# Electromagnetic scattering by a dielectric sphere in a diverging radiation field*

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The scattering of divergent electromagnetic radiation by a dielectric sphere is obtained by assuming that the radiation emanates from an oscillating dipole located an appropriate distance outside of the sphere. The field differs considerably from the case of parallel incident radiation when the dipole is within about ten particle radii, an effect which may be significant for multiple scattering in dense particulate systems.

Electromagnetic scattering by spheres is usually formulated for parallel incident radiation.[1,2] There is a related problem which arises in connection with the scattering of radio waves by a dipole source on the surface of the conducting Earth.[3] These two problems can be viewed as limiting cases where the dipole source is located either on the surface of the sphere or very distant from it. Here, we develop relevant formulas for the general case of divergent incident radiation where the dipole source is arbitrarily located outside the sphere. These reduce appropriately to each of the limiting cases. Computations of the scattered radiances, which were carried out to illustrate the effects, indicate that for dielectric spheres, considerable deviations from the case of parallel incident radiation begin to occur when the dipole is within about ten particle radii from the sphere.

This work was stimulated by the thought that the phase function which is utilized in multiple scattering theory is appropriate to parallel incident radiation even though the primary and higher order scattering is divergent. In closely packed systems (interparticle distance less than ten particle radii) the divergent nature of the radiation may affect the radiative transfer.

We note parenthetically that in the particular case of terrestrial scattering there is a problem of convergence because of the large size of the Earth ($ka \sim 10^4$), but that this can be obviated by utilization of the Watson transform.[4] This is not a problem in the usual optical case where the size parameter is smaller ($ka \sim 10$).

Consider an oscillating dipole of angular frequency $\omega$ and strength $\vec{p}$ located at $\vec{r}\,' = (0, 0, -d)$ outside a dielectric sphere of dielectric constant $\epsilon$, magnetic permeability $\mu$, and radius $a$ centered at the origin (see Fig. 1). If $d$ is much larger than $a$, the fields reaching the sphere approach those of a plane wave with electric vector along $\vec{p}$, and the scattered fields approach those given by the Lorenz-Mie (LM) theory.[1,2] When the dipole is near the sphere, however, significant departures from this theory may be expected owing to the divergent character of the radiation reaching the sphere. These indeed have been found at certain angles and will be discussed later.

The calculation of the scattered fields for arbitrary $d$ is very similar to that for the corresponding fields in our model for Raman and fluorescent scattering.[5] The main difference is that the dipole source is now outside the sphere so that there are two fields outside (the dipole and the scattered) and only one (the transmitted) field inside. This necessitates the interchange of $j_l$ and $h_l^{(1)}$ in the expansion of the dipole fields in the results of Ref. 5. Since the dipole lies on the negative $z$ axis, the dipole coefficients take on a much simpler form than the general case considered in Ref. 5, and are different from zero only for $m = \pm 1$ and 0. In the notation of that reference, we have for $m = 1$,

$$\vec{\epsilon}^{\,-} = (-1)^{l-1}\left(\frac{(2l-1)l(l+1)}{4\pi}\right)^{1/2}(1, -i, 0)\,,$$

$$\vec{\epsilon}^{\,+} = (-1)^{l}\left(\frac{(2l+3)l(l+1)}{4\pi}\right)^{1/2}(1, -i, 0)\,, \qquad \text{(1a)}$$

$$\vec{M} = (-1)^{l}\left(\frac{(2l+1)l(l+1)}{4\pi}\right)^{1/2}(1, -i, 0)\,;$$

for $m = 0$,

$$\vec{\epsilon}^{\,-} = 2l(-1)^{l}\left(\frac{2l-1}{4\pi}\right)^{1/2}(0, 0, 1)\,,$$

$$\vec{\epsilon}^{\,+} = 2(l+1)(-1)^{l}\left(\frac{2l+3}{4\pi}\right)^{1/2}(0, 0, 1)\,, \qquad \text{(1b)}$$

$$\vec{M} = 0\,;$$

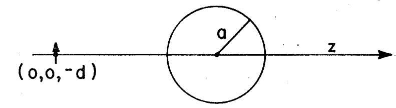


FIG. 1. Sphere, with radius $a$, centered at the origin. Dipole located at (0, 0, -d).

for $m=-1$,

$$\vec{\epsilon}^{-}=(-1)^l\left(\frac{(2l-1)l(l+1)}{4\pi}\right)^{1/2}(1,i,0)\,,$$

$$\vec{\epsilon}^{+}=(-1)^{l+1}\left(\frac{(2l+3)l(l+1)}{4\pi}\right)^{1/2}(1,i,0)\,, \qquad (1c)$$

$$\vec{\mathrm{M}}=(-1)^l\left(\frac{(2l+1)l(l+1)}{4\pi}\right)^{1/2}(1,i,0)\,.$$

The dipole expansion coefficients are given by

$$a_E^d(l,\pm 1)=\pm ik^3\sqrt{\pi}\,(-1)^l(2l+1)^{-1/2}(p_x\pm ip_y)\times[lh_{l+1}^{(1)}(kd)-(l+1)h_{l-1}^{(1)}(kd)]\,,$$

$$a_E^d(l,0)=ik^3(-1)^l p_z[4\pi(2l+1)l(l+1)]^{1/2}\times h_l^{(1)}(kd)/kd\,, \qquad (2)$$

$$a_M^d(l,\pm 1)=ik^3(-1)^l[\pi(2l+1)]^{1/2}\times(p_x\pm ip_y)h_l^{(1)}(kd)\,,$$

$$a_M^d(l0)=0\,.$$

For definiteness, consider the case when the dipole oscillates along the $x$ axis so that $\vec{p}=(p,0,0)$. When the dipole is far from the sphere, its field at the sphere reduces to that of a plane wave polarized along the $x$ axis. The magnetic field of the scattered radiation in the $y$-$z$ plane is then given by

$$\vec{B}_{sc}=\frac{e^{ikr}}{kr}\sum_{l=1}^{\infty}(-i)^{l+1}\times\sum_{m=\pm 1}[\beta_E(l,m)\vec{Y}_{llm}+\beta_M(l,m)\hat{r}\times\vec{Y}_{llm}]\,, \qquad (3)$$

where[1,2,5]

$$\beta_E(l,\pm 1)=-b_l a_E^d(l,\pm 1),\qquad \beta_M(l,\pm 1)=-a_l a_M^d(l,\pm 1)\,,$$

so that[6]

$$\vec{B}_{sc}=-\frac{pk^2e^{ikr}}{r}\times\sum_{l=1}^{\infty}\frac{i^l}{l(l+1)}\{(2l+1)a_l h_l^{(1)}(kd)\tau_l(\theta)+ib_l[lh_{l+1}^{(1)}(kd)-(l+1)h_{l-1}^{(1)}(kd)]\,\pi_l(\theta)\}\,\hat{\theta}$$

$$\sim ipk\,\frac{e^{ik(r+d)}}{rd}\sum\frac{(2l+1)}{l(l+1)}(b_l\pi_l+a_l\tau_l)\hat{\theta}\quad\text{for large } kd. \qquad (3')$$

For a dipole source near the sphere, there is no generally accepted definition of the scattering cross section. We choose to define the differential scattering cross section as the scattered energy flux entering a solid angle $d\Omega$ divided by the total dipole energy flux incident on the sphere. The latter is obtained by integrating the normal component of the Poynting vector of the dipole source over the solid angle subtended by the sphere and has the value

$$F=\tfrac{1}{12}ck^4p^2[1-(1-a^2/d^2)^{3/2}]\,.$$

Accordingly we have, in the $y$-$z$ plane,

$$\frac{d\sigma}{d\Omega}=\frac{cr^2}{8\pi F}\,|\vec{B}_{sc}|^2=\frac{3I_1}{2\pi[1-(1-a^2/d^2)^{3/2}]}\,, \qquad (4)$$

where

$$I_1=\left|\sum\frac{i^{l+1}}{l(l+1)}\{b_l[lh_{l+1}^{(1)}(kd)-(l+1)h_{l-1}^{(1)}(kd)]\tau_l-(2l+1)ia_lh_l^{(1)}(kd)\pi_l\}\right|^2.$$

When $kd$ is large, Eq. (4) reduces to the differential cross section as usually defined divided by $\pi a^2$. If the dipole oscillates along the $y$ axis, Eq. (4) is replaced by

$$\left(\frac{d\sigma}{d\Omega}\right)_2=\frac{3I_2}{2\pi[1-(1-a^2/d^2)^{3/2}]} \qquad (4')$$

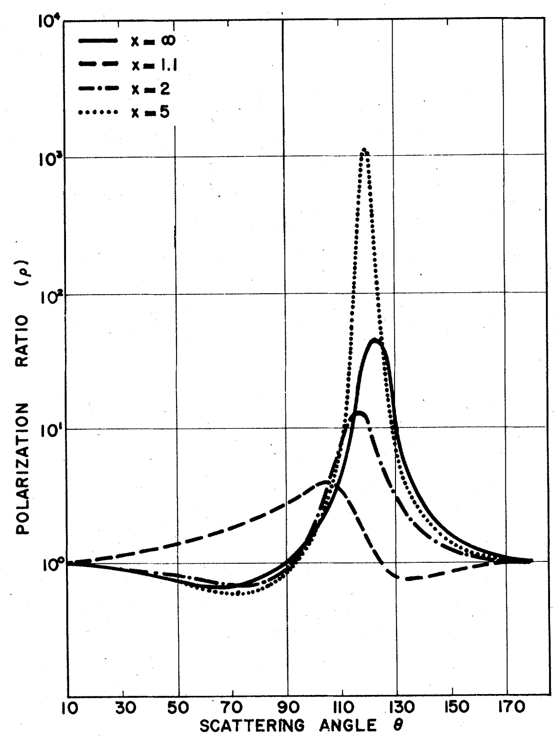


FIG. 2. Polarization ratio versus scattering angle for $\alpha=2$ and various values of $x=d/a$.

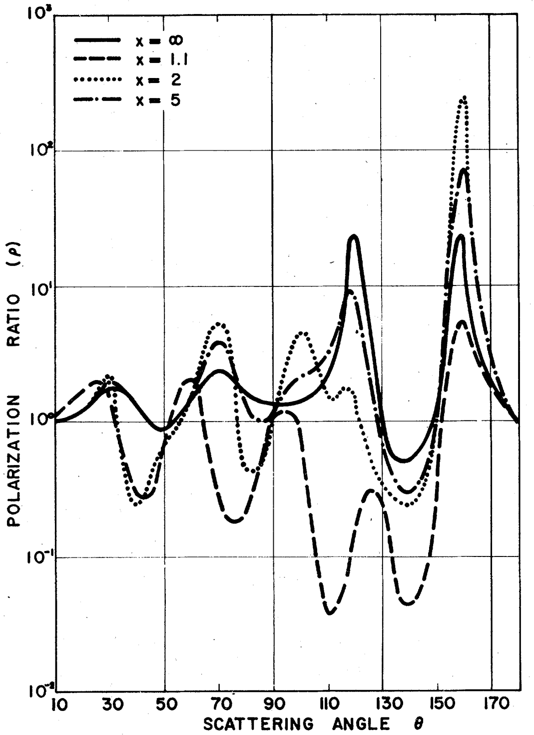


FIG. 3. Polarization ratio versus scattering angle for $\alpha = 5$ and various values of $x = d/a$.

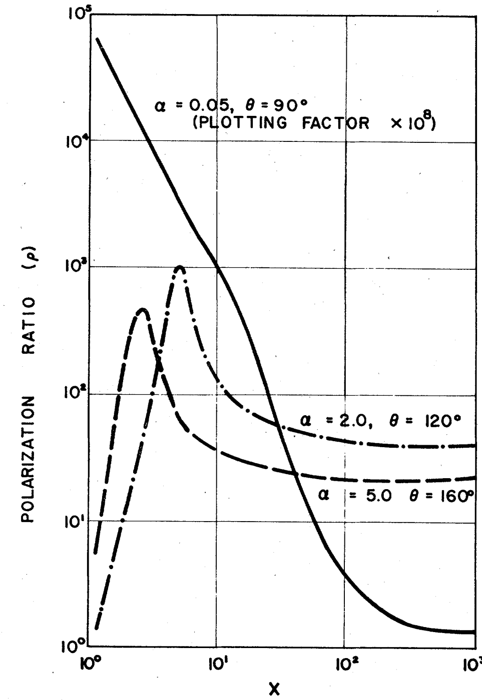


FIG. 4. Polarization ratio versus $x$ for various values of $\alpha$ and $\theta$.

with

$$I_2 = \left| \sum_{l=1}^{\infty} \frac{i^{l+1}}{l(l+1)} \{ b_l [l h_{l+1}^{(1)}(kd) - (l+1) h_{l-1}^{(1)}(kd)] \pi_l - (2l+1) i a_l h_l^{(1)}(kd) \tau_l \} \right|^2 .$$

We have computed the angular distributions given by (4) and (4′) for refractive index $m = 1.5$ and for values of $ka$ ranging from 0.05 to 5, and of $x = d/a$ from 1.1 to 1 000. In order to compare the results directly with the LM results with plane-wave incidence in a way independent of normalization, the polarization ratio

$$\rho = \frac{d\sigma/d\Omega_2}{d\sigma/d\Omega_1} = \frac{I_2}{I_1}$$

has also been calculated as a function of scattering angle. This was done both for the dipole source ($\rho_{\rm dip}$) and for plane-wave incidence ($\rho_{\rm LM}$).

Some sample results are plotted in Figs. 2 and 3 for $ka = \alpha = 2.0$ and 5.0, respectively, and for various dimensionless distances of the dipole, $x$. As expected, $\rho_{\rm dip}$ differs from $\rho_{\rm LM}$ for small $x$. The differences are greater in the backward directions than forward and are more pronounced for the larger particle size. More specifically, the Rayleigh limit ($ka \to 0$) is the same for all $x$, although even for $ka = 0.05$, the minimum in $\rho_{\rm dip}$ near 90° is much less sharp than the corresponding one in $\rho_{\rm LM}$. This is shown in Fig. 4 where $\rho_{\rm dip}$ at 90° for $\alpha = 0.05$ is plotted against $x$. Even at $x = 100$, the polarization ratio at 90° is 2.5 times greater than for parallel irradiance. This effect is due to the fact that for a dipole source at a finite distance from the scatterer, the electric vector of the radiation reaching the latter is not exactly along the $y$ axis for the polarization labeled 2 [Eq. (4′)], owing to the finite angle subtended by the sphere.

Also shown in Fig. 4 are plots of $\rho$ versus $x$ for $\alpha = 2$ at $\theta = 120°$ and $\alpha = 5$ at $\theta = 160°$ where peak values of $\rho_{\rm dip}$ occur. Significant deviations occur for values of $x$ as large as 50.

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[6]The $a_l$ and $b_l$ in Ref. 1 equal $-b_l^*$ and $-a_l^*$, respectively, in Ref. 2.