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Quantum theory of a one-dimensional laser with output coupling. Linear theory

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Subthreshold operation of a one-dimensional laser having output coupling is analyzed quantum mechanically under the assumption of constant population inversion. The conventional quantum Langevin theory of the laser is improved in that the spatial behavior of the field is treated exactly by use of a multimode formulation of the field of the universe. This formulation allows the thermal noise to be derived automatically without introducing any artificial damping mechanism for the field; this noise comes from the statistical nature of the initial state of the field. The analysis shows that the cavity quasimodes are excited, that the linewidth formula differs from that of a conventional theory by a factor determined by the field distribution excited in the cavity, and that, outside the cavity, the quantum-mechanical coherence function is propagated with the velocity of light.

I. INTRODUCTION

This paper is a continuation of previous papers,[1-3] wherein the quantum theory of a one-dimensional optical cavity having output coupling and linear semiclassical and nonlinear semiclassical theories of a laser incorporating such a cavity, were developed on the basis of a multimode expansion of the field of the universe. We present here a linear quantum-mechanical theory of a laser having output coupling. The purpose of this series of papers is to establish a method to determine the field distribution inside and outside of the laser cavity, and to give a more rigorous foundation for the quasimode theories, i.e., those that assume well-defined cavity modes (as an idealization of the actual quasimodes) and treat the coupling loss phenomenologically.[4-6]

Closely related to our work are those papers by Lang *et al.*, who gave a semiclassical multimode theory of the laser[7] taking into account the many modes of the universe and derived the thermal noise from the initial condition of the field.[8] Our semiclassical analyses[2,3] improved their results, making it possible to determine the spatial behavior of the laser field. Their derivation of the thermal noise is revived in this paper in a more rigorous way. The advantage of our formulation over that by Lang *et al.* lies in that we use the exact mode functions of the universe, which allows us to determine the spatial (as well as temporal) dependence of the field. This advantage is effective also in the quantum theory presented here.

In this paper, we extend the linear semiclassical analysis in Ref. 2 (hereafter referred to as II) to include noise and develop a fully quantum-mechanical theory of the laser having output coupling. We assume again a constant population inversion (linear theory) for the two-level atoms located in the optical cavity analyzed in Ref. 1, and consider the subthreshold operation. We treat the atomic fluctuation by means of the quantum Langevin method[4,5,9]: we introduce a phenomenological damping term (as in II) and a fluctuating term for the atomic polarization. In contrast, we introduce no damping term for the field, yet the coupling loss and the thermal noise are shown to be derived automatically. Except for the inclusion of noise terms the present calculation is completely parallel to that in II.

II. LASER EQUATION OF MOTION AND THE LANGEVIN FORCES

Our optical cavity has the cavity resonant modes (quasimodes) given by[1]

$$u_{ck}(z) = \sin\left(\Omega_k \frac{z+d}{c^1}\right) \quad -d \leq z \leq 0\,, \tag{1a}$$

$$= \sin\left(\Omega_k \frac{d}{c^1}\right) \exp\left(i\Omega_k \frac{z}{c^0}\right), \quad 0<z\,, \tag{1b}$$

$$\Omega_k = \omega_{ck} - i\gamma_c\,, \tag{2a}$$

$$\omega_{ck} = (2k+1)\pi(c^1/2d), \quad k=0,1,2,\ldots, \tag{2b}$$

$$\gamma_c = (c^1/2d)\ln(1/r)\,, \tag{2c}$$

where Eq. (1a) is for inside the cavity and Eq. (1b) for outside. The normal modes of our universe used in the following formulation are introduced by setting an imaginary boundary at $z=L$:

$$U_j(z) = \left(\frac{2}{\epsilon^1 L}\ \frac{1}{1-K\sin^2 k_j^1 d}\right)^{1/2} \times \begin{cases} \sin k_j^1(z+d), & -d\leq z\leq 0\,, \\ (c^0/c^1)\cos k_j^1 d \sin k_j^0 z + \sin k_j^1 d\cos k_j^0 z, & 0<z\leq L\,. \end{cases} \tag{3a, 3b}$$

Here, the superscripts 1 and 0 refer to the inside and the outside of the cavity, respectively. The letter r denotes the amplitude reflectivity at the coupling surface at $z=0$, k_j is equal to ω_j/c (where ω_j is the frequency of the jth mode of the universe), and $K=1-\epsilon^0/\epsilon^1$. Other letters have their usual meanings.

In Appendix A are shown our starting equations which are subject to the dipole and rotating-wave approximations. We added a fluctuating term for the atomic polarization in accordance with the quantum Langevin method.[4,5,10] Assuming that the coherent interaction between the field and the atoms begins at $t=0$ and using the slowly varying amplitude approximation, we get the following equation for the electric field operator [cf., Eq. (29) of II]:

$$E^{+}(z,t)=\sum_m \frac{|p_m|^2\nu_m^2\sigma_m}{2\hbar\omega}\int_0^t \sum_j U_j(z)U_j(z_m)e^{i\omega_j(t'-t)}\int_0^{t'}\exp[(i\nu_m+\gamma_m)(t''-t')]E^{+}(z_m,t'')\,dt''\,dt'+F_t(z,t)\,, \tag{4}$$

where

$$F_t(z,t)\equiv F_1(z,t)+F_2(z,t)+F_3(z,t)\,, \tag{5}$$

$$F_1(z,t)=\sum_j i\left(\frac{\hbar\omega_j}{2}\right)^{1/2}U_j(z)a_j(0)e^{-i\omega_j t}\,, \tag{6a}$$

$$F_2(z,t)=\sum_m \frac{i\nu_m p_m}{2}(a_{m1}^\dagger a_{m2})(0)\int_0^t \sum_j U_j(z)U_j(z_m)\exp[i\omega_j(t'-t)-(i\nu_m+\gamma_m)t']\,dt'\,, \tag{6b}$$

$$F_3(z,t)=\sum_m \frac{i\nu_m p_m}{2}\int_0^t \sum_j U_j(z)U_j(z_m)\exp[i\omega_j(t'-t)-(i\nu_m+\gamma_m)t']\int_0^{t'} e^{\gamma_m t''}\Gamma_m(t'')\,dt''\,dt'\,. \tag{6c}$$

The quantity ω is the assumed center frequency of oscillation and a_j is the annihilation operator for the jth mode of the universe. The letter m refers to the mth atom; p_m is the component of the electric dipole matrix element along the polarization of the field (e.g., in the x direction), ν_m is the resonance frequency, σ_m is the population inversion, γ_m is the damping constant for the polarization $(a_{m1}^\dagger a_{m2})$, and z_m is the location. Γ_m is the fluctuation for the polarization amplitude.

Later, we consider the statistical properties of the resultant field, when those of the fluctuating terms appearing in Eq. (6) are required. In this regard, we assume the existence of an initial radiation field which is in thermal equilibrium with the passive cavity and with the imaginary boundary at $z=L$. For the fluctuation of the polarization of each atom we follow the treatment due to the quantum Langevin method.[4,5] Thus we assume the following equations to hold[11]:

$$\langle a_j(0)\rangle=0\,, \tag{7a}$$

$$\langle (a_{m1}^\dagger a_{m2})(0)\rangle=0\,, \tag{7b}$$

$$\langle \Gamma_m(t)\rangle=0\,, \tag{7c}$$

$$\langle a_i^\dagger(0)a_j(0)\rangle=\delta_{ij}n_j(\Theta)\,, \tag{8a}$$

$$\langle (a_{n2}^\dagger a_{n1})(0)(a_{m1}^\dagger a_{m2})(0)\rangle=\delta_{nm}\langle (a_{m2}^\dagger a_{m2})(0)\rangle\,,$$

$$=\tfrac{1}{2}\delta_{nm}(1+\sigma_m)\,, \tag{8b}$$

$$\langle \Gamma_n^\dagger(t')\Gamma_m(t)\rangle=\delta_{nm}\gamma_m(1+\sigma_m)\delta(t'-t)\,, \tag{8c}$$

where the angular brackets denote the ensemble average with respect to the initial field and to the heat baths responsible for the pumping and damping of the atoms. The quantity $n_j(\Theta)$ is the Planck distribution for the jth mode of the universe at the initial temperature Θ.[12] Note that we are assuming that the modes of the universe are statistically independent of each other at $t=0$ and so are the atoms' heat baths at any time.

With the above specification, $F_1(z,t)$, the free oscillation of the field in the absence of the atoms, describes the thermal noise. Note that this fluctuating term appears without introducing any heat bath for the field. $F_3(z,t)$ accounts for the quantum noise. The second fluctuating term, $F_2(z,t)$, describes a switching-on effect which is insignificant for large t.

III. SOLUTION OF THE LASER EQUATION OF MOTION

Hereafter, we limit our consideration to atoms with identical properties and write ν_0, p_a, σ, and γ instead of ν_m, p_m, σ_m, and γ_m, respectively, as in II. However, we preserve the statistical independence of the atomic fluctuations (i.e., of the heat baths) and write $(a_{m1}^\dagger a_{m2})(0)$ and $\Gamma_m(t)$ as in Sec. II. We solve Eq. (4) following the procedure described in II. In doing this, we replace $\theta^{\pm}(z_m)$ in Eq. (C1)[13] of II by $V^{\pm}(z_m,s)$ which are the Laplace transforms of the portions of $F_t(z,t)\exp(i\omega t)$ propagated to the positive (+) and to the negative (−) z direction, respectively. The explicit forms of $V^{\pm}(z_m,s)$ are given in Appendix B. After some algebra, we have for the principal part of the laser field that is characterized by a small decay constant s_0,[14]

$$E^+(z,t)=\sum_k \frac{u_{ck}(z)}{\gamma+\gamma_c+i(\nu_0+\omega_{ck}-2\omega)}\Big[\sum_j c_{kj}a_j(0)\int_0^t e^{-i\omega_j\tau}\exp[(s_0-i\omega)(t-\tau)]\,d\tau+\frac{i\nu_0 p_a}{\epsilon^1 d}\sum_m u_{ck}(z_m)$$

$$\times\Big((a^\dagger_{m1}a_{m2})(0)e^{(s_0-i\omega)t}+\int_0^t \Gamma_m(\tau)e^{-i\nu_0\tau}\exp[(s_0-i\omega)(t-\tau)]\,d\tau\Big)\Big],$$

$$-d\le z\le 0, \qquad (9a)$$

$$E^+(z,t)=\sum_k \frac{\frac{1}{2}(1+r)\exp(i\Omega_k d/c^1)}{\gamma+\gamma_c+i(\nu_0+\omega_{ck}-2\omega)}\Big[-\sum_j ic_{kj}a_j(0)\int_0^t e^{-i\omega_j\tau}\exp[(s_0-i\omega)(t-z/c^0-\tau)]H(t-z/c^0-\tau)\,d\tau$$

$$+\sum_m \frac{\nu_0 p_a}{\epsilon^1 d}u_{ck}(z_m)\Big((a^\dagger_{m1}a_{m2})(0)\exp[(s_0-i\omega)(t-z/c^0)]H(t-z/c^0)$$

$$+\int_0^t \Gamma_m(\tau)e^{-i\nu_0\tau}\exp[(s_0-i\omega)(t-z/c^0-\tau)]H(t-z/c^0-\tau)\,d\tau\Big)\Big],\quad 0<z, \qquad (9b)$$

$$c_{kj}=-i\left(\frac{\hbar\omega_j}{\epsilon^1 L}\,\frac{1}{1-K\sin^2 k^1_j d}\right)^{1/2}\frac{GN(c^1)^2}{d}\,\frac{\exp(i\omega_{ck}d/c^1)\cosh[(\gamma_c-i\omega_j)d/c^1]}{\gamma_c+i(\omega_{ck}-\omega_j)}, \qquad (9c)$$

$$G=|p_a|^2\nu_0^{\,2}\sigma/2\hbar\omega\epsilon^1c^1, \qquad (9d)$$

where N is the number of atoms per unit distance in the z direction (which is assumed to be independent of z). In Eq. (9a) Heaviside functions of the form $H(t-\tau_m)$, where τ_m is of the order of the cavity round-trip time $2d/c^1$, are omitted and in Eq. (9b) factors of the same order are omitted in the arguments of the Heaviside functions. The complex decay constant s_0 is given by

$$s_0=-\frac{\gamma\gamma_c+(\nu_0-\omega)(\omega-\omega_{ck})-GNc^1-i\{\gamma(\omega-\omega_{ck})+\gamma_c(\omega-\nu_0)\}}{\gamma+\gamma_c+i(\nu_0+\omega_{ck}-2\omega)}, \qquad (10)$$

which agrees, formally, with that of a quasimode theory [see below Eqs. (18)].

The appearance of the quasimode functions, Eq. (1a), in Eqs. (9a) and (9b) implies that the cavity quasimodes are excited. The second terms in the square brackets in Eqs. (9a) and (9b), i.e., the terms of the initial polarization, vanish for large t. The other terms due to thermal noise and quantum noise are lasting, being convolutions of the noisy driving functions and the decaying function obtained by the semiclassical theory in II. This is to be expected since we are working with linearized differential equations and since the result obtained in II is the response of the laser to a perturbation which is a δ function of time.

IV. QUANTUM-MECHANICAL COHERENCE FUNCTION AND THE LINEWIDTH

Using the statistical properties of the driving forces given by Eqs. (7) and (8), we can calculate the quantum-mechanical coherence function.[15] For a single quasimode (e.g., the kth mode) of the cavity, we have

$$\langle E^-(z',t')E^+(z,t)\rangle=Ru^*_{ck}(z')u_{ck}(z)\begin{cases}e^{(s_0-i\omega)(t-t')}, & t>t',\\ e^{(s_0^*+i\omega)(t'-t)}, & t<t',\end{cases} \qquad (11a)$$

$$-d\le z'\le 0,\quad -d\le z\le 0$$

$$\langle E^-(z',t')E^+(z,t)\rangle=\frac{(1+r)^2}{4r}R\times\begin{cases}\exp\Big\{(s_0-i\omega)\Big[\Big(t-\frac{z}{c^0}\Big)-\Big(t'-\frac{z'}{c^0}\Big)\Big]\Big\}, & \Big(t-\frac{z}{c^0}\Big)>\Big(t'-\frac{z'}{c^0}\Big)\\ \exp\Big\{(s_0^*+i\omega)\Big[\Big(t'-\frac{z'}{c^0}\Big)-\Big(t-\frac{z}{c^0}\Big)\Big]\Big\}, & \Big(t-\frac{z}{c^0}\Big)<\Big(t'-\frac{z'}{c^0}\Big)\end{cases} \qquad (11b)$$

$$0<z',\quad 0<z$$

$$R=\frac{2\hbar\omega\gamma\,\beta_c/\gamma_c}{\epsilon^1 d(\gamma+\gamma_c)(1-\sigma/\sigma_{th})}\Big[\Big(\frac{\sigma^2}{\sigma_{th}\sigma^0_{th}}\Big)n(\Theta)+\frac{N_2}{N\sigma_{th}}\Big], \qquad (11c)$$

where

$$N_2 = \frac{1}{2}(1+\sigma)N, \tag{12}$$

$$\beta_c = \frac{c^1}{2d}\,\frac{1-r^2}{2r}. \tag{13}$$

The threshold population inversion σ_{th} and its value at zero detuning σ^0_{th} are given by

$$\sigma_{th} = \frac{2\epsilon^1\hbar\gamma\gamma_c}{N|p_a|^2\omega}\{1+\delta^2\},\quad \sigma^0_{th} = \sigma_{th}|_{\delta=0}, \tag{14a}$$

$$\delta^2 = [(\nu_0 - \omega_{ck})/(\gamma+\gamma_c)]^2. \tag{14b}$$

In deriving Eqs. (11), we ignored terms that vanish for large t or t', replaced the summation over z_m by an integration, and used Eqs. (2b) and (2c) repeatedly. Also, we made the approximation

$$\sum_j \omega_j n_j(\Theta)\,\frac{\exp[i\omega_j(\tau'-\tau)]}{\gamma_c^2+(\omega_{ck}-\omega_j)^2} = \frac{L}{c^0\pi}\,\omega n(\Theta)\,\frac{2\pi}{\gamma_c^2}\,\delta(\tau'-\tau), \tag{15}$$

where $L/c^0\pi$ is the density of modes of the field of the universe[1] and $n(\Theta)$ is the average Planck distribution around ω. Equation (15) is valid[16] only for time differences greater than the reciprocal cavity half-width $1/\gamma_c$. Further, we made the approximation

$$\omega = (\gamma\omega_{ck} + \gamma_c\nu_0)/(\gamma+\gamma_c) \tag{16}$$

for comparison with the results of Haken.[17] This is by no means the unique expression for ω as was discussed in II.

The linewidth (half-width at half-maximum) is given by the absolute value of the real part of the decay constant s_0 in the coherence function. In order to express it in terms of power output, we calculate the latter from the power flow outside the cavity or from the energy stored in the cavity multiplied by $2\gamma_c$ using Eq. (11b) or (11a), respectively. (These give the same output power, showing that the decay constant γ_c appearing in the quasimode analysis gives a correct damping factor also in the multimode analysis as far as the linear theory is concerned.) Denoting the linewidth and the power output by $\Delta\omega$ and by P, respectively, we have

$$\Delta\omega = \frac{(\gamma+\gamma_c)[\gamma\gamma_c(1+\delta^2) - GNc^1]}{(\gamma+\gamma_c)^2+\delta^2(\gamma-\gamma_c)^2} \tag{17a}$$

$$= \frac{\hbar\omega}{P}\,\frac{2\gamma^2\beta_c^2}{(\gamma+\gamma_c)^2+\delta^2(\gamma-\gamma_c)^2}\left[\left(\frac{\sigma^2}{\sigma_{th}\sigma^0_{th}}\right)n(\Theta) + \frac{N_2}{N\sigma_{th}}\right] \times(1+\delta^2). \tag{17b}$$

If we replace β_c by γ_c and the factor before $n(\Theta)$ by unity, Eq. (17b) reduces to the result of Haken,[17] derived assuming three-level atoms. As will be shown later, the factor before $n(\Theta)$ should indeed be close to unity within the range of validity of our analysis.

V. DISCUSSION

In Secs. I–IV, we have analyzed a laser having output coupling incorporating homogeneously broadened two-level atoms which have a constant population inversion. The present multimode (of the universe) analysis has several advantages over conventional quasimode theories. First, the present theory directly yields the field outside of the cavity which we can observe. In particular, the quantum-mechanical coherence function, Eq. (11b), is seen to be propagated with the velocity of light, which is a special case of the propagation of the mutual coherence function of partially coherent light.[18] Also our theory yields the linewidth in k space, which can easily be shown to be equal to $\Delta\omega/c^0$. Second, the thermal noise was derived without introducing any heat bath for the field[4,6,19]; it was derived directly from the initial thermal radiation field. Note that the thermal driving function,

$$\sum_j c_{kj}\,a_j(0)\exp(-i\omega_j t),$$

in Eqs. (9) is Markoffian only on a time scale greater than the reciprocal cavity half-width γ_c^{-1} as was discussed below Eq. (15), whereas in a quasimode Langevin theory corresponding driving force, e.g., the $F_q(t)$ in Eq. (18a) below, is usually assumed to be "exactly" Markoffian. (Thus a quasimode theory leads to an exorbitant range of validity.) Third, the linewidth formula, Eq. (17b), gives a correction[20] to that of a quasimode theory by a factor $(\beta_c/\gamma_c)^2$.

Here, we briefly discuss on the origin of the last correction. For direct comparison of our theory with the quasimode-Langevin theory, we consider a one-dimensional laser within the framework of the quasimode theory[21] rather than the generalized model of Haken.[17] We use a model which is identical to the one analyzed above except that the coupling surface is replaced by a perfectly conducting wall and that a damping constant γ_{cq} $(=\gamma_c)$ and the corresponding fluctuating term, $F_q(t)$, for the field are introduced. Here, the subscript q denotes the quasimode theory. For the basic equations for this model see Appendix A. The field of such a laser can easily be obtained, which reads

$$E_q^+(z,t) = -i\left(\frac{\hbar\omega}{2}\right)^{1/2} U_{kq}(z)a(0)e^{(s_0-i\omega)t} - i\left(\frac{\hbar\omega}{2}\right)^{1/2}\frac{U_{kq}(z)}{\gamma_{cq}+\gamma}\int_0^t \exp[(s_0-i\omega)(t-\tau)]\left[\frac{d}{d\tau}F_q(\tau)+\gamma F_q(\tau)\right]e^{-i\omega\tau}d\tau$$
$$- i\sum_m \frac{\nu_0 p_a}{2(\gamma_{cq}+\gamma)}\,U_{kq}(z)U_{kq}(z_m)\int_0^t \exp[(s_0-i\omega)(t-\tau)]\Gamma_m(\tau)e^{-i\omega\tau}d\tau, \tag{18a}$$

where

$$U_{kq}(z)=(2/\epsilon^1 d)^{1/2}\sin[(\omega_{kq}/c^1)(z+d)],\quad \omega_{kq}=k(\pi c^1/d),\quad k=1,2,3,\ldots . \tag{18b}$$

Here, for simplicity, homogeneous broadening of atoms, single-mode operation and zero detuning ($\omega=\omega_{kq}$) are assumed. Also, the slowly varying amplitude approximation is made as in the derivation of Eqs. (9). The factor s_0 is given by Eq. (10) with γ_c being replaced by γ_{cq}. Comparing the terms of quantum noise, i.e., terms of Γ_m, in Eqs. (9a) and (18a) and calculating the stored energies in the cavity, we have for the ratio of the power output in our theory to that in the above model:

$$\frac{P}{P_q}=\frac{2\gamma_c\int_{-d}^{0}|\sin[\Omega_{ck}(z+d)/c^1]|^2dz\sum_m|\sin[\Omega_{ck}(z_m+d)/c^1]|^2}{2\gamma_{cq}\int_{-d}^{0}\sin^2[\omega_{kq}(z+d)/c^1]dz\sum_m\sin^2[\omega_{kq}(z_m+d)/c^1]}=\left(\frac{\beta_c}{\gamma_c}\right)^2 . \tag{19}$$

Note that $2\gamma_c$ is the correct cavity damping factor also in this multimode analysis as noted earlier. This ratio is exactly the correction factor obtained in Eq. (17b) against a conventional formula.[17] Similar arguments can be given for thermal noise terms. Thus we conclude that the quasimode theory underestimates the power output because of the ignorance of the correct field distribution consistent with the output coupling. This is the implication of the correction obtained for the linewidth formula. The correction factor Eq. (19) is unity if the amplitude reflectivity r is unity, but it amounts to 2.2 if r is 0.2. Such a small value of r, or strong output coupling, is allowed in our theory as long as the criteria which follow are satisfied.

Finally, we examine the range of validity of the results obtained in Secs. III and IV. Allowed values of the linewidth $\Delta\omega$ are limited by: (a) the slowly varying amplitude approximation used in deriving Eqs. (9) [see Eq. (64) of II and Eq. (17a)]; (b) the neglect of small arguments in the Heaviside functions in Eqs. (9); (c) the approximation made in deriving Eqs. (9) [see Eq. (48) of II] and Eq. (15); and (d) the neglect of terms corresponding to the essential singular point $s=-\gamma'$ in Eq. (C1), II. These require that $\Delta\omega\ll(\gamma+\gamma_c)$, $\Delta\omega\ll\Delta\omega_c$, $\Delta\omega\ll\gamma_c$, and $\Delta\omega\ll\gamma$, respectively. Thus we should have[22]

$$\Delta\omega\ll\min(\Delta\omega_c,\gamma_c,\gamma) . \tag{20}$$

Here, $\Delta\omega_c$ is the cavity (quasi)mode spacing, γ_c the cavity decay constant, and γ the atomic halfwidth. By virtue of Eq. (17a) this reduces to

$$1-\frac{\sigma}{\sigma_{\text{th}}}\ll\frac{\min(\Delta\omega_c,\gamma_c,\gamma)[(\gamma+\gamma_c)^2+\delta^2(\gamma-\gamma_c)^2]}{\gamma\gamma_c(\gamma+\gamma_c)(1+\delta^2)} . \tag{21}$$

Here δ is the relative detuning defined by Eq. (14b). Since the quantity on the right is at most 2 (the maximum being reached when $\gamma=\gamma_c\leqslant\Delta\omega_c$ and $\delta=0$), the relative population inversion $\sigma/\sigma_{\text{th}}$ should be close to unity. (Especially, if both γ_c and γ are much greater than $\Delta\omega_c$, it should be very close to unity.) Therefore, the factor $\sigma^2/\sigma_{\text{th}}\sigma_{\text{th}}^0$ before $n(\Theta)$ in Eq. (17b) can safely be dropped.

Our basic assumption of constant population inversion imposes another limitation: the maximum value of the field intensity, $\langle E^-(z,t)E^+(z,t)\rangle$, in the cavity should be much smaller than the saturation parameter, $|E_s|^2$, in Eq. (29), Ref. 3. Using Eq. (11a) with $t=t'$, we have

$$\frac{4\alpha\gamma\gamma_c f(r)(1+\sigma)}{\Gamma(\gamma+\gamma_c)Nd\sigma_{\text{th}}^2}\ll1-\frac{\sigma}{\sigma_{\text{th}}} , \tag{22a}$$

$$f(r)=\frac{(1-r)(1+r)^3}{4r^2\ln(1/r)},\quad \alpha=1+\frac{2\sigma^2(1+\delta^2)}{\sigma_{\text{th}}(1+\sigma)}n(\Theta)\approx1 , \tag{22b}$$

where Γ on the left-hand side is the time constant of the incoherent pumping process and the function $f(r)$ takes a value of the order of unity for $r\gtrsim e^{-1}$.

The results of our analysis are valid under the limitations of Eqs. (21) and (22). Even if we ignore the thermal noise as compared with the quantum noise, these limitations cannot be relieved.

APPENDIX A: THE HEISENBERG EQUATIONS OF MOTION

The Heisenberg equations of motion leading to Eqs. (4)–(6) of the text read as follows. (For notations see the text.)

$$\frac{d}{dt}a_j=-i\omega_j a_j-i\sum_m\kappa_{jm}(a_{m1}^\dagger a_{m2}) , \tag{A1}$$

$$\frac{d}{dt}(a_{m1}^\dagger a_{m2})=-(i\nu_m+\gamma_m)(a_{m1}^\dagger a_{m2})+i\sum_j\kappa_{jm}^* a_j\sigma_m+\Gamma_m(t)e^{-i\nu_m t} , \tag{A2}$$

$$\kappa_{jm}=i\nu_m\,p_m(1/2\hbar\omega_j)^{1/2}U_j(z_m) . \tag{A3}$$

The Heisenberg equations for the conventional model that lead to Eq. (18a) of the text read as follows:

$$\frac{d}{dt}a_k=-i\omega_{kq}a_k-\gamma_{cq}a_k-i\sum_m\kappa_{km}(a_{m1}^\dagger a_{m2})+F_{kq}(t)e^{-i\omega_{kq}t} , \tag{A4}$$

$$\frac{d}{dt}(a_{m1}^\dagger a_{m2}) = -(i\nu_m + \gamma_m)(a_{m1}^\dagger a_{m2}) + i\sum_k \kappa_{km}^* a_k \sigma_m + \Gamma_m(t) e^{-i\nu_m t}, \quad \text{(A5)}$$

$$\kappa_{km} = i\nu_m p_m (1/2\hbar\omega_{kq})^{1/2} U_{kq}(z_m). \quad \text{(A6)}$$

Note that a damping term and a fluctuating term are added in Eq. (A4) and that the mode functions $U_{kq}(z)$, defined by Eq. (18b), are orthogonal functions *within the cavity*.

APPENDIX B: LAPLACE TRANSFORMS OF THE DRIVING TERMS

The Laplace transforms $V_i^\pm(z,s)$ of the portions of $F_i(z,t)\exp(i\omega t)$ $(i=1,2,3)$ propagated to the positive and to the negative z direction read

$$V_1^\pm(z,s) = \pm \sum_j \frac{1}{2}\left(\frac{\hbar\omega_j}{2}\,\frac{1}{1-K\sin^2 k_j^1 d}\right)^{1/2} \exp\left(\pm i(\omega_j-\omega)\,\frac{z+d}{c^1}\right) \frac{a_j(0)}{s+i(\omega_j-\omega)}, \quad \text{(B1)}$$

$$V_2^+(z,s) + V_3^+(z,s) = \sum_m \frac{i\nu_m p_m}{2\epsilon^1 c^1}\,\frac{(a_{m1}^\dagger a_{m2})(0)+\Delta_m(s)}{\gamma_m + i(\omega-\nu_m)+s} \times\left[\delta_{z>z_m}\exp\left((i\omega-s)\,\frac{z-z_m}{c^1}\right) + \frac{R(s)}{1-R(s)}\exp\left((i\omega-s)\,\frac{z-z_m}{c^1}\right) - \frac{1}{1-R(s)}\exp\left((i\omega-s)\,\frac{2d+z+z_m}{c^1}\right)\right], \quad \text{(B2)}$$

$$V_2^-(z,s) + V_3^-(z,s) = \sum_m \frac{i\nu_m p_m}{2\epsilon^1 c^1}\,\frac{(a_{m1}^\dagger a_{m2})(0)+\Delta_m(s)}{\gamma_m + i(\omega-\nu_m)+s} \times\left\{\delta_{z<z_m}\exp\left((i\omega-s)\,\frac{z_m-z}{c^1}\right) + \frac{R(s)}{1-R(s)}\left[\exp\left((i\omega-s)\,\frac{z_m-z}{c^1}\right) - \exp\left(-(i\omega-s)\,\frac{2d+z+z_m}{c^1}\right)\right]\right\}, \quad \text{(B3)}$$

where

$$\delta_{z\gtrless z_m} = \begin{cases} 1, & z \gtrless z_m, \\ 0, & z \lessgtr z_m, \end{cases} \quad \text{(B4)}$$

$$R(s) = -r\exp[2(i\omega-s)d/c^1], \quad \text{(B5)}$$

and $\Delta_m(s)$ is the Laplace transform of $\Gamma_m(t)\exp[i(\omega-\nu_m)t]$. The procedure of derivation is similar to that of Eq. (49) in II.

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[9]I. R. Senitzky, Phys. Rev. 119, 670 (1960).
[10]We assume that the atoms have been coupled to their respective pumping (and damping) mechanisms since $t=-\infty$, so that at $t=0$ the population inversion σ_m is at its unsaturated steady-state value.

[11]The diffusion coefficient $\gamma_m(1+\sigma_m)$ can be derived (Ref. 4 and 5) by allowing at first the variation of σ_m and making it constant afterwards. For the variation of σ_m see, e.g., Eqs. (9)–(11) in Ref. 3. [The last equation should be rewritten separately for $(a_{m2}^\dagger a_{m2})$ and for $(a_{m1}^\dagger a_{m1})$ for application of the quantum Langevin method.] The general expression for the coefficient is given, in the notations of Ref. 3, by

$$[2\gamma_m - \Gamma_m(1-\sigma_m^0)]\langle(a_{m2}^\dagger a_{m2})\rangle + \Gamma_m(1+\sigma_m^0)\langle(a_{m1}^\dagger a_{m1})\rangle.$$

[12]W. H. Louisell, *Radiation and Noise in Quantum Electronics* (McGraw-Hill, New York, 1964), p. 219.
[13]In Eq. (C1), II the first and the third terms should be multiplied by $\frac{1}{2}$ and the first term in the first integrand should be multiplied by r'.

[14]Other terms become important as we go far below

threshold. Therefore, the analysis applies to an operation below but near threshold.

[15]R. J.Glauber, Phys. Rev. 130, 2529 (1963).

[16]A more rigorous calculation of the summation in Eq. (15) adds, to Eqs. (11a) and (11b), terms with a decay constant $[\gamma_c + i(\omega_{ck} - \omega)]$ (instead of s_0), which can be important far below threshold.

[17]H. Haken, in Ref. 4, p. 99.

[18]M. Born and E. Wolf, *Principles of Optics*, 4th ed. (Pergamon, London, 1970).

[19]W. H. Louisell, in Ref. 10, p. 253.

[20]Equation (17a), giving $\Delta\omega$ in terms of the inversion σ, gives no new result.

[21]For the definition of the "quasimode theory" see Sec. I in this paper.

[22]Another slowly varying amplitude approximation was used in deriving Eq. (4) [or Eq. (29) of II]. This requires that $\Delta\omega \ll \omega$, which does not affect the condition Eq. (20) since the optical frequency ω is usually much greater than $\Delta\omega_c$, γ_c, and γ.