Drift solitons and their two-dimensional stability

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The nonlinear equation governing low-frequency drift waves is considered. Utilizing the linear dispersion relation for such waves, it is shown that there exists a parameter range for which the drift waves are governed by a modified Korteweg–de Vries equation having a solitary solution in one and two dimensions. The one-dimensional solitons are unstable with respect to perturbations in the direction perpendicular to their motion.

I. INTRODUCTION

Solitary solutions of the Korteweg–de Vries (KdV) equation and its modified forms has been of considerable interest for the last few years.[1] Beside the many interesting mathematical properties of these equations, they have been found to describe a wide range of one-dimensional nonlinear waves which are of considerable physical interest, including shallow water waves, ion acoustic waves, Alfvén waves etc. Experimental observations of solitons have been reported[2] and are supported by computer simulations.

The importance of the KdV equation in plasma was emphasized by Washimi and Tanuiti who have shown[3] that ion acoustic waves are governed by the KdV equation and that one should expect ion acoustic solitons. Starting from the fluid equations of a plasma, expanding n, v_z, and ψ (where n stands for the density, v_z for the velocity in the z direction, and ψ is the potential) in a small parameter ϵ, stretching the independent variables, and equating coefficients with the same power in ϵ, one arrives at a set of equations which can be reduced to the KdV equation. Using a somewhat more generalized perturbation method (named reductive perturbation method by Taniuti and Wei[4]) one can also show that Alfvén waves are governed by a modified Korteweg–de Vries (MKdV) equation having soliton solutions. Using this reductive perturbation method, a large number of nonlinear waves have been investigated. The reader is referred to a recent review of this approach for more details.[5]

Extensive investigations of Langmuir solitons using computer codes have revealed that solitons have remarkable stability properties with regard to perturbations in the direction of their motion. However for perturbations in a direction perpendicular to their motion the solitons seem to be unstable.[6] That ion acoustic solitons are stable to perturbations in the direction of their motion has been known for quite a while,[7] and the higher dimensional stability problem has recently been looked at by a few authors.[8,9]

In order to investigate analytically the stability properties of solitary solutions of the KdV equation and its modified forms, it is convenient to use an approach for deriving these equations somewhat different from the one mentioned above. This approach uses the linear dispersion relation as an ingredient in deriving the KdV and MKdV equations, and has been extensively used.[10] This approach will be explained and used in this paper.

The use of the dispersion relation approach also has the advantage of hinting at waves which might be described by a KdV or MKdV equations. One can expect that waves having dispersion relations similar (in some parametric range) to waves which are known to be described by KdV or MKdV equation could also be described by such equations. As an example we consider low-frequency drift waves. As is well known, the dispersion relation for such waves is

$$\omega^2 - \omega\omega_{*e} - K_z^2 C_s^2 = 0,$$

where $C_s = (T_e/m_i)^{1/2}$ is the sound velocity,

$$\omega_{*e} = \frac{K_y T_e}{eB_0}\frac{n_0'}{n_0},$$

T_e and T_i are the temperature of the electrons and ions, respectively, K_y and K_z are the wave numbers in the y and z direction, respectively, n_0' is the x derivative of the x-dependent density, B_0 is the z directional uniform magnetic field, and we assume that the drift waves are moving in the yz plane. When $C_s^2 K_z^2 \gg \omega_{*e}^2$, the two branches of solutions turn into the ordinary ion acoustic waves. As one moves toward smaller K_z domain, the upper branch of the dispersion relation departs from the acoustic mode and approaches ω_{*e} at $K_z = 0$. In the intermediate region between these two limits one may expect that the nonlinear equation governing these drift waves might not be too different from the equations describing ion acoustic waves. Now since we know that ion acoustic waves are described by the KdV equation, we expect that drift waves in what we call the "intermediate K_z range" might also be described by the KdV or MKdV equation.

Motivated in this way we consider in this paper the nonlinear equations governing low-frequency drift waves. Using the dispersion approach, we rederive in Sec. II the KdV equation for ion acoustic waves moving in one dimension, and the MKdV equation for ion acoustic waves moving in a two-dimensional magnetized plasma. Establishing in this way the procedure for deriving the relevant equation we turn in Sec. III to consider drift waves. We show that they obey a MKdV equation and then proceed using the stability analysis introduced by Kadomtsev and Petviashvili[11] to show that one-dimensional solitary solutions of the MKdV equations describing drift waves are unstable with respect to perturbations perpendicular to their motions.

II. KORTEWEG–de VRIES EQUATION FOR ION ACOUSTIC WAVES

We start from a fluid model of magnetized plasma

$$\frac{\partial n}{\partial t}+\vec{\nabla}\cdot(n\vec{v})=0, \tag{1}$$

$$\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\vec{\nabla})\vec{v}=-\frac{e}{m}\vec{\nabla}\psi+\Omega_i\vec{v}\times\hat{z}, \tag{2}$$

where we have assumed that $\vec{E}=-\vec{\nabla}\psi$, and that the magnetic field is in the z direction, $\hat{z}$ is a unit vector in the z direction, $\Omega_i=eB_0/m$, B_0 being the magnitude of the magnetic field, and v is the ion flow velocity. Furthermore we assume that the ions are cold and that the electrons follow a Boltzmann distribution. The Poisson equation will thus read

$$\vec{\nabla}^2\psi=-4\pi e(n-n_0e^{e\psi/T_e}), \tag{3}$$

where n is the ion density, n_0 is the unperturbed plasma density, and T_e is the electrons temperature.

For ψ, n_i, n_e we assume the following form

$$\psi(r,t)=\psi_0e^{i(K_yy+K_zz-\omega t)},$$
$$n_{i,e}(r,t)=n_0+\delta n_{i,e}e^{i(K_yy+K_zz-\omega t)}. \tag{4}$$

Assuming $e\psi/T_e\ll 1$, we find

$$\delta n_e\approx n_0(e\psi_0/T_e).$$

From Eq. (3) we have

$$(-\vec{\nabla}^2+4\pi e^2n_0/T_e)\psi(r,t)=4\pi e\delta n_ie^{i(K_yy+K_zz-\omega t)}. \tag{5}$$

Linearizing Eq. (2) we obtain for $(\omega/\Omega_i)^2\ll 1$

$$v_x\approx-\frac{e}{m\Omega_i}\frac{\partial\psi}{\partial y}, \tag{6a}$$

$$v_y\approx i\frac{e\omega}{m\Omega_i^2}\frac{\partial\psi}{\partial y}, \tag{6b}$$

$$\omega v_z=-i\frac{e}{m}\frac{\partial\psi}{\partial z}. \tag{6c}$$

Linearizing Eq. (1) and using Eqs. (6) we find

$$\delta n_ie^{i(K_yy+K_zz-\omega t)}=\frac{n_0e}{m\omega^2}\left(\frac{\omega^2}{\Omega_i^2}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}\right). \tag{7}$$

Inserting (7) in (5) we find

$$(-\lambda_D^2\vec{\nabla}^2+1)\psi=\frac{C_s^2}{\omega^2}\left(\frac{\omega^2}{\Omega_i^2}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}\right), \tag{8}$$

where $\lambda_D=(T_e/4\pi e^2n_0)^{1/2}$ is the Debye length.

In the limit $K^2\lambda_D^2\ll 1$, $(\omega/\Omega_i)^2\ll 1$, we have as a first approximation

$$\omega^2\psi=-C_s^2\frac{\partial^2\psi}{\partial z^2}. \tag{9}$$

Inserting this relation in (8) we find

$$\omega^2(-\lambda_D^2\vec{\nabla}^2+1)\psi=-C_s^2\frac{\partial^2}{\partial z^2}\left(1+r_H^2\frac{\partial^2}{\partial y^2}\right)\psi, \tag{10}$$

where $r_H=C_s/\Omega_i$. Assuming that terms like $\lambda_D^2K^2r_H^2K_y^2$ are negligibly small we find

$$\omega\approx C_sK_z\left[1-\frac{(r_H^2+\lambda_D^2)}{2}K_y^2-\frac{\lambda_D^2}{2}K_z^2\right]. \tag{11}$$

The corresponding equation in terms of the variables x and t applied to Eq. (6c) will result in

$$\frac{e}{m}\frac{\partial\psi}{\partial z}=C_s\left[\frac{\partial v_z}{\partial z}+\frac{(r_H^2+\lambda_D^2)}{2}\frac{\partial^3v_z}{\partial z\partial y^2}+\frac{\lambda_D^2}{2}\frac{\partial^3v_z}{\partial z^3}\right]. \tag{12}$$

Inserting this result in the z component of Eq. (2) we find

$$\frac{\partial v_z}{\partial t}+v_z\frac{\partial v_z}{\partial z}+C_s\left[\frac{\partial v_z}{\partial z}+\frac{(r_H^2+\lambda_D^2)}{2}\frac{\partial^3v_z}{\partial z\partial y^2}+\frac{\lambda_D^2}{2}\frac{\partial^3v_z}{\partial z^3}\right]=0. \tag{13}$$

This is the result of Zakharov and Kuznetsov.[8]

Considering motion with respect to a coordinate system moving with the sound speed C_s and denoting

$$u\equiv\frac{2v_z}{C_s},\quad \bar{y}=\frac{y}{(r_H^2+\lambda_D^2)^{1/2}},$$
$$\bar{z}=\frac{z}{\lambda_D},\quad \bar{t}=\frac{\omega_{Pi}t}{2},$$

where ω_{Pi} is the ion-plasma frequency, we find

$$\frac{\partial u}{\partial\bar{t}}+u\frac{\partial u}{\partial\bar{z}}+\frac{\partial^3u}{\partial\bar{z}^3}=-\frac{\partial^3u}{\partial\bar{z}\partial\bar{y}^2}. \tag{14}$$

This is a modified Korteweg–de Vries equation. For a one dimensional system moving in the z direction the right-hand side vanishes and we have the KdV equation for ion sound waves.

III. DRIFT SOLITONS

Having established the procedure for obtaining nonlinear equations through the use of linear dispersion relations, we proceed to derive a nonlinear equation for drift waves. Customarily, we assume that the unperturbed density is x dependent and the wave is propagating in the yz plane. We proceed as in Sec. II; however, the continuity equation will now have an additional term including the derivative of n_0 with respect to x. Instead of Eq. (8) we now have

$$(-\lambda_D^2\vec{\nabla}^2+1)\psi=\frac{C_s^2}{\omega^2}\left(\frac{i\omega\alpha}{\Omega_i}\frac{\partial\psi}{\partial y}+\frac{\omega^2}{\Omega_i^2}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}\right), \quad (15)$$

where

$$\alpha\equiv\frac{1}{n_0}\frac{dn_0}{dx}.$$

Considering the intermediate parametric range

$$\lambda_D^2K^2\ll 1, \quad (\omega/\Omega_i)^2\ll 1$$

and

$$\left(\frac{\omega K_y}{\Omega_i K_z}\right)^2\ll 1, \quad \left|\frac{\omega}{\Omega_i}\frac{\alpha K_y}{K_z^2}\right|\ll 1.$$

We have as a first approximation $\omega^2\psi=-C_s^2(\partial^2\psi/\partial z^2)$. Inserting this in (15) we find

$$\omega^2(-\lambda_D^2\vec{\nabla}^2+1)\psi=-C_s^2\left(\alpha r_H\frac{\partial^2\psi}{\partial z\partial y}+r_H^2\frac{\partial^4\psi}{\partial z^2\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right), \quad (16)$$

and the dispersion relation will read

$$\omega\approx C_sK_z\left[1-\frac{(r_H^2+\lambda_D^2)}{2}K_y^2-\frac{\lambda_D^2}{2}K_z^2\right]+\frac{C_sr_H\alpha}{2}K_y. \quad (17)$$

The corresponding equation in terms of the variables x and t applied to Eq. (6c) will result in

$$\frac{e}{m}\frac{\partial\psi}{\partial z}=C_s\left[\frac{\partial v_z}{\partial z}+\frac{\alpha r_H}{2}\frac{\partial v_z}{\partial y}+\frac{(r_H^2+\lambda_D^2)}{2}\frac{\partial^3 v_z}{\partial z\partial y^2}+\frac{\lambda_D^2}{2}\frac{\partial^3 v_z}{\partial z^3}\right]. \quad (18)$$

Inserting in the z component of Eq. (2) we get,

$$\frac{\partial v_z}{\partial t}+v_z\frac{\partial v_z}{\partial z}+C_s\left(\frac{\partial v_z}{\partial z}+\frac{\alpha r_H}{2}\frac{\partial v_z}{\partial y}+\frac{(r_H^2+\lambda_D^2)}{2}\frac{\partial^3 v_z}{\partial z\partial y^2}+\frac{\lambda_D^2}{2}\frac{\partial^3 v_z}{\partial z^3}\right)=0. \quad (19)$$

Transforming to new variables

$$z\to z-C_st, \quad y\to y-C_s\frac{\alpha r_H}{2}t,$$

and defining u, $\bar{y}$, $\bar{z}$, and $\bar{t}$ as in Sec. II we find

$$\frac{\partial u}{\partial\bar{t}}+u\frac{\partial u}{\partial\bar{z}}+\frac{\partial^3 u}{\partial\bar{z}^3}=-\frac{\partial^3 u}{\partial\bar{z}\partial\bar{y}}, \quad (20)$$

which is the modified Korteweg–de Vries equation, Eq. (14).[12]

Nonlinear equations for drift waves were considered by Nozaki and Taniuti,[13] using reductive perturbation methods. However, their approach to deriving the equation is somewhat less convenient for a stability analysis which they recommended performing. The way the equation was derived above in which u is expandable in a smallness parameter ϵ,

$$u=u_0+\epsilon u_1+\epsilon^2u_2+\cdots, \quad (21)$$

leads to a quite straight-forward stability analysis of the solution to this MKdV equation.[14] The way to proceed is to use the Krylov-Bogolyubov-Mitropolsky perturbation method in the manner used in the analysis of the phenomenological MKdV equation proposed by Kadomtsev and Petviashvili,[11] and for ion-acoustic solitons in Ref. 9. For the sake of completeness this analysis is briefly outlined. We consider a one-dimensional solitary solution to (20).

$$u_0(\bar{z},\bar{t})=3A\ \mathrm{sech}^2[A^{1/2}(\bar{z}-\bar{z}_0)/2],$$

where $\bar{z}_0=A\bar{t}$ is the phase. We introduce a variable $\xi=A^{1/2}(\bar{z}-\bar{z}_0)$, where A and $\bar{z}_0-A\bar{t}$ are slowly varying functions of $\bar{y}$ and $\bar{t}$. Using the expansion (21) we separate the terms in Eq. (20) in two groups, those in which the perturbation in A and $\bar{z}_0$ is explicitly exhibited and the other in which it is not. Following Ref. 9 we next assume that perturbations of the amplitude are of order ϵ smaller than of the phase, and derivatives with respect to $\bar{y}$ and $\partial\bar{z}_0/\partial\bar{t}-A$ are quantities of order ϵ. Using the variable ξ and choosing properly the amplitude or the phase such that $\partial\bar{z}_0/\partial\bar{t}=A$, one finds for the ϵ^2 terms of Eq. (20) the relation[9]

$$\frac{\partial}{\partial\xi}(-A^{3/2}u_2+A^{1/2}u_0u_2)+A^{3/2}\frac{\partial^3u_2}{\partial\xi^3}=-\frac{1}{A}\frac{\partial A}{\partial\bar{t}}\left(u_0+\xi\frac{u_0'}{2}\right)+Au_0''\frac{\partial^2\bar{z}_0}{\partial\bar{y}^2}-A^{3/2}u_0'''\left(\frac{\partial\bar{z}_0}{\partial\bar{y}}\right)^2, \quad (22)$$

where primes denote differentiation with respect to ξ. Multiplying Eq. (22) by u_0, integrating over ξ, and substituting $\partial^2z_0/\partial\bar{t}^2$ for $\partial A/\partial\bar{t}$, results in the relation

$$\frac{3}{4}\langle u_0^2\rangle\frac{\partial^2\bar{z}_0}{\partial\bar{t}^2}+A^2\langle u_0'^2\rangle\frac{\partial^2\bar{z}_0}{\partial\bar{y}^2}=0, \quad (23)$$

where

$$\langle\cdots\rangle=\int_{-\infty}^{\infty}\cdots d\xi.$$

The solution of Eq. (23) is unstable. We have thus shown that in the "intermediate parametric range." Drift waves are governed by a modified Korteweg-de Vries wave equation having both one- and two-dimensional soliton solutions. The one-dimensional drift solitons are unstable with respect to perturbations perpendicular to their motion. Finally, it may be worth noting that the effects of inhomogeneities in the magnetic field on the nonlinear equation governing drift waves and the stability of its solution may be of considerable interest, as they are for linear drift waves. These effects are presently under investigation.

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[12] It should be noticed that although the structure of the equations governing ion acoustic waves and drift waves is the same, the dependence of the solitons on the physical variables will of course be different.

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