCall-Hahn area theorem, Eq. (4.1).

Consequently, whenever the initial pulse is on resonance, is unchirped, and the inhomogeneous broadening is symmetric, we have that the area theorem of McCall and Hahn is contained in (4.6), due to (4.5). One now notes that (4.6) is more general than the simple result of McCall-Hahn, since (4.6) is valid for any integrable initial profile (even chirped) and for any inhomogeneous broadening, $g(\Delta\omega) \propto \alpha(\Delta\omega)$. Although $\bar{b}/a$ at $\zeta = 0$ cannot be simply expressed in terms of "areas" in the general case, nevertheless (4.6) is the natural generalization. For example, if we allow the initial pulse to be chirped, and consider the small area limit of Crisp,[19] then we can obtain his area theorem, when the transverse relaxation vanishes. From (2.12)–(2.15), for small areas we find

$$\frac{\bar{b}}{a}(\zeta = 0, \chi) \simeq -\tfrac{1}{2}\kappa \int_{-\infty}^{\infty} \epsilon(\chi, \tau)\, d\tau$$

$$\simeq -\tfrac{1}{2}\kappa \int_{-\infty}^{\infty} \mathcal{E}(\chi, \tau)e^{i\psi(\chi,\tau)}\, d\tau\,. \tag{4.8}$$

Inserting (4.8) into (4.6) and upon taking $g(\Delta\omega)$ to be a Lorentzian as Crisp did[19] will then give his result. Note that (4.8) agrees with (4.5) when θ is small and $\psi = 0$.

But this is not all. Consider the McCall-Hahn area theorem when the pulse is real and has positive and negative parts, such that the total area vanishes. Now (4.1) gives a trivial result for an important class of pulse shapes, which includes the $0-\pi$ pulse.[5] To find an analogy for this class of pulse shapes, one needs to only reconsider the (linear or nonlinear) Fourier transform at $\zeta = 0$. Although zero area implies the vanishing of this quantity at $\zeta = 0$, its slope need not vanish. And, for the linear Fourier transform, its slope at $\zeta = 0$ is simply related to its first moment. For the nonlinear Fourier transform, $\bar{b}/a$, there is also a corresponding quantity μ_1 which is like a "nonlinear first moment," and is given by

$$\mu_1 = \kappa \int_{-\infty}^{\infty} \tau\mathcal{E}\cos\mathfrak{a}_+\, d\tau\,, \tag{4.9}$$

where

$$\mathfrak{a}_+ = \kappa \int_{\tau}^{\infty} \mathcal{E}\, d\tau\,. \tag{4.10}$$

As in the case of the McCall-Hahn area theorem, (2.27) then implies that for the attenuator case, μ_1 must evolve according to

$$\mu_{1,\chi} = -\tfrac{1}{2}\alpha_0\mu_1\,, \tag{4.11}$$

when $\theta=0$ and the inhomogeneous broadening is symmetric. The details of this are given in the Appendix, where we also show how to handle the general case of $\theta\neq 0$, and also the "second nonlinear moment." Naturally, there is an infinity of these moments, all of which must vanish as $\chi\to\infty$ for the attenuator case.

V. NONLINEAR TRANSMISSION

One of the more striking features of SIT is the dramatic deviation from Beer's law which occurs whenever the area of the initial pulse becomes greater than π. By simulations, McCall and Hahn[3] were the first to describe this nonlinear transmission by the empirical rate equation

$$\frac{d\mathcal{T}}{d\chi}=-\alpha_0\mathcal{T}F(\theta,\ \mathcal{T},\text{"pulse shape"}), \tag{5.1}$$

where the pulse energy $\mathcal{T}$ is given by (2.39). In this form, one could interpret F to be just the nonlinear correction to Beer's law. But still, (5.1) was found to be valid in general only for reasonably small areas of less than about 3π. For larger areas, the deviation from Beer's law becomes so large that an expression such as (5.1) becomes almost useless.

The reason for this is easily found in the complete theory, where we find that the pulse energy is actually composed of two parts: one which is

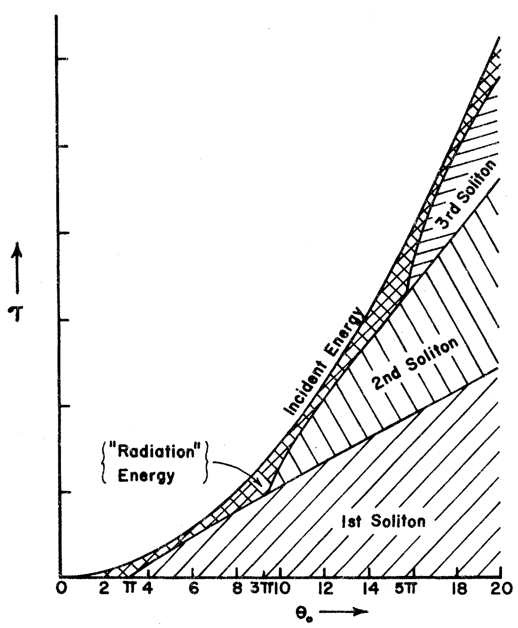

FIG. 3. Incident energy, radiation energy, and energy content of the various solitions as a function of θ_0 for the square profile. The units of energy are $16nc/\pi\tau_p$.

transmitted losslessly (solitons) and one which is absorbed according to Beer's law (the continuous spectrum, or radiation). The transmission of these two parts is so dramatically different that an approximation based on the transmission properties of one part can never fully describe the transmission properties of the other part, as in the case of (5.1). If we consider the problem of nonlinear transmission in the complete theory, we find that indeed we do not need (5.1), since one can obtain a closed-form solution for $\mathcal{T}(\chi)$ in terms of the initial scattering data.

From (2.35) and (2.36) we have

$$\mathcal{T}(\chi)=\frac{4nc}{\pi\kappa^2}\left(\sum_{j=1}^{J}\eta_j+\frac{1}{4\pi}\int_{-\infty}^{\infty}d\xi\ln[1+\Gamma_0(\xi)e^{\pm\alpha(2\xi)\chi}]\right), \tag{5.2}$$

where $\Gamma_0(\xi)=\Gamma(\xi,\chi=0)$ and $\{\eta_j\}_{j=1}^{J}$ are the imaginary parts of the bound-state eigenvalues of (2.12), which by (2.29) are χ independent.

Before continuing on, it will be well worthwhile to consider (5.2) in various limits, and to point out some implications. First, we start with the linear limit where $\kappa\int|\epsilon|d\tau\ll 1$, so that there are no bound states (solitons), and $\Gamma_0(\xi)\ll 1$. Then in this limit

$$\mathcal{T}(\chi)\simeq\frac{nc}{\pi^2\kappa^2}\int_{-\infty}^{\infty}d\xi\,\Gamma_0(\xi)e^{\pm\alpha(2\xi)\chi}, \tag{5.3}$$

which is just the Beer's limit result, as one would expect, since now $\Gamma_0(\xi)$ is simply the square of the magnitude of the linear Fourier transform of $-\frac{1}{2}\kappa\epsilon$ [see (3.8) and (3.9)] at $\chi=0$. As the initial area is increased up to an order of unity, then nonlinear deviations from Beer's law start to occur, as is indicated by (5.1). In terms of the scattering data, these deviations are very simple in that $\Gamma_0e^{\pm\alpha\chi}$ in (5.3) is simply replaced by $\ln(1+\Gamma_0e^{\pm\alpha\chi})$. When the initial area becomes greater than π, the bound-state spectrum of (2.12) has appeared and these then contribute to (5.2). If we consider the case of an attenuator, then as $\chi\to+\infty$, $\Gamma(\xi,\chi)$ vanishes in (5.2), which leaves only the soliton (2π pulse) part of the energy. On the other hand, for the amplifier case, as χ becomes large, the continuous spectrum $\Gamma(\xi,\chi)$ becomes larger and will eventually dominate.

In order to understand how this decomposition of the initial pulse energy into soliton energy and radiation energy is related to the shape of the initial pulse profile, let us consider Fig. 3 where we have plotted the initial energy, for the model discussed in Sec. III, and its transmitted energy in the attenuator case (for $\chi=\infty$) versus the *initial* pulse area. Here, one can clearly see that as the initial area increases and crosses $(2n+1)\pi$, another soliton (2π pulse) appears in the spectrum. Furthermore, one should note that as the initial area is increasing, each soliton is increasing its own energy content, in such a manner as to allow as much of the initial pulse energy as practical to be transmitted. (If we had plotted the ratio of the transmitted energy to the incident energy instead, we would have obtained a curve similar to Fig. 13 of Ref. 17.) Consideration of other profiles which are also slowly varying (such as the $\tau e^{-\tau}$ and Gaussian profiles) gives similar results, except that the profile in Fig. 3 for the box model tends to have the largest amount of "radiation energy," due to the sharp discontinuities at the edges.

Of course, one can also have profiles where the radiation part does dominate over the soliton part, and a simple example of this would be the amplifier solution for χ large. Here, the pulse energy is essentially all radiation, and the profile is a sharp spike followed by decaying oscillations (ringing). Thus from the above we can conclude that as a general rule, when the profile is slowly varying and has sufficient (absolute) area, one can expect the soliton spectrum to dominate. On the other hand, if the (absolute) area is small or the profile is not slowly varying (has sharp discontinuities or rapid oscillations) then we can expect the radiation part to become relatively more important.

Unfortunately, at the present there is no general simple criteria for solitons to exist for an arbitrary initial profile. In general, it is felt that the absolute area $\int \kappa|\epsilon|d\tau$ must be at least of order π. Of course, for unchirped profiles with only one extremum, the McCall-Hahn result gives the simple criteria that $\theta_0 > \pi$. Otherwise, the only other general criteria is that the total change in the phase of $a(\zeta)$ from $\zeta=-\infty$ to $\zeta=+\infty$ must be nonzero.[1]

VI. SOLITON SPECTRUM

After a pulse enters an attenuating resonant medium, it undergoes a reshaping whereby the continuous spectrum of (2.12) is absorbed by the medium within a few Beer's lengths, and the N solitons and M bions are formed, which eventually break away and propagate losslessly in the medium. Thus, after this initial transition region, the solution rapidly approaches what is known as an N-soliton solution,[8] which is completely specified upon knowing the bound-state parameters $\{\zeta_j, D_j\}_{j=1}^{N}$ of (2.12).

Up until 1972, this final state could only be determined via computer simulations of the MB equations. In 1972, Lamb, Scully, and Hopf[7] showed how one could approximately determine the pulse heights from the infinity of conserved quantities, which was later extended by Schnack and Lamb[10] in 1973. This was a reasonably successful method, and why it is, can be seen by reconsidering Fig. 3. In Fig. 3, we see for large areas, that the energy of the pulse (which is the first one of the infinity of conserved quantities) is almost entirely soliton energy. Thus, without much error, one can ignore the energy retained by the continuous spectrum and assume that all of the initial pulse energy is distributed only among the solitons. Similarly with the higher conserved quantities, one can assume that essentially all of their initial values will be distributed also among the solitons. Of course, this procedure will only be valid when the soliton part contains the major part of these conserved quantities. In particular, as indicated earlier, whenever the initial profile is not slowly varying and has discontinuities and/or rapid oscillations (as in a strongly chirped pulse) the soliton part can be decreased significantly, which causes the above method to become much less accurate.

In 1973 and again in 1974 when Lamb[11,13] indicated how the MB equations might be related to the IST theory, he did not point out how one can determine *exactly* the final pulse heights directly from the initial pulse profile. From his eigenvalue equations, Eq. (8) of Ref. 11 and Eq. (2.10) of Ref. 13, one had that the 2π h.s. pulse heights would be related to the bound-state eigenvalues of these eigenvalues problems, as shown by his inverse scattering analysis. Although it was clear that these bound-state eigenvalues were χ independent in the *absence* of the continuous spectrum (which was the only case for which he obtained the χ dependence), one could not be certain that they were also χ independent in the presence of the continuous spectrum, until the full problem was solved. This was finally accomplished in 1974 by Ablowitz, Kaup, and Newell,[14] who showed that indeed these eigenvalues were *always* χ independent, and that they (and therefore the final 2π h.s. pulse heights) could be determined directly from the initial pulse profile.

These eigenvalues can always be found fairly rapidly by numerically searching for the bound-state eigenvalues of (2.12). (Some "tricks" are discussed in Ref. 1.) In Fig. 4, we give the results for the following three initial pulse profiles: (i) the model in Sec. III; (ii) the $\tau e^{-\tau}$ profile given explicitly by

$$\kappa\epsilon(0,\tau)=\begin{cases}(16\theta_0/\tau_p^2)\tau e^{-4\tau/\tau_p} & \text{if } 0\le\tau\\ 0 \quad \text{if } \tau\le 0\end{cases}; \tag{6.1}$$

(iii) the Gaussian profile given by

$$\kappa\epsilon(0,\tau)=\sqrt{2}\,(\theta_0/\tau_p)e^{-2\pi\tau^2/\tau_p^2}. \tag{6.2}$$

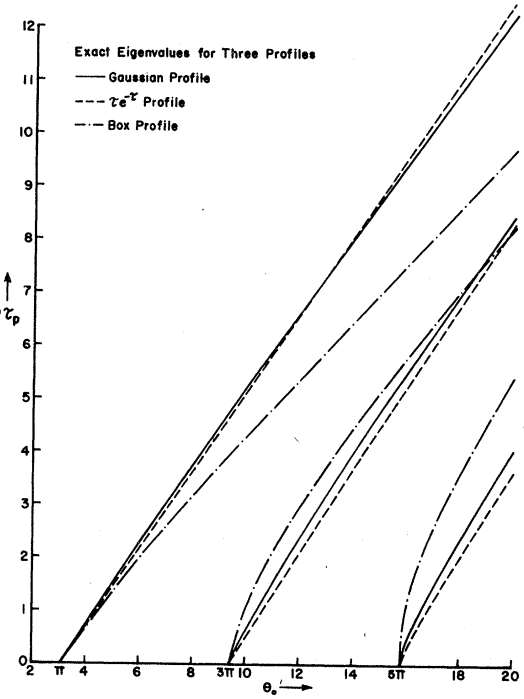

FIG. 4. Comparison of the exact eigenvalues for the square profile, the $\tau e^{-\tau}$ profile, and the Gaussian profile.

In (6.1) and (6.2), θ_0 is the initial area, and τ_p has been defined such that the initial energy of all three initial profiles will be the same, namely,

$$\mathcal{T}_0 = \frac{nc}{4\pi\kappa^2} \frac{\theta_0^2}{\tau_p}. \tag{6.3}$$

As one can see in Fig. 4, there is very little difference between the $\tau e^{-\tau}$ and the Gaussian profiles, while the results for the box profile does differ quantitatively, but not qualitatively.

From such a plot as Fig. 4, one can readily see what pulse heights would emerge for a given initial profile and area. For example, if the initial profile was the $\tau e^{-\tau}$ profile with an initial area of 12, then from Fig. 4 we would see that the two 2π h.s. pulses which would emerge would have the eigenvalues $\eta\tau_p$ equal to 2.1 and 6.6, respectively, which by (3.17) gives the pulse heights as being $8.4(\kappa\tau_p)^{-1}$ and $26.4(\kappa\tau_p)^{-1}$, respectively. Of course, knowing the pulse heights and the inhomogeneous broadening factor also gives us the respective velocities but not the absolute time delays τ_{0j}. But these would follow after determining the D_j's, as indicated in Sec. III.

When the initial pulse profile is slowly varying, unchirped, and has only one extremum, then one may use the WKB approximation to determine the bound-state eigenvalues of (2.12). Following Zakharov and Shabat,[24] we then have

$$\int_{\tau_1}^{\tau_2} [\tfrac{1}{4}\kappa^2\epsilon^2 - \eta^2]^{1/2}\, d\tau = (n - \tfrac{1}{2})\pi\,, \tag{6.4}$$

as the condition for a bound state to occur, where τ_1 and τ_2 are the two turning points. For the square profile, (6.4) can be exactly evaluated and gives

$$\eta_j\,\tau_p = [\tfrac{1}{4}\theta_0^2 - (j - \tfrac{1}{2})^2\pi^2]^{1/2}\,, \tag{6.5}$$

when $\theta_0 \geq (2j-1)\pi$. These values are compared against the exact values in Fig. 1, where we see excellent qualitatively agreement, with the only quantitative disagreement being a shift upward of the WKB solution. For the $\tau e^{-\tau}$ and the Gaussian profiles, comparisons of the WKB results with the exact results are shown in Figs. 5 and 6, respectively. We note that here the agreement is much better, and is obviously due to the smoother profiles. For the $\tau e^{-\tau}$ profile, which has only a discontinuity in the slope at $\tau = 0$, the agreement is much better than for the box profile, while for the Gaussian profile, which has no discontinuities, the agreement is excellent with a maximum difference of only about 0.11.

It also should be noted that the WKB method gives the threshold condition exactly [$n_j = 0$ at $\theta_0 = (2j-1)\pi$] in contrast to the method of using the conserved quantities.[7,10] The latter method

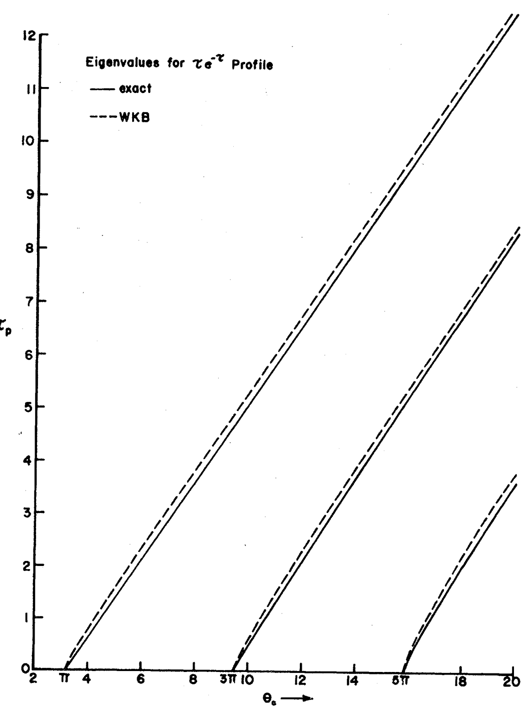

FIG. 5. Comparison of the exact and WKB eigenvalues for the $\tau e^{-\tau}$ profile.

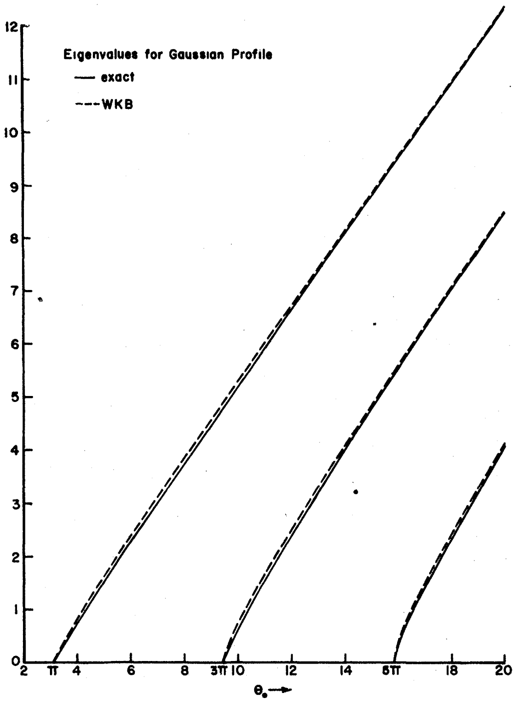

FIG. 6. Comparison of the exact and WKB eigenvalues for the Gaussian profile.

has its largest error when the initial area is close to these threshold values.

VII. ABSOLUTE TIME DELAY OF 2π h.s. PULSES

As pointed out by McCall and Hahn,[3] a 2π h.s. pulse undergoes a constant delay time *per unit length* as the pulse propagates through a resonant medium. However, what is usually measured is instead the absolute delay time, which contains the effects of τ_{0j}, which is given for the box profile by (3.16). As seen in Fig. 2, this contribution can be a significant amount when the pulse height is small,[28] although for such a wide pulse, the relaxation effects will certainly be important also.

Deviations from the McCall-Hahn value when the initial area is close to π are already known to occur from computer simulations,[18] where as the initial area decreases toward π, the time delay at first rises, reaches a maximum, then quickly drops toward zero. We find this behavior also present in the model of Sec. III, and will proceed to describe it.

Considering only the one-soliton case ($\theta_0 < 3\pi$) we have from (3.1), (3.17), and (2.9), that the absolute pulse delay t_D for the box profile in the attenuator case is

$$t_D = t_{0j} - \tfrac{1}{2}\tau_p + \frac{L}{4\pi}\int_{-\infty}^{\infty}\frac{\alpha(\Delta\omega)\,d\Delta\omega}{\eta_j^2 + (\Delta\omega)^2}, \tag{7.1}$$

where L is the sample length, assumed to be larger than several Beer's lengths. Consider now the limit where the initial area θ_0 approaches π from above so that $\eta_j \to 0^+$ ($j=1$). Then from (3.16) we find

$$t_D \simeq (1/2\eta_j)[\tfrac{1}{2}L\alpha_0 - \ln(\pi/4\eta_j\tau_p)]. \tag{7.2}$$

Thus, for small pulses, the effect of τ_{0j} is to *decrease* the McCall-Hahn value of the time delay by reducing the effective value of $L\alpha_0$. This is not unreasonable, since one could interpret (7.2) as if these small pulses were not formed until the initial pulse had penetrated a Beer distance of $2\ln(\pi/4\eta_j\tau_p)$.

One can easily argue that (7.2) is also qualitatively correct for a general pulse profile, since in general, from (3.15), $\tau_{0j} = (1/2\eta_j)\ln(\frac{1}{2}D_j/\eta_j)$. As $\eta_j \to 0, D_j$ should approach a nonzero constant value, giving the limit of (7.1) again being of the form of (7.2).

VIII. OFF-RESONANCE EFFECTS

When the central frequency of the initial profile is not exactly matched to the central frequency of the atomic line, the McCall-Hahn area theory is no longer valid,[3] the effective Beer's length changes,[17] 2π h.s. pulses form at a slower rate,[17,21,29] their central frequency becomes equal to the central frequency of the initial profile,[21,22] and when one is far off resonance, one can recover[30] the usual linear dispersion theory if the initial areas remain sufficiently small. All of these effects can be explained with the complete theory.

As noted by Lamb,[11] the off-resonant case is equivalent to the on-resonant case upon introducing a frequency shift. To see how this arises, take ω in (2.1) to be equal to the central frequency of the atomic line so that $g(\Delta\omega)$ is symmetric. If the initial profile is unchirped but off resonance by the amount $\delta\omega$, then at $\chi=0, \psi=\delta\omega\tau$, and by (2.2), ϵ is now complex. Turning to (2.12), one finds that upon defining, at $\chi=0$,

$$\hat{v}_1(\tau,0) = e^{-i\delta\omega\tau/2}v_1(\tau,0), \tag{8.1a}$$

$$\hat{v}_2(\tau,0) = e^{i\delta\omega\tau/2}v_2(\tau,0), \tag{8.1b}$$

then (2.12) becomes

$$\partial_\tau \hat{v}_1 + i\hat{\xi}\hat{v}_1 = \tfrac{1}{2}\kappa\epsilon\hat{v}_2, \tag{8.2a}$$

$$\partial_\tau \hat{v}_2 - i\hat{\xi}\hat{v}_2 = -\tfrac{1}{2}\kappa\epsilon\hat{v}_1, \tag{8.2b}$$

where

$$\hat{\xi} = \xi + \tfrac{1}{2}\delta\omega, \tag{8.3}$$

and ϵ is real. Equation (8.2) is just (2.11) rewritten. Thus determining the scattering data in the off-resonance case is equivalent to the same as in the on-resonance case, except for the eigenvalue (frequency) shift given by (8.3). From (8.1) and (2.13), if we let $\hat{a}(\hat{\zeta})$ and $\hat{b}(\hat{\zeta})$ be the scattering coefficients for (8.2), then

$$a(\zeta)=\hat{a}(\hat{\zeta})=\hat{a}(\zeta+\tfrac{1}{2}\delta\omega)\,, \tag{8.4a}$$

$$b(\zeta)=\hat{b}(\hat{\zeta})=b(\zeta+\tfrac{1}{2}\delta\omega)\,, \tag{8.4b}$$

and for the bound states

$$\zeta_j=\hat{\zeta}_j-\tfrac{1}{2}\delta\omega\,,\quad D_j=\hat{D}_j\quad (j=1,2,\ldots,J)\,. \tag{8.5}$$

First from (2.22)–(2.26) and (8.5) for a one-soliton state, we have

$$\kappa\epsilon(\chi,\tau)=\frac{4\eta_j e^{i\beta_j}e^{-2i\xi_j\tau}}{\cosh[4\eta_j(\tau-\tau_{0j})]}\,, \tag{8.6}$$

where β_j and τ_{0j} are the only functions of χ through

$$D_j\equiv 2\eta_j e^{i\beta_j}e^{-2\tau_{0j}\eta_j}\,, \tag{8.7}$$

and (2.28), and we have set

$$\zeta_j=\xi_j+i\eta_j\,. \tag{8.8}$$

When ϵ in (8.2) is the box profile of Sec. III, then $\hat{\xi}_j$ in (8.5) is real, which by (8.8) gives

$$\xi_j=-\tfrac{1}{2}\delta\omega \tag{8.9}$$

for all j values. Thus all the soliton solutions as given by (8.6) will contain the factor $e^{i\delta\omega\tau}$, and thus will oscillate at the central frequency of the initial pulse profile.

One should also note that due to (8.5), moving a pulse off-resonance will *not* allow solitons to form for initial absolute areas less than π, contrary to that suggested by Diels and Hahn.[21] In (8.2), if the absolute area is less than π, by the area theorem (since ϵ is real), no bound states can occur.

For the radiation part of the spectrum, if $\hat{b}(\hat{\zeta})$ is centered at $\hat{\zeta}=0$, then $b(\zeta)$ will be centered on $\zeta=-\frac{1}{2}\delta\omega$. Thus in (4.10), the radiation will decay at a rate of $\alpha(-\delta\omega)$ instead of α_0, and we may expect solitons to form at a slower rate. However, due to $D_j=\hat{D}_j$ in (8.5), the value of τ_{0j} for each soliton will be the same as if it were on resonance. Thus for not too large thicknesses, the time delay will not be affected by moving off resonance.[31]

Finally, we note that for off-resonant profiles, the natural generalization of the McCall-Hahn area theorem would be the evolution in χ of $\hat{\bar{b}}/\hat{a}(\hat{\zeta}=0)$ $=\bar{b}/a(\zeta=-\frac{1}{2}\delta\omega)$. From (2.27), we have

$$\partial_\chi \ln[\bar{b}/a(\zeta=-\tfrac{1}{2}\delta\omega)]$$
$$=(\pm)\left(\tfrac{1}{2}\alpha(-\delta\omega)+\frac{1}{2\pi i}P\int_{-\infty}^{\infty}\frac{\alpha(\Delta\omega)}{\Delta\omega+\delta\omega}d\Delta\omega\right), \tag{8.10}$$

which has a decay rate of $\frac{1}{2}\alpha(-\delta\omega)$ for the magnitude. However, $\bar{b}/a$ for $\zeta\neq 0$ cannot be expressed simply in terms of areas, but (8.10) does indicate again the slower decay rate. Also, it is known that for off resonance, the absolute area does not monotonically approach the values of $2\pi n$.[21,29]

IX. CONTINUOUS SPECTRUM

Contrary to the soliton spectrum, the continuous spectrum is almost linear in its nature, at least qualitatively.[32,33] In SIT, due to the exponential decay of the radiation, as given by (2.27), this part of the spectrum usually dies away fairly rapidly,[14] except for small blips or "precursors".[18,21,22] The solution for thin attenuators has been given by Burnham and Chia[34] and Crisp[18,19] has also obtained solutions for small areas.

When the sample length is finite, that part of the energy or electric field which corresponds to the continuous spectrum need not decay exponentially, due to a variety of reasons. As pointed out by Crisp,[19] for the $\tau e^{-\tau}$ profile given by (6.1), the energy decays according to

$$\mathcal{T}(\chi)=\mathcal{T}(0)e^{-\alpha_0\chi}I_0(\alpha_0\chi)\,, \tag{9.1}$$

for small areas, where $I_0(z)$ is a Bessel function. For χ large, (9.1) reduces to

$$\mathcal{T}(\chi)\simeq\mathcal{T}(0)/(2\pi\alpha_0\chi)^{1/2} \tag{9.2}$$

which decays instead only algebraically. Briefly stated, the reason for this is due to the radiation in the wings of the (nonlinear) Fourier spectrum, which is attenuated very slowly due to the small value of $g(2\xi)$ for ξ large. Thus if a significant amount of the radiation is contained in the wings $(4T_2^{*2}\xi^2\gtrsim 1)$, this will not be absorbed until χ becomes very large.

For the profile given by (6.1) and used by Crisp, we have

$$\frac{\bar{b}}{a}(\xi)\cong-\tfrac{1}{2}\kappa\hat{\epsilon}(2\xi)\cong\frac{2\kappa\theta_0/\tau_p^2}{(\xi-2i/\tau_p)^2}\,, \tag{9.3}$$

for small θ_0. Here we see $\bar{b}/a\simeq\xi^{-2}$ as $\xi\to\infty$ giving a significant amount of radiation in the wings. Of course, this limiting form is due to the discontinuity in the slope of (6.1) at $\tau=0$, and this again illustrates how we can consider $\bar{b}/a$ like a nonlinear Fourier transform.

Anytime the linear Fourier transform of the initial profile has significant components in the wings, we cannot expect an exponential decay. This will occur whenever the initial profile has discontinuities, is chirped, or is off resonance.[18,21,22] Although this may give the *appearance* of transparency, it is not true SIT, since the propagation is not completely lossless as is the case for solitons.

X. FIRST-ORDER EFFECTS ON 2π h.s. PULSE PROPAGATION

In their second paper[3] McCall and Hahn derived some of the first-order effects of relaxations on 2π h.s. pulse propagation. The same can be done also in the inverse scattering framework, where one seeks to determine the deviations of the scattering data from the solutions of (2.27)–(2.29). This can be done by using a recently developed perturbation expansion[23] for the Zakharov-Shabat problem. Although this expansion was derived for inversion about $x=+\infty$, it is a simple matter to obtain the corresponding result for inversion about $x=-\infty$, as is the case here. For (2.12), we have

$$i\left(\frac{\bar{b}}{a}\right)_\chi = -\frac{1}{a^2} I[\psi,\psi;\zeta], \tag{10.1}$$

$$\zeta_{j,\chi} = -\frac{I[\psi,\psi;\zeta_j]}{D_j(a_j')^2}, \tag{10.2}$$

$$D_{j,\chi} = \frac{a_j''}{(a_j')^2} I[\psi,\psi;\zeta_j] - \frac{J[\Psi(\zeta_j)]}{(a_j')^2}, \tag{10.3}$$

where the functionals I and J are defined by

$$I[u,v;\zeta] = \frac{i\kappa}{2}\int_{-\infty}^{\infty} d\tau\,[\epsilon_\chi u_2 v_2 + \epsilon_\chi^* v_1 u_1], \tag{10.4}$$

$$J[U] = \frac{i\kappa}{2}\int_{-\infty}^{\infty} d\tau\,[\epsilon_\chi U_2 + \epsilon_\chi^* U_1], \tag{10.5}$$

and for $n=1$ or 2,

$$\Psi_n(\zeta_j) = \left.\frac{\partial \psi_n^2(\zeta,\tau)}{\partial\zeta}\right|_{\zeta=\zeta_j}. \tag{10.6}$$

In (10.1)–(10.3), ψ is the solution of (2.12) satisfying the boundary condition

$$\psi \to \begin{pmatrix}0\\1\end{pmatrix} e^{i\zeta\tau} \quad \text{as } \tau\to+\infty, \tag{10.7}$$

which in terms of ϕ and $\bar{\phi}$ is

$$\psi = \bar{b}\phi - a\bar{\phi}. \tag{10.8}$$

In (10.2) and (10.3), a_j' (a_j'') is the first (second) derivative of $a(\zeta)$ with respect to ζ, evaluated at $\zeta=\zeta_j$. Also, note that due to (10.6), $J[\Psi(\zeta_j)]$ is in general *almost* the ζ derivative of $I[\psi,\psi;\zeta]$, and it *is* the derivative whenever ϵ is on compact support. However, in general the integration over τ and the differentiation with respect to ζ do not commute, so in evaluating (10.5), one must first evaluate (10.6), and then (10.5).

From (2.6) and (2.10), one finds that

$$I[u,v;\zeta] = f(\zeta,\tau)\Big|_{\tau=-\infty}^{+\infty} + \int_{-\infty}^{\infty} h(\zeta,\tau)\,d\tau, \tag{10.9}$$

where

$$f(\zeta,\tau) = \frac{\pi\kappa^2\omega}{nc\omega_0}\left[\frac{\omega_0}{\kappa}\left(v_2u_2\left\langle\frac{\Lambda}{2\zeta-\Delta\omega}\right\rangle - v_1u_1\left\langle\frac{\Lambda^*}{2\zeta-\Delta\omega}\right\rangle\right) - (u_1v_2+u_2v_1)\left\langle\frac{W}{2\zeta-\Delta\omega}\right\rangle\right], \tag{10.10a}$$

$$\begin{aligned} h(\zeta,\tau) = {} & \frac{\pi\kappa}{nc}v_2u_2\left(\omega(1/T_2+\gamma_{bc})\left\langle\frac{\Lambda}{2\zeta-\Delta\omega}\right\rangle + \frac{\omega}{2}(\gamma_{ac}-\gamma_{bc})\left\langle\frac{\Lambda(1+W/X)}{2\zeta-\Delta\omega}\right\rangle - \frac{1}{2}\left\langle\frac{c}{n}\Lambda_\chi+\Lambda_\tau\right\rangle - i\sigma\epsilon\right) \\ & + \frac{\pi\kappa}{nc}v_1u_1\left(-\omega(1/T_2+\gamma_{bc})\left\langle\frac{\Lambda^*}{2\zeta-\Delta\omega}\right\rangle - \tfrac{1}{2}\omega(\gamma_{ac}-\gamma_{bc})\left\langle\frac{\Lambda^*(1+W/X)}{2\zeta-\Delta\omega}\right\rangle + \frac{1}{2}\left\langle\frac{c}{n}\Lambda_\chi^*+\Lambda_\tau^*\right\rangle - i\sigma\epsilon^*\right) \\ & - \frac{\pi\kappa^2\omega\hbar N}{4nc}(u_1v_2+u_2v_1)\left((\gamma_{ac}-\gamma_{bc})\left\langle\frac{(1+W/X)^2}{2\zeta-\Delta\omega}\right\rangle + (2/T_1-\gamma_{ac}+4\gamma_{bc})\left\langle\frac{(1+W/X)}{2\zeta-\Delta\omega}\right\rangle - 4\gamma_{bc}\left\langle\frac{1}{2\zeta-\Delta\omega}\right\rangle\right) \end{aligned} \tag{10.10b}$$

We shall now evaluate (10.2) and (10.3) to first order in the perturbing terms for a single 2π h.s. pulse. In (10.10), we therefore only need g, Λ, W, X, and ψ to only zeroth order. In (10.9), since f is to be evaluated only at $\tau=\pm\infty$, we only need to know what ψ will be at these limits, which to all orders follow from (2.13), (10.7), and (10.8). Furthermore, in (10.2) and (10.3), we shall only need to evaluate f at $\tau=-\infty$, at which point, g, Λ, and W are known. [However, in the evaluation of (10.1) for ζ real, it would be necessary to know also Λ at $\tau=+\infty$ to first order. This we shall not do here, and thus we will not discuss how these first-order effects affect the area theorem.]

For a 2π h.s. pulse in zeroth order, from (2.22) we have

$$G = -D_1 e^{-i\zeta_1\tau}. \tag{10.11}$$

Then solving (2.25) for $\bar{L}_1(\tau,\theta)$, one may construct ϕ and $\bar{\phi}$ [Eqs. (4.36), (4.37), and (4.39) in Ref. 1), which from (10.8) gives

$$\psi = \frac{e^{i\zeta\tau}}{1+Z^*Z}\begin{pmatrix} Z(a-1) \\ a+Z^*Z \end{pmatrix}, \tag{10.12}$$

where

$$Z \equiv (D_1/2\eta_1)e^{-2i\zeta_1\tau}, \tag{10.13}$$

$$a \equiv (\zeta - \zeta_1)/(\zeta - \zeta_1^*). \tag{10.14}$$

Also, from (2.19), (2.20), and (2.26), and since $X = \frac{1}{2}\hbar\omega_0 N$ in zeroth order,[17] we have

$$1 + \frac{W}{X} = -2\frac{Z^*Za^*(1-a)^2}{(1+Z^*Z)}\bigg|_{\zeta=(\Delta\omega)/2}, \tag{10.15}$$

$$\Lambda = \hbar\kappa N(1-a)Z\frac{a^*+Z^*Z}{(1+Z^*Z)^2}\bigg|_{\zeta=(\Delta\omega)/2}, \tag{10.16}$$

$$\kappa\epsilon = 8\eta_1 Z/(1+Z^*Z). \tag{10.17}$$

From the above, one may then evaluate $I[\psi,\psi;\zeta_1]$, and find that

$$I[\psi,\psi;\zeta_1] = -i\frac{\pi D_1\sigma}{nc\eta_1} - \frac{iD_1}{48\pi\eta_1^3}\times\left[\frac{3}{T_2}A_1 + 2A_2\left(\frac{1}{T_1} - \frac{1}{T_2} + \tfrac{1}{2}\gamma_{ac}\right) + 3i\gamma_{bc}B_1 - 2iB_2\left(\frac{1}{T_1} - \frac{1}{T_2} - \tfrac{1}{2}\gamma_{ac} + \gamma_{bc}\right)\right], \tag{10.18}$$

where

$$A_n \equiv \int_{-\infty}^{\infty}\frac{\alpha(\Delta\omega)\,d\Delta\omega}{\{[(\Delta\omega - 2\xi_1)/(2\eta_1)]^2+1\}^n}, \tag{10.19a}$$

$$B_n \equiv \int_{-\infty}^{\infty}\frac{\alpha(\Delta\omega)[(\Delta\omega - 2\xi_1)/(2\eta_1)]\,d\Delta\omega}{\{[(\Delta\omega - 2\xi_1)/(2\eta_1)]^2+1\}^n}. \tag{10.19b}$$

For the evaluation of $J[\Psi(\zeta_k)]$, one can note that this can be done by letting $u = v = \psi$ in (10.10), then we have

$$J[\Psi(\zeta_1)] = \left(\frac{\partial f}{\partial\zeta}\bigg|_{\zeta=\zeta_1}\right)\bigg|_{\tau=-\infty}^{+\infty} + \int_{-\infty}^{\infty}\left(\frac{\partial h}{\partial\zeta}\bigg|_{\zeta=\zeta_1}\right)d\tau, \tag{10.20}$$

or

$$\begin{aligned} J[\Psi(\zeta_1)] = &-\frac{D_1}{32\pi\eta_1^3}(A_1 - iB_1) + \frac{2\pi D_1\sigma}{nc\eta_1}\tau_{01} + \frac{D_1}{48\pi\eta_1^4} \\ &\times\Bigg[-6\gamma_{bc}\eta_1\tau_0A_1 - 3A_2\left(\frac{1}{T_1} - \frac{2}{T_2} - \tfrac{1}{2}\gamma_{ac} + 2\gamma_{bc}\right) + 4A_3\left(\frac{1}{T_1} - \frac{1}{T_2} + \tfrac{1}{2}\gamma_{bc}\right) + 6\eta_1\tau_{01}A_1\left(\frac{1}{T_2} + \gamma_{bc}\right) \\ &+ iB_2\left(\frac{1}{T_1} - \frac{4}{T_2} - \tfrac{1}{2}\gamma_{ac} + 4\gamma_{bc}\right) - 4iB_3\left(\frac{1}{T_1} - \frac{1}{T_2} + \tfrac{1}{2}\gamma_{bc}\right) - 4i\eta_1\tau_{01}B_2\left(\frac{1}{T_1} - \frac{1}{T_2} - \tfrac{1}{2}\gamma_{ac} + \gamma_{bc}\right) \\ &+ \frac{3c}{8\pi n\omega}[2B_1A_1 - i(A_1^2 + B_1^2)] - \frac{3\eta_1}{\omega}[A_1\xi_1 + B_1\eta_1 - i(A_1\eta_1 + B_1\xi_1)] \\ &+ 4\eta_1\tau_{01}A_2\left(\frac{1}{T_1} + \tfrac{1}{2}\gamma_{ac} - \frac{1}{T_2}\right) + 3i\gamma_{bc}B_1(1 + 2\eta_1\tau_0)\Bigg]. \end{aligned} \tag{10.21}$$

In (10.21), τ_{01} is the central position of the 2π pulse, which is a function of χ, and is given by (8.7), or

$$\tau_{01} = -(1/4\eta_1)\ln(4\eta_1^2/D_1^*D_1). \tag{10.22}$$

Also, the quantity $\tau_0(\chi)$ is a "turning-on" time. When $\gamma_{bc} = 0$, it does not occur. But when $\gamma_{bc} \neq 0$, the ground state of the two-level system is unstable. Thus at some finite time in the past $\tau_0(\chi)$ it must have been prepared, and we have assumed that it was prepared well before the pulse arrived, or

$$(\tau_{01} - \tau_0)\eta_1 \gg 1, \tag{10.23}$$

when $\gamma_{bc} \neq 0$.

Decomposing D_1 and ζ_1 as given in (8.7) and (8.8), and using (10.2), (10.3), and (10.14), we have from (10.18) and (10.21),

$$\eta_{1,\chi} = -\frac{4\pi\sigma}{nc}\eta_1 - \frac{1}{12\pi\eta_1}\left[\frac{3}{T_2}A_1 + 2\left(\frac{1}{T_1} - \frac{1}{T_2} + \tfrac{1}{2}\gamma_{ac}\right)A_2\right], \tag{10.24}$$

$$\xi_{1,\chi} = \frac{1}{12\pi\eta_1}\left[3B_1\gamma_{bc} - 2B_2\left(\frac{1}{T_1} - \frac{1}{T_2} - \tfrac{1}{2}\gamma_{ac} + \gamma_{bc}\right)\right], \tag{10.25}$$

$$\begin{aligned} \beta_{1,\chi} = \frac{1}{8\pi\eta_1}B_1 + \frac{1}{12\pi\eta_1^2}\Bigg[&6\gamma_{bc}B_1\eta_1\tau_0 + 3B_2\left(\frac{1}{T_1} - \frac{2}{T_2} - \tfrac{1}{2}\gamma_{ac} + 2\gamma_{bc}\right) - 4B_3\left(\frac{1}{T_1} - \frac{1}{T_2} + \tfrac{1}{2}\gamma_{bc}\right) \\ &- 4B_2\eta_1\tau_{01}\left(\frac{1}{T_1} - \frac{1}{T_2} - \tfrac{1}{2}\gamma_{ac} + \gamma_{bc}\right) - \frac{3c}{8\pi n\omega}(A_1^2 + B_1^2) + \frac{3\eta_1}{\omega}(A_1\eta_1 + B_1\xi_1)\Bigg] \end{aligned} \tag{10.26}$$

$$\tau_{01,\chi}=\frac{1}{16\pi\eta_1^2}+\frac{1}{24\pi\eta_1^3}\Big[-6\gamma_{bc}A_1\eta_1(\tau_{01}-\tau_0)+3A_2\Big(\frac{1}{T_1}-\frac{2}{T_2}-\tfrac{1}{2}\gamma_{ac}+2\gamma_{bc}\Big)-4A_3\Big(\frac{1}{T_1}-\frac{1}{T_2}+\tfrac{1}{2}\gamma_{bc}\Big)-\frac{3cB_1A_1}{4\pi n\omega}+\frac{3\eta_1}{\omega}(A_1\xi_1+B_1\eta_1)\Big]. \qquad (10.27)$$

In Eqs. (10.24)–(10.27), we have the first-order equations of motion for the parameters of the 2π h.s. pulse, as given by (8.6). From (8.6), (2.8), and (2.9), we see that $4\eta_1/\kappa$ is the pulse height, $\beta_1+2\eta\chi/c$ is the phase at fixed χ, $\omega-2\xi_1$ is the instantaneous frequency, and τ_{01} is the (retarded) central position of the 2π h.s. pulse. One should note that due to (10.19), $A_2<A_1$. Thus $\eta_{1,\chi}<0$ if $\sigma>0$, and the pulse must then always decay.

Even in first order, the effect of the second derivatives in Maxwell's equation [the last two terms of (2.5)] is in general very small, and is given by the last two terms in (10.26) and (10.27). (They affect the eigenvalues only in second order.[35]) These terms would need to be considered only for pulses much shorter than those in current usage.

We note that (10.24), upon setting $\sigma=\gamma_{ac}=0$, is exactly the same as that obtained by McCall and Hahn[3] for the decay of the pulse energy, since $\mathcal{T}=(4nc/\pi\kappa^2)\eta_1$ due to (2.36). From this, one can determine the lifetime of the pulse, since when η_1 becomes zero, the 2π h.s. pulse is no longer present, and its energy is then rapidly absorbed by the medium.

More interesting is the implication of (10.25). When $\gamma_{bc}=0$ and if $\eta_1T_2^*\ll 1$, we have

$$\xi_{1,\chi}\simeq-\tfrac{1}{3}\eta_1\alpha'(2\xi_1)\Big(\frac{1}{T_1}-\frac{1}{T_2}-\tfrac{1}{2}\gamma_{ac}\Big), \qquad (10.28)$$

since

$$B_2\simeq 2\pi\eta_1^2\alpha'(2\xi_1), \qquad (10.29)$$

for $\alpha(\Delta\omega)$ symmetric about $\Delta\omega=0$. For the atomic rubidium system, $(T_1^{-1}-T_2^{-1}-\frac{1}{2}\gamma_{ac})$ is positive, and for a Gaussian inhomogeneous broadening, $\alpha'(2\xi_1)\approx-\xi_1T_2^{*2}\alpha(2\xi_1)$. Thus we then have

$$\xi_{1,\chi}\approx\alpha(2\xi_1)[\tfrac{1}{3}\eta_1T_2^{*2}(T_1^{-1}-T_2^{-1}-\tfrac{1}{2}\gamma_{ac})]\xi_1, \qquad (10.30)$$

showing that a 2π h.s. pulse off resonance will move *away* from resonance. One should note that this result is only valid *after* the 2π h.s. pulse is formed. The frequency shifts occurring during the initial formation of a 2π h.s. pulse have been discussed by Diels and Hahn.[21,22] Since the path lengths used by Slusher and Gibbs[17] were not very long, the shifts they observed were probably mostly due to initial formation of the 2π h.s. pulse. However, over longer path lengths, Eq. (10.30) shows that if a 2π h.s. pulse is not exactly on resonance initially, it will usually slowly drift away from resonance. This follows since (10.30) and (2.4) shows that if we want the stable case where an off-resonance pulse will move toward resonance for $\gamma_{bc}=0$, then $\gamma_{ab}<\gamma_{ac}+2\Gamma_{\text{phase}}$. But this requires decays from the upper level a to all other states to dominate over decays to the lower state b.

Finally, (10.27) gives the equation of motion of the time delay. Although another expression has also been given for this by McCall and Hahn,[3] for small pulse heights,[28] they neglected the most important terms. As given by them, they only considered the first term (10.27), which is the zeroth-order term. [In truth, one probably cannot directly obtain these additional terms from the MB equations, since these terms arise from the affect of the perturbations on D_j $(\chi=0)$, a quantity which does not occur in the McCall-Hahn theory, but which does naturally occur in the inverse scattering formulation.] If we consider (10.27) for $\gamma_{bc}=0$, neglect the terms inversely proportional to ω, and take the small pulse height limit, $\eta_1T_2^*\ll 1$, then we have

$$\tau_{01,\chi}\simeq\frac{\alpha(2\xi_1)}{8\eta_1}\Big(1-\frac{1}{T_s\eta_1}\Big), \qquad (10.31)$$

where $T_s^{-1}=T_2^{-1}+\frac{1}{2}\gamma_{ac}$, and we have used the approximation (for $n\geq 1$)

$$A_n\simeq 2\eta_1\alpha(2\xi_1)\frac{\pi(2n-2)!}{4^{n-1}[(n-1)!]^2}. \qquad (10.32)$$

In order to integrate (10.31), we need $\eta_1(\chi)$, which follows from (10.24) and (10.32), and for $\sigma=0$, this is

$$\eta_1(\chi)=\eta_1(0)-\tfrac{1}{8}\alpha(2\xi_1)\chi/T_e, \qquad (10.33)$$

where $T_e^{-1}=\frac{4}{3}(T_1^{-1}+2T_2^{-1}+\frac{1}{2}\gamma_{ac})$. Then the solution of (10.31) is

$$\tau_{01}(\chi)\simeq\tau_{01}(0)+T_e\Big[\ln[\eta_1(0)/\eta_1(\chi)]-T_s^{-1}\Big(\frac{1}{\eta_1(\chi)}-\frac{1}{\eta_1(0)}\Big)\Big]. \qquad (10.34)$$

As shown by (10.34), the delay time will at first rise due to the ln term, but as the pulse height decreases, this rise will eventually stop, with the delay time then decreasing as the pulse starts to decay more strongly where $\eta_1T_s\sim 1$. Thus when the pulse width $\frac{1}{2}\eta_1^{-1}$ becomes of the order of magnitude of the relaxation times, deviation from the McCall-Hahn value of $\alpha\chi/(8\eta_1)$ will occur.

APPENDIX: NONLINEAR MOMENTS

The nonlinear moments will be defined as the coefficients of a power series expansion of $\bar{b}/a$ in $2i\zeta$. In the linear limit (Ref. 1, pp. 266–271), we have

$$\frac{\bar{b}}{a}(\zeta) \simeq -\frac{1}{2}\int_{-\infty}^{\infty} \kappa\mathcal{E}(\tau,\chi)e^{2i\zeta\tau}\,d\tau\,, \tag{A1}$$

so we will define our nonlinear moments μ_n by

$$\frac{\bar{b}}{a}(\zeta) \equiv -\frac{1}{2}\sum_{n=0}^{\infty}\frac{\mu_n}{n!}(2i\zeta)^n\,, \tag{A2}$$

so that in the linear limit

$$\mu_n(\chi) \simeq \int_{-\infty}^{\infty} \kappa\mathcal{E}(\tau,\chi)\tau^n\,d\tau\,. \tag{A3}$$

First, we note that the solution ψ (Ref. 1) of (2.12) satisfies the boundary conditions

$$\psi \to \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\zeta\tau} \quad \text{as } \tau \to +\infty\,, \tag{A4a}$$

$$\psi \to \begin{pmatrix} \bar{b}e^{-i\zeta\tau} \\ ae^{i\zeta\tau} \end{pmatrix} \quad \text{as } \tau \to -\infty\,. \tag{A4b}$$

Thus

$$\lim_{\tau\to-\infty}(\psi_1/\psi_2)e^{2i\zeta\tau} = \bar{b}/a\,. \tag{A5}$$

To find ψ_1 and ψ_2, we let

$$\psi_1 = (f\cos\tfrac{1}{2}\mathcal{A}_+ - h\sin\tfrac{1}{2}\mathcal{A}_+)e^{-i\zeta\tau}\,, \tag{A6a}$$

$$\psi_2 = (f\sin\tfrac{1}{2}\mathcal{A}_+ + h\cos\tfrac{1}{2}\mathcal{A}_+)e^{i\zeta\tau}\,, \tag{A6b}$$

where

$$\mathcal{A}_+ \equiv \kappa\int_{\tau}^{\infty}\mathcal{E}\,d\tau\,. \tag{A7}$$

It then follows that the functions f and h will satisfy

$$f_\tau - \tfrac{1}{2}if\kappa\mathcal{E}\sin\mathcal{A}_+\sin 2\zeta\tau + h\kappa\mathcal{E}(\sin^2\zeta\tau - \tfrac{1}{2}i\cos\mathcal{A}_+\sin 2\zeta\tau) = 0\,, \tag{A8a}$$

$$h_\tau + \tfrac{1}{2}ih\kappa\mathcal{E}\sin\mathcal{A}_+\sin 2\zeta\tau - f\kappa\mathcal{E}(\sin^2\zeta\tau + \tfrac{1}{2}i\cos\mathcal{A}_+\sin 2\zeta\tau) = 0\,, \tag{A8b}$$

subject to the boundary conditions

$$\left.\begin{matrix} h(\tau,\chi,\zeta) = 1 \\ f(\tau,\chi,\zeta) = 0 \end{matrix}\right\} \text{ as } \tau \to +\infty\,. \tag{A9}$$

We may solve (A8) and (A9) in a Taylor series expansion of $2i\zeta$, given by

$$h = \sum_{n=0}^{\infty} h_n(2i\zeta)^n\,, \tag{A10a}$$

$$f = \sum_{n=0}^{\infty} f_n(2i\zeta)^n\,, \tag{A10b}$$

then the solution for the first three terms is

$$h_0 = 1\,, \tag{A11a}$$

$$f_0 = 0\,, \tag{A11b}$$

$$h_1 = \tfrac{1}{2}\kappa\int_{\tau}^{\infty}\tau\mathcal{E}\sin\mathcal{A}_+\,d\tau\,, \tag{A11c}$$

$$f_1 = -\tfrac{1}{2}\kappa\int_{\tau}^{\infty}\tau\mathcal{E}\cos\mathcal{A}_+\,d\tau\,, \tag{A11d}$$

$$h_2 = \tfrac{1}{4}\kappa\int_{\tau}^{\infty}\mathcal{E}(2\tau h_1\sin\mathcal{A}_+ - 2\tau f_1\cos\mathcal{A}_+)\,d\tau\,, \tag{A11e}$$

$$f_2 = -\tfrac{1}{4}\kappa\int_{\tau}^{\infty}\mathcal{E}(\tau^2 + 2\tau f_1\sin\mathcal{A}_+ + 2\tau h_1\cos\mathcal{A}_+)\,d\tau\,. \tag{A11f}$$

From (A5)–(A7) and (4.2) we have

$$\bar{b}/a = -(h\tan\tfrac{1}{2}\theta - f)/(h + f\tan\tfrac{1}{2}\theta)\,. \tag{A12}$$

From (A2), (A10), and (A12), we find

$$\mu_0 = 2\tan\tfrac{1}{2}\theta\,, \tag{A13a}$$

$$\mu_1 = -2f_1/\cos^2\tfrac{1}{2}\theta\,, \tag{A13b}$$

$$\mu_2 = 4(f_1h_1 + f_1^2\tan\tfrac{1}{2}\theta - f_2)/\cos^2\tfrac{1}{2}\theta\,, \tag{A13c}$$

where in (A13) it is understood that the f_n's and h_n's are to be evaluated at $\tau = -\infty$.

From (2.27), upon expanding the integral in a power series of $2i\zeta$, one obtains for the equations of motion of the first three nonlinear moments, the following

$$\mu_{0,\chi} = -\tfrac{1}{2}\mu_0\alpha_0\,, \tag{A14a}$$

$$\mu_{1,\chi} = -\tfrac{1}{2}\mu_1\alpha_0 - \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty}\frac{d\Delta\omega}{\Delta\omega}\alpha'(\Delta\omega)\,, \tag{A14b}$$

$$\mu_{2,\chi} = -\tfrac{1}{2}\mu_2\alpha_0 - \frac{\mu_1}{\pi}\times\int_{-\infty}^{\infty}\frac{d\Delta\omega}{\Delta\omega}\alpha'(\Delta\omega) + \tfrac{1}{2}\mu_0\alpha_0''\,, \tag{A14c}$$

when $\alpha(\Delta\omega)$ is an even function, and where $\alpha_0 = \alpha(0)$, $\alpha_0'' = [d^2\alpha(\Delta\omega)/d\Delta\omega^2]|_{\Delta\omega=0}$, and $\alpha'(\Delta\omega)$ is the first derivative. Of course (A14a) is just the McCall-Hahn area theorem, and when $\theta = 0$, (A14b) gives (4.11).

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