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Modulations and degrees of coherence of optical fields

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By using the concept of degree of coherence of quantum optical fields, the properties of fields which can be modulated at a complex rate are studied.

In a recent paper[1] the concept of modulation of quantum optical fields was introduced in order to compare classical and quantum theories of optical coherence. Almost at the same time the translation of a short paper[2] appeared, in which the statistics of fields with equal degrees of coherence was studied. In the present paper we compare the concepts of modulations and degrees of coherence of optical fields. These two concepts are, in some cases, closely related.

Let us first recall very shortly the concept of modulation of a quantum optical field F statistically defined by a density matrix ρ. This field is said to be modulable at the rate λ if it is possible to find a new field F_λ, i.e., a new density matrix ρ_λ, such that for every x_i, n, and m

$$G_\lambda^{(n,m)}[\{x_i\}] = \lambda^{n+m} G^{(n,m)}[\{x_i\}] , \tag{1}$$

where the G functions are the quantum coherence functions of the two fields F and F_λ.

Because the results presented in Ref. 2 are applied to the monomode case only, the following discussion will be restricted to that case. Thus the condition (1) becomes identical to Eq. (3.11) of Ref. 1:

$$\mathrm{Tr}(\rho_\lambda a^{\dagger n} a^m) = \lambda^{n+m}\,\mathrm{Tr}(\rho a^{\dagger n} a^m) . \tag{2}$$

In the discussion of Ref. 1 it was sufficient to consider modulation at *real rates* λ. However, it is straightforward to generalize the result to complex rates λ. In this case Eq. (2) becomes evidently

$$\mathrm{Tr}(\rho_\lambda a^{\dagger n} a^m) = \lambda^{*n}\lambda^m\,\mathrm{Tr}(\rho a^{\dagger n} a^m) . \tag{3}$$

Using exactly the same method as in Sec. III of Ref. 1 one can show that, if Eq. (3) is valid for every n and m and if the density matrix ρ has a P representation $P(\alpha)$, then ρ_λ can be written

$$\rho_\lambda = \int P(\alpha)\,|\lambda\alpha\rangle\langle\lambda\alpha|\, d^2\alpha . \tag{4}$$

If $P(\alpha) > 0$, i.e., if the field F is strictly equivalent to a classical one, the matrix ρ_λ defined by Eq. (4) is a density matrix for any value of λ. This means that F is modulable at any rate. In that case we say that F is *consistent for modulation*. Conversely if the field is consistent for modulation, then $P(\alpha)$ must be positive and that was one of the main results of Ref. 1.

In other words, if $P(\alpha) \ngtr 0$ the domain of values of λ for which ρ_λ is effectively a density matrix is limited. We shall examine this point in more detail. For this purpose we recall that the only condition for ρ_λ to be a density matrix is that

$$\langle f|\rho_\lambda|f\rangle \geq 0 \tag{5}$$

for any state vector $|f\rangle$. We can expand this vector in terms of the n-quantum states which are complete and orthonormal. We thus obtain

$$|f\rangle = \sum_{n=0}^{\infty} c_n |n\rangle , \tag{6}$$

and we can write Eq. (5) in the form

$$\langle f|\rho_\lambda|f\rangle = \int P(\alpha) e^{-|\lambda|^2|\alpha|^2} \times \sum_{n,m} (c_m\lambda^{*m})(c_n\lambda^{*n})^* \frac{\alpha^n\alpha^{*m}}{(n!m!)^{1/2}}\, d^2\alpha . \tag{7}$$

Because $P(\alpha)$, which has negative values, is the P representation of the density matrix ρ, we have evidently for $\lambda = 1$, $\langle f|\rho_\lambda|f\rangle \geq 0$. Indeed, as a consequence of Eq. (4), $\rho_1 = \rho$.

Let us now see that if $|\lambda| \leq 1$, then $\langle f|\rho_\lambda|f\rangle \geq 0$. For this purpose we write $|\lambda|^2 = 1 - |\mu|^2$, with $|\mu| \leq 1$, and

$$e^{-|\lambda|^2|\alpha|^2} = e^{-|\alpha|^2} e^{|\mu|^2|\alpha|^2} . \tag{8}$$

If we expand $e^{|\mu|^2|\alpha|^2}$ we obtain a sum of terms of the general form $(k!)^{-1}(\mu\alpha)^k(\mu^*\alpha^*)^k$. Each such term gives in Eq. (7) a contribution proportional to $\langle u|\rho|u\rangle$, where $|u\rangle$ is a state deduced from $|f\rangle$. Because ρ is a density matrix, $\langle u|\rho|u\rangle \geq 0$, and we obtain in Eq. (7) a sum of positive terms. Then Eq. (5) is satisfied.

This result is not necessarily true of $|\lambda| > 1$ because, in this case, $|\lambda|^2 = 1 + |\mu|^2$ and we obtain in Eq. (8) the term $e^{-|\mu|^2|\alpha|^2}$. Its expansion has negative terms and we cannot deduce that $\langle f|\rho_\lambda|f\rangle$ is a sum of positive terms.

To conclude, if $P(\alpha) \ngtr 0$, it is always possible to modulate F at a rate λ such that $|\lambda| \leq 1$. In other words *it is always possible to decrease the mean number of photons in the field without changing its statistics*.

From that result and the fact that the field F is not consistent for modulation because $P(\alpha) \not> 0$, we conclude that there exists a positive value r, $(r \geq 1)$, such that ρ_λ is a density matrix only if

$$0 \leq |\lambda| \leq r \ . \tag{9}$$

This limiting value of the modulation rate evidently depends on the function $P(\alpha)$. It becomes infinite if $P(\alpha) > 0$. A particular example of such a limiting value was presented in Sec. IV of Ref. 1, Eqs. (4.12), (4.14).

Finally let us introduce the class $M(F)$ of fields which can be obtained from F by modulation. Any element of this class has a density matrix defined by Eq. (4) where the value of λ is restricted by Eq. (9).

Following the line of the discussion presented in Ref. 2 we shall introduce the nth-order degrees of coherence of the field which in the monomode case are defined by

$$g_n = \frac{\mathrm{Tr}(\rho a^{\dagger\, n} a^n)}{[\mathrm{Tr}(\rho a^\dagger a^n)]^n} \ . \tag{10}$$

These coefficients were introduced and studied in earlier papers[3–5] not cited in Ref. 2.

Let us first point out that *all the fields of the class $M(F)$ have the same degrees of coherence*. Indeed if we modulate F at a rate λ, we obtain the field F_λ and we conclude readily from Eq. (3) that $g_n(\lambda) = g_n$. This is the main result of Ref. 2. In that reference the authors have studied the set of fields whose density operators are defined as ρ_λ in Eq. (4). This set is exactly our class $M(F)$.

We will now study in more detail the class $E(F)$ of fields which have the same degrees of coherence as F. Any element F' of this class is defined by

$$g_n' = g_n \tag{11}$$

for every n. By using Eq. (9) this property can be expressed by the relation

$$\mathrm{Tr}(\rho' a^{\dagger\, n} a^n) = |\lambda|^{2n}\, \mathrm{Tr}(\rho a^{\dagger\, n} a^n) \ , \tag{12}$$

where $|\lambda|$ is defined by

$$|\lambda|^2 = \frac{\mathrm{Tr}(\rho' a^\dagger a)}{\mathrm{Tr}(\rho a^\dagger a)} \ . \tag{13}$$

We see that Eq. (12) is a particular case of Eq. (3), obtained for $n = m$. This is evidently due to the structure of the degrees of coherence which are deduced from the coherence function calculated for $n = m$.

In conclusion we can say that $M(F)$ is the class of fields which satisfy Eq. (3) for every n and m, while $E(F)$ is the class of fields which satisfy this equation only for $n = m$.

Now let us call $E_S(F)$ the class of all *stationary fields* belonging to $E(F)$ and $E_{\not S}(F)$ the class of all *nonstationary fields* belonging to $E(F)$. These two classes evidently have no common elements, $E_S(F) \cap E_{\not S}(F) = \phi$ (the nullset), and moreover

$$E(F) = E_S(F) \cup E_{\not S}(F) \ . \tag{14}$$

If the initial field F is stationary, i.e., if $P(\alpha)$ is a function only of $|\alpha|$, all the coherence functions $G^{n,m}$ are equal to zero for $n \neq m$. In that case it is clear that all the fields of $M(F)$ are stationary and we deduce easily that

$$E_S(F) = M(F) \ . \tag{15}$$

If the initial field is nonstationary, a circumstance which appears in some interesting physical situation, all the elements of $M(F)$ are nonstationary. It nevertheless remains possible to define precisely the structure of $E_S(F)$.

For this purpose let us associate to any nonstationary field F a *stationary equivalent field* F_S. The P representation of F_S is deduced from $P(\alpha)$ by integration over the phase θ of the complex number α and can be written

$$P_S(|\alpha|) = \frac{1}{2\pi} \int_0^{2\pi} P(\alpha)\, d\theta \ . \tag{16}$$

In all experiments of interference, photon-counting, and photon coincidences, the fields F and F_S give exactly the same results[6,7] because their coherence functions are the same for $n = m$. Evidently they have also the same degrees of coherence and thus F_S belongs to the class $E_S(F)$. It is nevertheless possible to distinguish F and F_S, for example, in beat experiments by using a reference beam.[8] It is clear that all the elements of $M(F_S)$ are stationary and have the same degrees of coherence as F, and we deduce easily that

$$E_S(F) = M(F_S) \ . \tag{17}$$

But $M(F)$ is evidently a subclass of $E_{\not S}(F)$, $M(F) \subset E_{\not S}(F)$. Indeed the elements of $E_{\not S}(F)$ are nonstationary as F itself, but they satisfy Eq. (3) only for $n = m$.

Finally let us observe that the two parameters r and r_S appearing in the definition of $M(F)$ and $M(F_S)$, Eq. (18), are not necessarily the same.

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