Electron dynamics in a free-electron laser*

H. Al Abawi,† F. A. Hopf,‡ and P. Meystre
Department of Physics and Optical Sciences Center, University of Arizona, Tucson, Arizona 85721
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The behavior of the momentum distribution function is discussed theoretically in the small-signal regime of the free-electron laser (FEL). The distribution function is derived analytically following a series of approximations that reduce this problem to the Klein-Gordon equation. The distribution function changes in nontrivial ways that may play an important role in the efficiency of the FEL.

I. INTRODUCTION

Free-electron lasers have recently[1] been proposed as an alternative to more conventional laser devices in applications where a high tunability range and a high power are desired. One of them is presently being developed at Stanford, and Elias *et al.*[1,2] have observed gain at 10.6 μm. It is now known that free-electron lasers (FEL) are purely classical devices,[3,4] and that the gain is due to a bunching of the electron distribution, rather than to the (quantum-mechanical) Compton recoil, as was originally suggested.[5,6] Indeed, quantum-mechanical and classical theories give the same small-signal gain to within numerical factors, which can be attributed to the vastly different approximations used by various authors.

In order to achieve appreciable efficiency in the FEL, it has been suggested that one should recycle the electron beam from one shot to the next.[7] In a description of the problem taking into account the electron recoil only, as it is sometimes done in "*ad hoc*" quantum-mechanical theories, this would appear to be straightforward, since one could simply replace the energy lost through recoil in each cycle of the accelerator. With the description of the problem in terms of the coupled Maxwell-Boltzmann equations,[3,4] however, we have been able to compute the development of the electron distribution in detail, and we find that the picture is basically quite different. We show in this paper that the effects that lead to small-signal gain in a FEL are small compared to other effects which occur together with them.

These other effects have the consequence of spreading out the electron distribution in momentum space, and, for large fields, cause the momentum distribution to split into two parts. This presents no difficulty on a single-shot basis. It may, however, present difficulties in a recycling configuration, since it increases the emittance of the electron beam.

This article is divided into four sections. Section II is a formal development which leads from the coupled Maxwell-Boltzmann equations, which form the basis of the classical discussion developed earlier, to another more convenient description of the problem in terms of a set of generalized Bloch equations. Section III is the main part of this paper, and deals with the evolution of the electron momentum distribution function. By dropping from the generalized Bloch equations the small term which gives rise to gain, we obtain a Klein-Gordon equation which can be solved exactly, and which gives the modification of the electron distribution. The spread and eventual breakup of this distribution into two parts appears explicitly, and is shown to be the dominant process in the small-signal regime. This is compared to a numerical solution[4] of the generalized Bloch equations to confirm that the Klein-Gordon equation indeed describes the large effects that occur in the distribution function. Section IV is a conclusion and summary.

II. GENERALIZED BLOCH EQUATIONS

In a FEL, a beam of relativistic electrons of energy $E=\gamma mc^2$ is passed through a helictical magnetic field and produces stimulated emission of radiation.[8] In the highly relativistic limit of the FEL, the Weizsäcker-Williams approximation is used, in which the static magnetic field of period λ_q is simulated by a fictitious incident EM field of wavelength $\lambda_i=(1+v/c)\lambda_q$ propagating in the opposite direction of the electron beam.[9] In the electron rest frame, the problem can be understood as that of stimulated Thomson scattering. The relevant part of the emitted radiation is in the direction of the electron beam (backscattering), since the Doppler up shift is maximum under this configuration. In the laboratory frame the wavelength λ_s of the backscattered radiation is given for $\gamma \gg 1$ by

$$\lambda_s \simeq \lambda_i/4\gamma^2 , \tag{2.1}$$

and can be tuned continuously by changing the energy of the incident electron beam (i.e., by changing γ). We find it convenient to study this problem directly in the laboratory frame, and to stay in a space-time representation. The FEL is then described classically by the coupling of the relativistic Boltzmann equation to Maxwell's equations.

We take the vector potential to be of the form

$$\vec{A}(z,t)=\hat{e}_-[A_i e^{-i(\omega_i t+k_i z)} + A_s(t)e^{-i(\omega_s t-k_s z)}]+\text{c.c.}, \tag{2.2}$$

where the z axis is taken along the electron beam. We take A_i to be constant (i.e., we neglect the depletion of the static magnetic field), and A_s to be a function of time.

In Ref. 3, we showed that if one takes a field of this type, and if one neglects the transverse velocity spread in the electron beam, the coupled Maxwell-Boltzmann equations can be separated exactly into a transverse and a longitudinal set of equations. The transverse equations can be solved trivially, and the longitudinal Maxwell-Boltzmann equations give

$$\frac{\partial h}{\partial t}+\frac{p_z}{m\gamma}\frac{\partial h}{\partial z}=\frac{e^2}{m\gamma}\left(\frac{\vec{A}^2}{2}\right)\frac{\partial h}{\partial p_z}, \tag{2.3}$$

and

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A}=\frac{e^2F}{mc\epsilon_0}\vec{A}\int_{-\infty}^{\infty}dp_z\frac{h(z,p_z,t)}{\gamma}. \tag{2.4}$$

$h(z,p_z,t)$ is the longitudinal part of the Boltzmann distribution function. It is normalized to the number of electrons $N(t)$ inside the cavity by

$$N(t)=\pi a^2\int_{-\infty}^{\infty}dp_z\int_0^L dz\,h(z,p_z,t). \tag{2.5}$$

πa^2 is the area of the electron beam, L the length of the cavity, $p_z=\gamma m v_z$ is the z component of the electron (kinetic) momentum, ϵ_0 the dielectric constant of vacuum (MKS units), and the filling factor F is the ratio of the section of the electron beam to the section of the cavity. We note that for highly relativistic electrons ($\gamma\gg 1$) the laboratory fields, we can neglect the mass-shift[3] $(e\vec{A}/mc)^2$ of the electrons, and thus

$$\gamma\equiv 1/(1-\beta^2)^{1/2}\simeq[1+(p_z/mc)^2]^{1/2}.$$

We know of no manner in which the set of equations (2.4) and (2.5) can be solved exactly. If one wishes to find the small-signal gain,[3] it is sufficient to expand $h(z,p_z,t)$ in power of $|A_iA_s|$ and, following the procedure used by Lamb[10] for the laser, solve (2.4) and (2.5) self-consistently.[11] An alternative procedure that we used to analyze the saturation consists in expressing $h(z,p_z,t)$ as the harmonic expansion

$$h(z,p_z,t)=n(z,p_z,t)+\sum_{m=1}^{\infty}(ig_m e^{im(\Delta\omega t-Kz)}+\text{c.c.}), \tag{2.6}$$

where

$$K=k_i+k_s, \tag{2.7}$$

and

$$\Delta\omega=\omega_s-\omega_i. \tag{2.8}$$

This later expansion presents several advantages over the perturbative one. First, each term in (2.6) contains the saturation to all orders in the field, and we do not encounter the divergence problems associated with the power expansion. Second, a computer analysis[4] shows that the expansion (2.6) can be truncated at $m=1$ without introducing noticeable corrections in the small signal regime of the FEL. Finally, with this truncation the Boltzmann equation (2.3) can be reexpressed at steady state ($\partial_t n=\partial_t g_1=0$) in terms of the following set of equations[4]

$$\frac{\partial R_1}{\partial\zeta}+PR_2=-\frac{\partial R_3}{\partial P}, \tag{2.9a}$$

$$\frac{\partial R_2}{\partial\zeta}-PR_1=0, \tag{2.9b}$$

$$\frac{\partial R_3}{\partial\zeta}=-\frac{\partial R_1}{\partial P}. \tag{2.9c}$$

These equations contain, in addition, a perturbation expansion in P which will be explained later. The boundary conditions are

$$R_1(0,P)=R_1(0,P)=0, \quad R_3(0,P) \tag{2.10}$$

prescribed by the initial electron momentum distribution. The $R_i(\zeta,P)$ are dimensionless functions related to n and g_1 through

$$\begin{aligned} R_1&=(mc\sqrt{2}/4n_e\sigma\beta_s\gamma_s)(g_1+g_1^*),\\ R_2&=-i(mc\sqrt{2}/4n_e\sigma\beta_s\gamma_s)(g_1-g_1^*),\\ R_3&=(mc/2n_e\sigma\beta_s\gamma_s)n, \end{aligned} \tag{2.11}$$

where

$$\sigma^2=mc/4\sqrt{2}\gamma_s^4\beta_s^2e^2A_iA_s. \tag{2.12}$$

The value of β_s is defined as the velocity at which there is neither gain nor loss (center of the gain line)

$$\beta_s\equiv[1-1/\gamma_s^2]^{1/2}=\Delta\omega/Kc. \tag{2.13}$$

ζ is the dimensionless length

$$\zeta=z/l, \tag{2.14}$$

where the scale length l is

$$l=(mc)^2/2\sqrt{2}K\sigma e^2 A_i A_s\,, \tag{2.15}$$

in terms of which the saturation condition is given as $l \sim L$. The dimensionless energy P is measured relative to the zero gain condition such that

$$P=\sigma[(p_z/mc)^2-\beta_s^2\gamma_s^2]\,. \tag{2.16}$$

The set of equations (2.9) presents a striking resemblance to the usual optical Bloch equations, where R_3 would be the population inversion and R_1 the polarization. However, it differs from them in two respects. First, the signs on the RHS of Eqs. (2.9a) and (2.9c) are opposite in the usual Bloch equations, and second, the RHS of these equations contains derivatives of R_3 and R_1, respectively. The difference lies in the fact that in a free-electron laser, the gain is not proportional to the electron distribution function. Rather, it is its slope in momentum space which plays the role of an inversion.[2,3,6]

To complete this formal development, we still have to express Maxwell's equation (2.4) in terms of the new variables R_i. We first observe that since the small signal gain is finite only in the region[3]

$$-2\pi \lesssim \mu L \lesssim 2\pi\,, \tag{2.17}$$

where

$$\mu \equiv \Delta\omega/v_z - K = K(\beta_s-\beta)/\beta\,, \tag{2.18}$$

is a measure of the detuning between the cavity and the electron beam, we can expand[12] μ about $p_s = mc\beta_s\gamma_s$ and find that P is given by

$$P \simeq -\mu l\,. \tag{2.19}$$

It is consistent with this approximation to let

$$\gamma\frac{dP}{dp_z}\simeq\gamma_s\frac{dP}{dp_z}\bigg|_{p_z=p_s}=2\sigma\gamma_s^2\beta_s\,,$$

on the RHS of Maxwell's equation. We then obtain

$$\frac{d}{dt}|A_s|^2=\frac{-Fn_e e^4K^2L^2|A_i|^2|A_s|^2}{k_s\epsilon_0\gamma_s^5\beta_s^2(mc)^3}\times\left(\frac{l}{L}\right)^3\int_0^{L/l}d\zeta\int_{-\infty}^{\infty}dP R_1(\zeta,P)\,. \tag{2.20}$$

The double integral on the RHS of Eq. (2.20) can be reexpressed in a way that shows explicitly that the gain is due to the recoil of the electrons.[13] Integrating by parts and using the fact that $R_1(\zeta,-\infty) = R_1(\zeta,+\infty)=0$ we find, using Eqs. (2.9) that

$$\delta P \equiv \int_0^{L/l} d\zeta\int_{-\infty}^{\infty}dP R_1(\zeta,P) = \int_{-\infty}^{\infty}dP P[R_3(L/l,P)-R_3(0,P)]\,. \tag{2.21}$$

Since R_1 and R_2 contribute in a rapidly varying time-dependent fashion to the electron distribution function, the RHS of Eq. (2.21) can be understood as the time average of the difference between the average final energy and the average initial energy of the electrons. This recoil is computed for the small-signal regime of a FEL and in the small-cavity limit in Appendix A. In the next sections, we shall see that, although it is responsible for the gain, the electron recoil is a small effect when compared to other effects which occur simultaneously with it, namely a broadening and eventual splitting of the electron distribution into two parts.

III. DYNAMICS OF THE ELECTRON DISTRIBUTION

In the previous section, we developed the dynamics of the FEL in terms of a set of generalized Bloch equations. In this section, we develop an approximate analytic solution for the functions R_i ($i=1,2,3$) in Eq. (2.9) in the so-called "small-cavity" limit[14] of this problem. This limit, which is the optimum configuration for the FEL and the one used experimentally, is discussed extensively in other publications.[3-6] For our purposes this limit can be described as follows: We take P_0 to be the center of the initial distribution function [i.e., the center of $R_3(\zeta=0,P)$]. In the small-cavity limit, the range over which one can vary P_0 and still obtain finite gain or loss at a fixed wavelength λ_s is much wider than the initial width of the distribution function. This corresponds to the "homogeneously broadened" limit of the FEL and enables us to take the initial "inhomogeneous" distribution to be δ function

$$R_3(0,P)=\delta(P-P_0)\,. \tag{3.1}$$

The amount by which the distribution spreads out depends upon the magnitude of the field. In the small-signal regime, the spread due to the field is small compared to P_0, so that the domain of interest in P remains extremely small. This suggests that it might be interesting to replace P by P_0 in terms of Eq. (2.9) that do not contain derivatives. When that is done, Eq. (2.9) leads directly to the Klein-Gordon equation

$$\frac{\partial^2R_1}{\partial\zeta^2}-\frac{\partial^2R_1}{\partial P^2}+P_0^2R_1=0\,, \tag{3.2}$$

whose solution is well known.[15] With the boundary conditions in (2.10), we obtain

$$R_1(\zeta,P) = -\tfrac{1}{2}H(\zeta,P)\frac{\partial}{\partial P}J_0\{P_0[\zeta^2-(P-P_0)^2]^{1/2}\}$$
$$-\tfrac{1}{2}\{\delta[\zeta+(P-P_0)]-\delta[\zeta-(P-P_0)]\},$$
$$R_2(\zeta,P) = -\tfrac{1}{2}H(\zeta,P)\times\int_{|P_0(P-P_0)|}^{P_0\zeta}dx\,\frac{\partial}{\partial P}\times J_0\{[x^2-P_0^2(P-P_0)^2]^{1/2}\}+\tfrac{1}{2}P_0\tilde{H}(\zeta,P)$$
$$R_3(\zeta,P) = \tfrac{1}{2}P_0H(\zeta,P)\times\int_{|P_0(P-P_0)|}^{P_0\zeta}dx\,J_0\{[x^2-P_0^2(P-P_0)^2]^{1/2}\}$$
$$+\tfrac{1}{2}H(\zeta,P)\frac{\partial}{\partial\zeta}J_0\{P_0[\zeta^2-(P-P_0)^2]^{1/2}\}$$
$$+\tfrac{1}{2}\{\delta[\zeta+(P-P_0)]+\delta[\zeta-(P-P_0)]\}, \quad (3.3)$$

where

$$H(\zeta,P)=\begin{cases}1, & |P-P_0|<\zeta,\\ 0, & |P-P_0|>\zeta,\end{cases} \quad (3.4)$$

and

$$\tilde{H}(\zeta,P)=\begin{cases}H(\zeta,P), & P>P_0,\\ -H(\zeta,P), & P<P_0,\end{cases} \quad (3.5)$$

J_0 is the zeroth-order Bessel function.

There are four points worth noting immediately about these solutions: (i) We have compared these directly to the numerical solutions of (2.9) and found that they agree almost exactly in the small signal regime. (ii) The disagreement involves the terms which give the gain. These are smaller than the ones given here by $(S/S_{\rm sat})^{1/2}$. (iii) Since R_1 and R_2 contribute in a time-dependent fashion to the distribution function (giving a time-average contribution of zero), the most interesting term turns out to be $R_3(\zeta,P)$. (iv) As we shall show presently, one can get a reasonable qualitative picture of the distribution function by ignoring all contributions from the terms containing Bessel functions. In light of remark (iii), this amounts to considering only the two δ functions in $R_3(\zeta,P)$.

To quantify this remark we define the energy spread ΔP as the root of the second moment of the electron distribution function

$$|\Delta p| = \left[\int dP(P-P_0)^2R_3\right]^{1/2}, \quad (3.6)$$

where, as in the case of the recoil, we neglect the contribution from the rapidly varying terms R_1 and R_2. The δ functions alone give $\Delta P=\zeta$. In Appendix B, we compute the spread exactly and find

$$\Delta P=(2/P_0)\sin\tfrac{1}{2}P_0\zeta. \quad (3.7)$$

This spread reduces to the δ-function contribution for small ζ. The contribution from the background can be readily determined from the difference between the exact answer and the δ-function contribution. The point of maximum gain corresponds to the maximum of the sine function (i.e., $P_0L/2l=\frac{1}{2}\pi$), in which case the exact spread at the output is given by

$$\Delta P=(2/\pi)(L/l). \quad (3.8)$$

This is about 35% smaller than the spread obtained from the δ function alone. Thus the main part of $R_3(\zeta,P)$ is still the two δ functions, and the Bessel function contribution can be omitted if one is interested in the qualitative behavior of the free-electron laser only.

The splitting of $R_3(\zeta,P)$ is independent of the initial distribution, which is in complete agreement with the numerical results (in the numerical calculations the initial distribution has a finite width $\bar{E}$). In the case of finite widths, if $\Delta E<\bar{E}$, the output width is $\bar{E}+\Delta E$, and if $\bar{E}<\Delta E$, two peaks result, each with a width $\bar{E}$.

In Table I, we have summarized the results of the previous discussion by giving the spread ΔP (ΔE) in scaling (P) and real (E) units. The real

TABLE I. Summary of key parameters and results expressed in terms of optical fluxes (physical units) and scale lengths.

Process	Physical units	Scaled units
Center of initial electron distribution, relative to the zero gain condition	$E_0=\gamma^3mc^2(\lambda_s/2L)$	$P_0=\pi(l/L)$
Electron recoil	$\delta E=(\lambda_s/L)\gamma^3mc^2(S/S_{\rm sat})$	$\delta P=(4/\pi^3)(L/l)^3$
Electron spread	$\Delta E=(1/\sqrt{2})(\lambda_s/L)\gamma^3mc^2(S/S_{\rm sat})^{1/2}$	$\Delta P=(2/\pi)(L/l)$
Recoil to spread ratio	$\dfrac{\delta E}{\Delta E}=\sqrt{2}(S/S_{\rm sat})^{1/2}$	$\dfrac{\delta P}{\Delta P}=(2/\pi^2)(L/l)^2$

units are written in terms of S/S_{sat}, which is found from the ratio L/l in Eq. (19) in Ref. 4. All of the quantities are evaluated at the output ($z=L$), and for P_0 (E_0) evaluated at the point of maximum gain (see Appendix A). The last row shows the ratio of recoil to spread, and one sees that for $S<S_{sat}$ it is the spread which is dominant. The numerical calculations show that the effects are comparable for $S\sim S_{sat}$, so that the ratio is valid for large signals. In the small-signal regime, the dominant effect on the distribution function will be the spread.

In light of this, it is important to note that a reduction in gain can occur either from saturation (i.e., recoil) when $\delta E\sim E_0$, or from reaching the "large-cavity" limit when the total spread $\Delta E_{tot}\sim E_0$. If, in order to increase the efficiency of the FEL, one adopts a recycling configuration of the electron beam, one needs not only to replace the energy lost by recoil, but also to compensate for the spread effect. While the first requirement is rather easy to achieve, the second one involves subtle considerations on the physics of a storage ring. In view of our present discussion, this is the dominant problem, and it will have to be addressed in detail.

IV. CONCLUSIONS

In this paper, we have shown that the effects that lead to the gain in the FEL play a relatively minor role compared to other effects. Thus, unlike conventional lasers, it is incorrect in principle to identify the small-signal regime of the FEL with an unchanging population distribution. Instead, there is a substantial spreading, or, for large fields, a splitting, of the distribution function. Here, we develop a theory which allows us to compute analytically the development of the distribution function. The chief features of this theory are that it is nonperturbative (i.e., it does not involve an expansion in powers of the field), and that it leads to equations that are somewhat similar to the Bloch equations, with solutions that resemble those found in optical nutation and free induction decay.

The features of the small-signal solution turn out to be characterized by the same ratio of flux to saturation flux [in particular, $(S/S_{sat})^{1/2}$] found in the large-signal regime. The ratio of energy recoil, which characterizes the gain effects, to energy spread which we describe here, is given by this ratio. The spread may limit the number of times this can be done before the electrons can no longer contribute to the gain.

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APPENDIX A

In this Appendix, we compute the electron recoil δP of the electrons in the small-signal regime and small cavity limit. Since the small-signal gain can be obtained in the frame of a standard perturbation theory, it is sufficient here to take $R_3(\zeta,P)$ to be unperturbed by the field, i.e.,

$$R_3(\zeta,P)=R_3(0,P)=\delta(P-P_0)\,, \tag{A1}$$

where the last term of the equality can be understood as the definition of the small-cavity limit. Under this approximation, the set of Bloch equations (2.9) can be solved exactly, and we find

$$R_1(\zeta,P)=\frac{\sin\zeta P}{P}\frac{d}{dP}\delta(P-P_0)\,. \tag{A2}$$

The recoil is given by Eq. (2.21). This integral gives, with (A2) and after some minor algebra,

$$\delta P=-\frac{1}{4}\left(\frac{L}{l}\right)^3\frac{\partial}{\partial x}\left(\frac{\sin x}{x}\right)^2, \tag{A3}$$

where

$$x=PL/2l=-\tfrac{1}{2}\mu L\,. \tag{A4}$$

The last term of this equality comes from the Taylor expansion (2.19). We know from Ref. 3 and Eq. (2.20), that the gain will be maximum if $x\simeq\frac{1}{2}\pi$ is maximum, i.e., if the initial energy P_0 is chosen such that

$$P_0=\pi l/L\,.$$

The corresponding recoil is

$$\delta P=-(4/\pi^3)(L/l)^3\,. \tag{A5}$$

APPENDIX B

Here we compute the spread ΔP of the electron distribution. It is given by the second moment of the electron distribution defined in Eq. (3.6). Multiplying the third Bloch equation (2.9c) by $(P-P_0)^2$, integrating the RHS by parts, and using

$$R_i(\zeta,\pm\infty)=0\,,\quad i=1,\ 2,\ 3\,, \tag{B1}$$

we obtain

$$\frac{\partial}{\partial\zeta}\int_{-\infty}^{\infty}dP(P-P_0)^2R_3=2\int_{-\infty}^{\infty}dP(P-P_0)R_1\,. \tag{B2}$$

The first and second Bloch equations (2.9) can be combined to give

$$\frac{\partial R}{\partial\zeta}-iP_0R=-\frac{\partial R_3}{\partial P}\,, \tag{B3}$$

where

$$R \equiv R_1 + iR_2 . \tag{B4}$$

Multiplying (B3) by $(P - P_0)$ and integrating by parts gives

$$\frac{\partial}{\partial \zeta} \int_{-\infty}^{\infty} dP(P - P_0)R - iP_0 \int_{-\infty}^{\infty} dP(P - P_0)R = 1 , \tag{B5}$$

where we have also used the fact that

$$\frac{\partial}{\partial \zeta} \int_{-\infty}^{\infty} dPR_3(\zeta, P) = 0 , \tag{B6}$$

which is an immediate consequence of the third Bloch equation and the boundary condition (B1), to obtain the RHS; Eq. (B5) can be readily integrated to obtain $\int_{-\infty}^{\infty} dP(P - P_0)R$, and, combining the result with Eq. (B2), we find

$$\frac{\partial}{\partial \zeta} \int_{-\infty}^{\infty} dP(P - P_0)^2 R_3 = -2Re\left(\frac{1 - e^{iP_0\zeta}}{iP_0}\right). \tag{B7}$$

Finally, integrating with respect to ζ gives the energy spread

$$|\Delta P|^2 = (4/P_0^2)\sin^2(P_0\zeta/2) . \tag{B8}$$

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[11]Unlike the usual laser theory, one cannot continue with a perturbation expansion of these equations and obtain meaningful results.

[12]Note that the perturbation expansion indicated here is used in deriving Eq. (2.9).

[13]It is easy but somewhat lengthy to show that the limits of integration over P can be extended to $\pm\infty$.

[14]The nomenclature was first introduced in Ref. 6.

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