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Exact solution of the superradiance master equation. II. Arbitrary initial excitation*

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The cooperative spontaneous emission from a system of N identical two-level atoms, excited arbitrarily and interacting with a single mode of radiation, is studied by using the Bonifacio-Schwendimann-Haake approach. The evolution of the probability distribution of the atomic system is considered as a Markov process described by a time-evolution matrix (TEM). The exact analytic expressions for the elements of this TEM are obtained by solving the superradiance master equation. It is proved that the TEM satisfies the normalization condition and the Chapman-Kolmogorov equation as required. The exact general formulas for the expectation values of various physical observables, expressed in terms of an arbitrary initial distribution, are obtained. Finally, these results are applied to a specific example, the superradiant initial state.

I. INTRODUCTION

In a previous paper,[1] hereafter referred to as I, we presented the exact analytic solution of the superradiance master equation (SME) which describes the cooperative spontaneous emission from a completely excited system of N identical two-level atoms. We used the Dicke state representation and expressed the SME in the following form:

$$\dot{P}(r,m,t)=(r+m+1)(r-m)P(r,m+1,t)-(r+m)(r-m+1)P(r,m,t), \qquad (1.1)$$

where $P(r,m,t)$ is the probability that the atomic system is in the Dicke state $|r,m\rangle$ and the time t is rescaled to absorb the constant characterizing the emission rate of a single excited atom. The initial condition considered in I was

$$P(r,m,0)=\delta_{r,m}. \qquad (1.2)$$

Although this particular initial condition has drawn the most attention in the literature; it is too restrictive and too unrealistic. To use Eq. (1.2), we must assume much more knowledge about the system than we can possibly have. Even if we could prepare the atomic system in such a fully excited state, we still need to consider the following problem. Equation (1.1) is valid only when the spontaneous emission is confined to the end-fire mode and the cooperation number r is conserved. However, it is known[2] that the initial spontaneous emission is very much undirectional and that for extended systems r is not conserved. So, during the transient stage, Eq. (1.1) is not very reliable. However, once the cooperative effect is established, Eq. (1.1) will be valid.

In this paper, we will derive a $(2r+1)\times(2r+1)$ time-evolution matrix (TEM) by solving Eq. (1.1) with an arbitrary initial condition, as suggested by Bonifacio, Schwendimann, and Haake.[3] We can apply this TEM to any probability distribution at any time to yield the probability distribution at any later time. This will allow us very much flexibility. For example, to deal with the problem of initial irregularity mentioned previously, we may use an r-nonconserving equation such as proposed in Ref. 2 to describe the initial evolution to a certain point and then let the TEM derived here take over from that point on.

In Sec. II, we will use a Laplace transform to solve Eq. (1.1). As the result, we will obtain the exact analytic expressions for the elements of the TEM; there will be a series of exponential functions of time. The coefficients in these series must satisfy certain identities in order for the TEM to satisfy the normalization condition and the Chapman-Kolmogorov equation; the latter is required due to the fact that the SME describes a Markov process. These identities will be established independently in Sec. III. Then in Sec. IV we will derive exact general formulas for various expectation values of physical observables, expressed in terms of the arbitrary initial distribution so that these quantities can be calculated immediately when the initial state is given. Finally, in Sec. V, we will apply our results to the simplest possible specific case, namely, the Dicke superradiant state $|r,0\rangle$ as the initial state. In addition to exact expressions thus obtained, we will also introduce a systematic approximation to simplify the analytic expressions so that numerical evaluations can be carried out easily. We will compare our numerical result with that of Ref. 3.

II. TIME-EVOLUTION MATRIX

In conformity with I, we will again define a new variable n as

$$n\equiv r-m. \qquad (2.1)$$

Assuming N to be an even number, n can take the values $0,1,2,\ldots,2r$. Since r is conserved in our problem, we need only two variables, one discrete and one continuous, to describe the probability distribution. So we let

$$P_n(t) \equiv P(r, m, t). \tag{2.2}$$

We will further assume that $r = \frac{1}{2}N$ and will consider r and $\frac{1}{2}N$ as exchangeable from now on. Then Eq. (1.1) can be rewritten as

$$\frac{d}{dt}P_n(t) = (N-n+1)nP_{n-1}(t) - (N-n)(n+1)P_n(t). \tag{2.3}$$

To construct the analytic solution of Eq. (2.3), we first observe that it represents a set of $N+1$ simultaneous equations. The linearity of these equations allows the introduction of a time evolution matrix (TEM) $V_{n,l}(t)$ according to

$$P_n(t) = \sum_{l=0}^{N} V_{n,l}(t)P_l(0), \tag{2.4}$$

with

$$V_{n,l}(0) = \delta_{n,l}. \tag{2.5}$$

The Laplace transform of $V_{n,l}(t)$,

$$V_{n,l}(s) \equiv \int_0^\infty dt e^{-st} V_{n,l}(t), \tag{2.6}$$

may be found from the recurrence relation, obtained as the Laplace transform of Eq. (2.3), by iteration to read

$$V_{n,l}(s) = \frac{1}{s+(N-l)(l+1)} \prod_{i=l+1}^{n} \frac{(N-i+1)i}{s+(N-i)(i+1)}. \tag{2.7}$$

Now, in principle, we can easily obtain $V_{n,l}(t)$ by reducing the expression in Eq. (2.7) to partial fractions and then taking the inverse Laplace transformation. However, this procedure is complicated by the fact that we may have double poles as well as simple poles, since the root

$$s_i = -(N-i)(i+1) \tag{2.8}$$

remains unchanged when i is replaced by $N-i-1$. Hence, according to the way that the simple and double poles occur, the elements of the TEM are classified into regions as illustrated in Fig. 1. The final results are

$$V^{\mathrm{I}}_{n,l}(t) = \sum_{i=l}^{n} A^i_{n,l} e^{s_i t}, \tag{2.9}$$

$$V^{\mathrm{II}}_{n,l}(t) = \sum_{i=l}^{N-n-2} A^i_{n,l} e^{s_i t} + \sum_{i=N-n-1}^{r-1} [B^i_{n,l} + C^i_{n,l} t] e^{s_i t}, \tag{2.10}$$

$$V^{a}_{n,l}(t) = \sum_{i=l}^{r-1} [B^i_{N-l-1,l} + C^i_{N-l-1,l} t] e^{s_i t}, \tag{2.11}$$

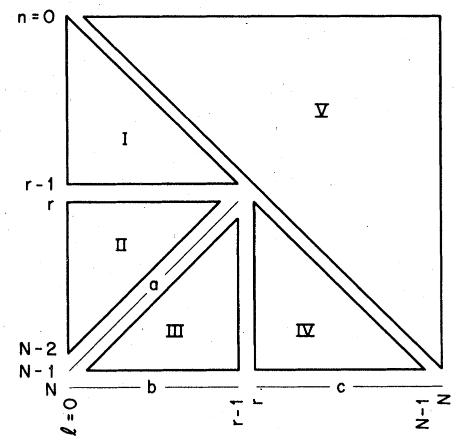

FIG. 1. Classification of the elements of the time evolution matrix according to the types of singularities of their Laplace transforms.

$$V^{\mathrm{III}}_{n,l}(t) = \sum_{i=N-n-1}^{l-1} D^i_{n,l} e^{s_i t} + \sum_{i=l}^{r-1} [B^i_{n,l} + C^i_{n,l} t] e^{s_i t}, \tag{2.12}$$

$$V^{b}_{N,l}(t) = 1 + \sum_{i=0}^{l-1} D^i_{N,l} e^{s_i t} + \sum_{i=l}^{r-1} [B^i_{N,l} + C^i_{N,l} t] e^{s_i t}, \tag{2.13}$$

$$V^{\mathrm{IV}}_{n,l}(t) = \sum_{i=N-n-1}^{N-l-1} D^i_{n,l} e^{s_i t}, \tag{2.14}$$

$$V^{c}_{N,l}(t) = 1 + \sum_{i=0}^{N-l-1} D^i_{N,l} e^{s_i t}, \tag{2.15}$$

$$V^{V}_{n,l}(t) = 0, \tag{2.16}$$

where the superscript on the matrix elements denotes the regions they belong to. The coefficients appearing in Eqs. (2.9)–(2.15) are defined as

$$A^i_{n,l} \equiv (-1)^{i+l}(N-2i-1)\binom{N-l}{i+1}\binom{i}{l}\binom{n}{i} \times B(N-n-i-1, i+2), \tag{2.17}$$

$$B^i_{n,l} \equiv F^i_{n,l}\left[2 - (N-2i-1)\sum_{k=i+1}^{N-i-1}\left(\frac{1}{k-l} + \frac{1}{n-k+1}\right)\right], \tag{2.18}$$

$$C^i_{n,l} \equiv F^i_{n,l}(N-2i-1)^2, \tag{2.19}$$

$$D^i_{n,l} \equiv (-1)^{N+n+i+1}(N-2i-1) \times \binom{N-l}{i+1}\binom{n}{i}\binom{i+1}{N-n} B(l-i, i+1), \tag{2.20}$$

where we have used the notations for binomial coefficient and beta function, and have defined the common factor of $B^i_{n,l}$ and $C^i_{n,l}$ as

$$F^i_{n,l} \equiv (-1)^{N+n+l+1}\binom{N-l}{i+1}\binom{i}{l}\binom{n}{i}\binom{i+1}{N-n} . \quad (2.21)$$

We note that the coefficients A^i_n, B^i_n, and C^i_n, which appeared in I, are particular cases of $A^i_{n,l}$, $B^i_{n,l}$, and $C^i_{n,l}$ defined here, corresponding to $l=0$. We also note that the first column of the TEM corresponding to $l=0$ is identical to the probability distribution obtained in I.

III. NORMALIZATION CONDITION AND CHAPMAN-KOLMOGOROV EQUATION

The TEM derived in the previous section is supposed to describe superradiance as a Markov process. Therefore, it must satisfy the normalization condition and the Chapman-Kolmogorov equation. We have used these conditions to check the correctness of our derivation. In this process, it reveals some very interesting identities involving the coefficients $A^i_{n,l}$, $B^i_{n,l}$, $C^i_{n,l}$, and $D^i_{n,l}$. The significance of these coefficients and their relations is enhanced when we realize that they also occur in the analytic solution[4] of the Bonifacio-Lugiato master equation for oscillatory superfluorescence[5] which is non-Markovian. Therefore, it is worthwhile to devote this section to showing how these identities occur and to giving independent proofs for some of them.

A. Normalization condition

The element of the TEM $V_{n,l}(t)$ represents the probability that the atomic system, initially excited to the Dicke state $|r,r-l\rangle$, has evolved to the state $|r,r-n\rangle$ at time t by emitting $n-l$ photons. Therefore, we must have

$$\sum_{n=l}^{N} V_{n,l}(t)=1 \quad (3.1)$$

as the normalization condition at any time.

For $0<l\le r-1$, Eq. (3.1) can be written as

$$\sum_{n=l}^{N} V_{n,l}(t)=\sum_{n=l}^{r-1} V^{\rm I}_{n,l}(t)+\sum_{n=r}^{N-l-2} V^{\rm II}_{n,l}(t)+V^{a}_{N-l-1,l}(t) + \sum_{n=N-l}^{N-1} V^{\rm III}_{n,l}(t)+V^{b}_{N,l}(t) \quad \text{for } 0<l\le r-1 . \quad (3.2)$$

Using Eqs. (2.9)–(2.13) and changing the order of summations, we obtain

$$\sum_{n=l}^{N} V_{n,l}(t)=1+\sum_{i=0}^{l-1} e^{s_i t}\sum_{n=N-i-1}^{N} D^i_{n,l} +\sum_{i=l}^{r-1} e^{s_i t}\left[t\sum_{n=N-i-1}^{N} C^i_{n,l} +\left(\sum_{n=i}^{N-i-2} A^i_{n,l}+\sum_{n=N-i-1}^{N} B^i_{n,l}\right)\right] . \quad (3.3)$$

For $r<l\le N$, we have

$$\sum_{n=l}^{N} V_{n,l}(t)=\sum_{n=l}^{N-1} V^{\rm IV}_{n,l}(t)+V^{c}_{N,l}(t) =1+\sum_{i=0}^{N-l-1} e^{s_i t}\sum_{n=N-i-1}^{N} D^i_{n,l} \quad \text{for } r\le l<N . \quad (3.4)$$

From Eqs. (3.3) and (3.4) we can see that the normalization condition will be satisfied if and only if the following identities are true:

$$\sum_{n=N-i-1}^{N} D^i_{n,l}=0 , \quad (3.5)$$

$$\sum_{n=N-i-1}^{N} C^i_{n,l}=0 , \quad (3.6)$$

$$\sum_{n=i}^{N-i-2} A^i_{n,l}+\sum_{n=N-i-1}^{N} B^i_{n,l}=0 . \quad (3.7)$$

1. Proof of Eqs. (3.5) and (3.6)

Using Eqs. (2.19)–(2.21) we can see immediately that both the summations in Eqs. (3.5) and (3.6) reduce to some factor multiplied by the following sum

$$\sum_{n=N-i-1}^{N}(-1)^{N+n}\binom{i+1}{N-n}\binom{n}{i}=\sum_{j=0}^{i+1}(-1)^j\binom{i+1}{j}\binom{N-j}{i} =0 , \quad (3.8)$$

where we have replaced n by $N-j$ and then the expression can be seen to be just the coefficient of x^{N-i} in the binomial expansion of $(1-x)^{i+1}/(1-x)^{i+1}$, hence, must vanish for $i<N$. Thus, the identities (3.5) and (3.6) have been easily established.

2. Proof of Eq. (3.7)

From Eqs. (2.18), (2.19), and (2.21) we have

$$\sum_{n=N-i-1}^{N} B^i_{n,l}=\left(\frac{2}{(N-2i-1)^2}-\frac{1}{(N-2i-1)}\sum_{k=i+l}^{N-i-1}\frac{1}{(k-l)}\right)\sum_{n=N-i-1}^{N} C^i_{n,l}+(-1)^l(N-2i-1)\binom{N-l}{i+1}\binom{i}{l}X_i , \quad (3.9)$$

with

$$X_i \equiv \sum_{n=N-i-1}^{N} (-1)^{N+n}\binom{i+1}{N-n}\binom{n}{i}\sum_{k=i+1}^{N-i-1}\frac{1}{(n-k+1)}$$

$$=\sum_{k=i+1}^{N-i-1}\sum_{\lambda=0}^{i+1}(-1)^{\lambda}\binom{i+1}{\lambda}\binom{N-\lambda}{i}\frac{1}{N-k-\lambda+1}$$

$$=(-1)^{i+1}\sum_{n=i}^{N-i-2}\binom{n}{i}B(N-n-i-1,i+2)\ , \qquad (3.10)$$

where, to obtain the second line, we have changed the order of summations and replaced the dummy index n by $N-\lambda$; to obtain the last line, we have used identity (A1) proven in the Appendix and replaced k by $n-1$.

Putting Eq. (3.10) into Eq. (3.9) and using Eqs. (3.6) and (2.17), we have

$$\sum_{n=N-i-1}^{N} B^i_{n,l} = -\sum_{n=i}^{N-i-2} A^i_{n,l}\ . \qquad (3.11)$$

Hence, Eq. (3.7) is proven.

B. Chapman-Kolmogorov equation

Since our SME is Markovian and homogeneous, we expect the TEM to satisfy the Chapman-Kolmogorov equation

$$\sum_{m=l}^{n} V_{n,m}(t_2)V_{m,l}(t_1)=V_{n,l}(t_1+t_2)\ , \qquad (3.12)$$

where the summation runs from l to n, instead of from 0 to N, because $V_{n,l}$ vanishes for $n<l$. The identities revealed by this condition are listed as follows:

$$\sum_{m=j}^{i} A^i_{n,m}A^j_{m,l}=\delta_{i,j}A^i_{n,l}\ , \qquad (3.13)$$

$$\sum_{m=j}^{i} C^i_{n,m}A^j_{m,l}=\delta_{i,j}C^i_{n,l}\ , \qquad (3.14)$$

$$\sum_{m=N-j-1}^{N-i-1} D^i_{n,m}C^j_{m,l}=\delta_{i,j}C^i_{n,l}\ , \qquad (3.15)$$

$$\sum_{m=N-j-1}^{N-i-1} D^i_{n,m}D^j_{m,l}=\delta_{i,j}D^i_{n,l}\ , \qquad (3.16)$$

$$\sum_{m=j}^{i} B^i_{n,m}A^j_{m,l}=-\sum_{m=i+1}^{N-i-1} D^i_{n,m}A^j_{m,l}\quad \text{for } i>j\ , \qquad (3.17)$$

$$\sum_{m=N-j-1}^{N-i-1} D^i_{n,m}B^j_{m,l}=-\sum_{m=j}^{N-j-2} D^i_{n,m}A_{m,l}\quad \text{for } i<j\ . \qquad (3.18)$$

1. Proof of identities (3.13) and (3.14)

Using Eqs. (2.17), (2.19), and (2.21) we have

$$\sum_{m=j}^{i} A^i_{n,m}A^j_{m,l}=(-1)^{i+l}(N-2i-1)(N-2j-1)\binom{n}{i}\binom{N-l}{j+1}\binom{j}{l}B(N-n-i-1,i+2)S_{i,j}, \qquad (3.19)$$

$$\sum_{m=j}^{i} C^i_{n,m}A^j_{m,l}=(-1)^{N+n+l+1}(N-2i-1)^2(N-2j-1)\binom{n}{i}\binom{i+1}{N-n}\binom{N-l}{j+1}\binom{j}{l}S_{i,j}, \qquad (3.20)$$

where

$$S_{i,j}\equiv\sum_{m=j}^{i}(-1)^{m+j}\binom{N-m}{i+1}\binom{i}{m}\binom{m}{j}B(N-m-j-1,j+2)$$

$$=\begin{cases}\dfrac{(j+1)}{(i+1)(i-j)}\displaystyle\sum_{k=0}^{i-j}(-1)^k\binom{N-k-2j-2}{i-j-1}\binom{i-j}{k} & \text{for } i>j\\ 1/(N-2i-1) & \text{for } i=j\end{cases}$$

$$=\delta_{i,j}/(N-2i-1). \qquad (3.21)$$

Substitution of Eq. (3.21) into Eqs. (3.19) and (3.20) gives identities (3.13) and (3.14), respectively. Identities (3.15) and (3.16) can be proven in similar way.

2. Proof of identity (3.17)

Using Eqs. (2.17)–(2.19) and (2.21) we have

$$\sum_{m=j}^{i} B^i_{n,m}A^j_{m,l}=\left[\frac{2}{(N-2i-1)^2}-\frac{1}{(N-2i-1)}\sum_{k=i+1}^{N-i-1}\frac{1}{(n-k+1)}\right]\sum_{m=j}^{i} C^i_{n,m}A^j_{m,l}$$

$$+(-1)^{N+n+l}(N-2i-1)(N-2j-1)\binom{n}{i}\binom{i+1}{N-n}\binom{N-l}{j+1}\binom{j}{l}\sum_{k=i+1}^{N-i-1} S^k_{i,j}, \qquad (3.22)$$

where

$$S^k_{i,j} \equiv \sum_{m=j}^{i} (-1)^{m+j} \binom{N-m}{i+1} \binom{i}{m} \binom{m}{j} B(N-m-j-1, j+2) \frac{1}{(k-m)}$$
$$= \frac{(j+1)}{(i+1)(i-j)} \sum_{\lambda=0}^{i-j} (-1)^{\lambda} \binom{i-j}{\lambda} \binom{N-2j-\lambda-2}{i-j-1} \frac{1}{k-j-\lambda}$$
$$= \frac{(j+1)}{(i+1)(i-j)} (-1)^{i+j} \binom{N-k-j-2}{i-j-1} B(k-i, i-j+1)$$
$$= (-1)^{i+j} \binom{N-k}{i+1} \binom{k}{j} B(k-i, i+1) B(N-k-j-1, j+2). \quad (3.23)$$

To obtain the third line in Eq. (3.23), we have used Eq. (A1).

Substituting Eqs. (3.14) and (3.23) into Eq. (3.22), replacing the dummy index k by m, and then comparing the result with Eqs. (2.17) and (2.20), we establish identity (3.17).

Identity (3.18) can be proven in similar way.

3. *Proof of Eq. (3.12)*

This equation is satisfied in different ways according to which region the matrix elements belong. To save space, only region II will be considered. The situation in region III is similar to region II, while it is much simpler in regions I and IV.

In region II, where $r \le n < N-1$ and $l < N-n-1$, we have

$$\sum_{m=l}^{n} V_{n,m}(t_2) V_{m,l}(t_1)$$
$$= \sum_{m=l}^{N-n-2} V^{\mathrm{II}}_{n,m}(t_2) V^{\mathrm{I}}_{m,l}(t_1) + V^{a}_{n,N-n-1}(t_2) V^{\mathrm{I}}_{N-n-1,l}(t_1)$$
$$+ \sum_{m=N-n}^{r-1} V^{\mathrm{III}}_{n,m}(t_2) V^{\mathrm{I}}_{m,l}(t_1) + \sum_{m=r}^{n} V^{\mathrm{IV}}_{n,m}(t_2) V^{\mathrm{II}}_{m,l}(t_1)$$
$$= \sum_{m=l}^{N-n-2} \sum_{i=m}^{N-n-2} \sum_{j=l}^{m} A^i_{n,m} A^j_{m,l} e^{s_i t_2 + s_j t_1}$$
$$+ \left\{ \sum_{m=l}^{N-n-1} \sum_{i=N-n-1}^{r-1} \sum_{j=l}^{m} + \sum_{m=N-n}^{r-1} \sum_{i=m}^{r-1} \sum_{j=l}^{m} \right\} [B^i_{n,m} A^j_{m,l} + t_2 C^i_{n,m} A^j_{m,l}] e^{s_i t_2 + s_j t_1}$$
$$+ \left\{ \sum_{m=N-n}^{r-1} \sum_{i=N-n-1}^{m-1} \sum_{j=l}^{m} + \sum_{m=r}^{n} \sum_{i=N-n-1}^{N-m-1} \sum_{j=l}^{N-m-2} \right\} D^i_{n,m} A^j_{m,l} e^{s_i t_2 + s_j t_1}$$
$$+ \sum_{m=r}^{n} \sum_{i=N-n-1}^{N-m-1} \sum_{j=N-m-1}^{r-1} \{ D^i_{n,m} B^j_{m,l} + t_1 D^i_{n,m} C^j_{m,l} \} e^{s_i t_2 + s_j t_1}, \quad (3.24)$$

where we have used Eqs. (2.9)–(2.12) and (2.14).

We should now change the order of summations in Eq. (3.24) so that we can carry out the summations over m first. Then it will be convenient to collect all the terms in the summands into three groups according to whether $i>j$, $i=j$, or $i<j$, namely,

$$\sum_{m=l}^{n} V_{n,m}(t_2) V_{m,l}(t_1) \equiv G_{i>j} + G_{i=j} + G_{i<j}. \quad (3.25)$$

We notice that the summations over m of the terms $A^i_{n,m} A^j_{m,l}$, $C^i_{n,m} A^j_{m,l}$, and $D^i_{n,m} C^j_{m,l}$ are of the forms of Eqs. (3.13)–(3.15); hence, they all disappear from $G_{i>j}$ and $G_{i<j}$. So we are left with

$$G_{i>j} = \sum_{i=N-n-1}^{r-1} \sum_{j=l}^{i-1} e^{s_i t_2 + s_j t_1} \left[\sum_{m=j}^{i} B^i_{n,m} A^j_{m,l} + \sum_{m=i+1}^{N-i-1} D^i_{n,m} A^j_{m,l} \right] = 0, \quad (3.26)$$

and

$$G_{i<j}=\sum_{i=N-n-1}^{r-2}\sum_{j=i+1}^{r-1}e^{s_i t_2+s_j t_1}\left[\sum_{m=N-j-1}^{N-i-1}D^i_{n,m}B^j_{m,l}+\sum_{m=j}^{N-j-2}D^i_{n,m}A^j_{m,l}\right]=0, \tag{3.27}$$

where we have used Eqs. (3.17) and (3.18).

Now we can concentrate our attention on

$$G_{i=j}=\sum_{i=l}^{N-n-2}A^i_{n,l}e^{s_i(t_1+t_2)}+(t_1+t_2)\sum_{i=N-n-1}^{r-1}C^i_{n,l}e^{s_i(t_1+t_2)}$$
$$+\sum_{i=N-n-1}^{r-1}e^{s_i(t_1+t_2)}[B^i_{n,i}A^i_{i,l}+D^i_{n,N-i-1}B^i_{N-i-1,l}]+\sum_{i=N-n-1}^{r-2}e^{s_i(t_1+t_2)}\sum_{m=i+1}^{N-i-2}D^i_{n,m}A^i_{m,l}. \tag{3.28}$$

Using Eqs. (2.17), (2.20), and (2.21) we have

$$\sum_{m=i+1}^{N-i-2}D^i_{n,m}A^i_{m,l}=F^i_{n,l}(N-2i-1)^2\sum_{m=i+1}^{N-i-2}\frac{1}{(m-i)(N-m-i-1)}$$
$$=F^i_{n,l}(N-2i-1)\left\{\sum_{k=i+1}^{N-i-2}\frac{1}{(k-i)}+\sum_{k=i+2}^{N-i-1}\frac{1}{(N-k-i)}\right\}; \tag{3.29}$$

using Eqs. (2.17), (2.18), and (2.21) we have

$$B^i_{n,i}A^i_{i,l}=\begin{cases}F^i_{n,l}\left[1-(N-2i-1)\left(\sum_{k=i+1}^{N-i-2}\frac{1}{(k-i)}+\sum_{k=i+1}^{N-i-1}\frac{1}{(n-k+1)}\right)\right] & \text{for } i<r-1\\ F^i_{n,l}\left(1-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{(n-k+1)}\right) & \text{for } i=r-1;\end{cases} \tag{3.30}$$

using Eqs. (2.18), (2.20), and (2.21) we have

$$D^i_{n,N-i-1}B^i_{N-i-1,l}=\begin{cases}F^i_{n,l}\left[1-(N-2i-1)\left(\sum_{k=i+1}^{N-i-1}\frac{1}{(k-l)}+\sum_{k=i+2}^{N-i-1}\frac{1}{(N-k-i)}\right)\right] & \text{for } i<r-1\\ F^i_{n,l}\left(1-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{(k-l)}\right) & \text{for } i=r-1.\end{cases} \tag{3.31}$$

Putting Eqs. (3.29)–(3.31) into Eq. (3.28) we obtain

$$G_{i=j}=\sum_{i=l}^{N-n-2}A^i_{n,l}e^{s_i(t_1+t_2)}+\sum_{i=N-n-1}^{r-1}[(t_1+t_2)C^i_{n,l}+B^i_{n,l}]e^{s_i(t_1+t_2)}=V^{\rm II}_{n,l}(t_1+t_2). \tag{3.32}$$

Now, substitution of Eqs. (3.26), (3.27), and (3.32) into Eq. (3.25) gives

$$\sum_{m=l}^{n}V_{n,m}(t_2)V_{m,l}(t_1)=V^{\rm II}_{n,l}(t_1+t_2)\ . \tag{3.12}$$

IV. CALCULATIONS OF EXPECTATION VALUES

The description of superradiance by the SME allows us to consider it as a Markov process. The final purpose of studying a stochastic process is to be able to calculate expectation values of relevant random variables. In this section, we will describe the procedure and derive general formulas for doing such calculations.

To begin with, let us recall the meanings of various key variables we used. The atomic system is initially excited to a Dicke state $|r,r-l\rangle$ with a probability distribution $P_l(0)$; at time t the atomic system will be in a state $|r,r-n\rangle$ with a probability distribution $P_n(t)$. The connection between $P_l(0)$ and $P_n(t)$ is through the TEM $V_{n,l}(t)$ according to Eq. (2.4). Both l and n can take integer values $0, 1, 2, \ldots, 2r=N$. The difference $n-l\equiv N_p$ is the number of photons emitted; $N-n$ measures the energy still stored in the atomic system; and

$$I\equiv\frac{d}{dt}\langle N_p\rangle\equiv-\frac{d}{dt}\langle N-n\rangle=\frac{d}{dt}\langle n\rangle$$

gives the emission rate or radiation intensity, in terms of photon number.

A physical quantity Q, as a random variable, can be considered as a function of the random variable n and, hence, denoted by $Q(n)$. The expectation value of Q can be calculated as follows:

$$\langle Q\rangle=\sum_{n=0}^{N} Q(n)P_n(t)=\sum_{n=0}^{N} Q(n)\sum_{l=0}^{n} V_{n,l}(t)P_l(0) . \quad (4.1)$$

Thus, the expectation value of a physical quantity depends on the initial distribution. Since we would like to leave the initial condition open, we should change the order of summations in Eq. (4.1) to read

$$\langle Q\rangle=\sum_{l=0}^{N}\sum_{n=l}^{N} Q(n)V_{n,l}(t)P_l(0)=\sum_{l=0}^{N}\langle Q\rangle_l P_l(0) , \quad (4.2)$$

where we have defined

$$\langle Q\rangle_l\equiv\sum_{n=l}^{N} Q(n)V_{n,l}(t) . \quad (4.3)$$

We can usually carry out the calculation of $\langle Q\rangle_l$ explicitly before $P_l(0)$ is specified.

A. Calculation of $\langle n\rangle_l$

Since n is the basic random variable of our problem, the calculation of

$$\langle n\rangle_l\equiv\sum_{n=l}^{N} nV_{n,l}(t)$$

can be a very convenient first step in the calculations of the expectation values of most of the physical quantities that we are interested in. This is what is done below.

1. $0<l<r$

For this range of l, we can write

$$\langle n\rangle_l=\sum_{n=l}^{r-1} nV^{\mathrm{I}}_{n,l}(t)+\sum_{n=r}^{N-l-2} nV^{\mathrm{II}}_{n,l}(t)+N-l-1)V^{a}_{N-l-1,l}(t)$$
$$+\sum_{n=N-l}^{N-1} nV^{\mathrm{III}}_{n,l}(t)+NV^{b}_{N,l}(t) . \quad (4.4)$$

Substituting Eqs. (2.9)–(2.13) into Eq. (4.4) and changing the order of summations, we obtain

$$\langle n\rangle_l=N+\sum_{i=0}^{l-1} e^{s_i t}\sum_{n=N-i-1}^{N} nD^{i}_{n,l}$$
$$+\sum_{i=l}^{r-1} e^{s_i t}\left\{\sum_{n=i}^{N-i-2} nA^{i}_{n,l}+\sum_{n=N-i-1}^{N} nB^{i}_{n,l}\right.$$
$$\left.+t\sum_{n=N-i-1}^{N} nC^{i}_{n,l}\right\}. \quad (4.5)$$

Using Eqs. (2.19)–(2.21) we have

$$\sum_{n=N-i-1}^{N} nD^{i}_{n,l}=(-1)^{i}(N-2i-1)\binom{N-l}{i+1}B(l-i,i+1)Y_i , \quad (4.6)$$

and

$$\sum_{n=N-i-1}^{N} nC^{i}_{n,l}=(-1)^{l}(N-2i-1)^2\binom{N-l}{i+1}\binom{i}{l}Y_i , \quad (4.7)$$

where

$$Y_i\equiv\sum_{n=N-i-1}^{N}(-1)^{N+n+1}n\binom{i+1}{N-n}\binom{n}{i}$$
$$=\sum_{\lambda=0}^{i+1}(-1)^{\lambda}\binom{i+1}{\lambda}\binom{N-\lambda}{i}$$
$$-(i+1)\sum_{\lambda=0}^{i+1}(-1)^{\lambda}\binom{i+1}{\lambda}\binom{N-\lambda+1}{i+1}$$
$$=-(i+1) . \quad (4.8)$$

Substitution of Eq. (4.8) into Eqs. (4.6) and (4.7), respectively, gives

$$\sum_{n=N-i-1}^{N} nD^{i}_{n,l}=(-1)^{i+1}(N-2i-1)\frac{(N-l)!\,(l-i-1)!}{l!\,(N-l-i-1)!} \quad (4.9)$$

and

$$\sum_{n=N-i-1}^{N} nC^{i}_{n,l}=(-1)^{l+1}\frac{(N-2i-1)^2(N-l)!}{l!\,(N-l-i-1)!\,(i-l)!} . \quad (4.10)$$

On the other hand, using Eqs. (2.18), (2.19), and (2.21), we have

$$\sum_{n=N-i-1}^{N} nB^{i}_{n,l}=\left\{\frac{2}{(N-2i-1)^2}-\frac{1}{N-2i-1}\sum_{k=i+1}^{N-i-1}\frac{1}{k-l}\right\}\sum_{n=N-i-1}^{N} nC^{i}_{n,l}+(-1)^{l}(N-2i-1)\binom{N-l}{i+1}\binom{i}{l}Z_i . \quad (4.11)$$

where

$$Z_i\equiv\sum_{n=N-i-1}^{N}(-1)^{N+n}\binom{i+1}{N-n}\binom{n}{i}\sum_{k=i+1}^{N-i-1}\frac{n}{n-k+1}$$
$$=(N-2i-1)\sum_{\lambda=0}^{i+1}(-1)^{\lambda}\binom{i+1}{\lambda}\binom{N-\lambda}{i}+\sum_{k=i+1}^{N-i-1}(k-1)\sum_{\lambda=0}^{i+1}(-1)^{\lambda}\binom{i+1}{\lambda}\binom{N-\lambda}{i}\frac{1}{N-k-\lambda+1}$$
$$=(-1)^{i+1}\sum_{n=i}^{N-i-2} n\binom{n}{i}B(N-n-i+1,i+2) . \quad (4.12)$$

To obtain the last line of Eq. (4.12), we have used Eq. (A1) and then replaced the dummy index k by $n-1$.

Putting Eq. (4.12) into Eq. (4.11) and using Eqs. (4.10) and (2.17), we have

$$\sum_{n=N-i-1}^{N} nB^i_{n,l} = \frac{(-1)^{l+1}(N-l)!}{l!(N-l-i-1)!(i-l)!}\left\{2-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{k-l}\right\} - \sum_{n=i}^{N-i-2} nA^i_{n,l} . \tag{4.13}$$

Now, substitution of Eqs. (4.9), (4.10), and (4.13) into Eq. (4.5) gives

$$\langle n\rangle_l = N - \frac{(N-l)!}{l!}\left\{\sum_{i=0}^{l-1}(-1)^i(N-2i-1)\frac{(l-i-1)!}{(N-l-i-1)!}e^{s_i t} + (-1)^l\sum_{i=l}^{r-1}\frac{e^{s_i t}}{(N-l-i-1)!(i-l)!}\left[2-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{k-l}+(N-2i-1)^2 t\right]\right\}$$

$$\text{for } 0<l<r . \tag{4.14a}$$

2. $r\leq l<N$

For values of l in this range, we have

$$\begin{aligned}\langle n\rangle_l &= \sum_{n=l}^{N-1} nV^{\mathrm{IV}}_{n,l}(t) + NV^c_{N,l}(t)\\ &= N + \sum_{i=0}^{N-l-1} e^{s_i t}\sum_{n=N-i-1}^{N} nD^i_{n,l}\\ &= N - \frac{(N-l)!}{l!}\sum_{i=0}^{N-l-1}(-1)^i(N-2i-1)\frac{(l-i-1)!}{(N-l-i-1)!}e^{s_i t} \quad \text{for } r\leq l<N ,\end{aligned} \tag{4.14b}$$

where we have used Eqs. (2.14), (2.15), and then (4.9).

3. $l=0$ or $l=N$

In the special case of $l=0$, $\langle n\rangle_0$ is exactly the same as $\langle n\rangle$ given in Eq. (3.8) of I, namely,

$$\langle n\rangle_l = N - \sum_{i=0}^{r-1}\frac{N!\,e^{s_i t}}{(N-i-1)!\,i!}\times\left(2-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{k}+(N-2i-1)^2 t\right) \quad \text{for } l=0 , \tag{4.14c}$$

which can be considered as the limiting case of Eq. (4.14a). In the special case of $l=N$, we simply have

$$\langle n\rangle_l = N \quad \text{for } l=N . \tag{4.14d}$$

B. Some expectation values

We are now in a position to calculate with ease the expectation values of some of the important physical quantities in the problem of superradiance. They will serve as examples for this kind of calculations.

First of all, using Eqs. (4.14a)–(4.14d) and (4.2), we can obtain

$$\begin{aligned}\langle n\rangle &= \sum_{l=0}^{N}\langle n\rangle_l P_l(0)\\ &= N - \sum_{i=0}^{r-1} e^{-(N-i)(i+1)t}\left\{(-1)^i(N-2i-1)\sum_{l=i+1}^{N-i-1}P_l(0)\frac{(N-l)!(l-i-1)!}{l!(N-l-i-1)!} + \sum_{l=0}^{i}\frac{P_l(0)(-1)^l(N-l)!}{l!(N-l-i-1)!(i-l)!}\left(2-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{k-l}+(N-2i-1)^2 t\right)\right\} .\end{aligned} \tag{4.15}$$

Then the expectation value of the number of photon emitted is simply

$$\langle N_p\rangle = \langle n\rangle - \langle l\rangle = \langle n\rangle - \sum_{l=1}^{N} lP_l(0) . \tag{4.16}$$

And the derivative of Eq. (4.15) with respect to time gives the emission rate as

$$I=\sum_{i=0}^{r-1} e^{-(N-i)(i+1)t}\Bigg\{(-1)^i(N-i)(i+1)(N-2i-1)\sum_{l=i+1}^{N-i-1} P_l(0)\frac{(N-l)!\,(l-i-1)!}{l!\,(N-l-i-1)!} + \sum_{l=0}^{i}\frac{P_l(0)(-1)^l(N-l)!}{l!\,(N-l-i-1)!\,(i-l)!}\left[(N-i)(i+1)\left(2-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{k-l}\right) + (N-2i-1)^2[(N-i)(i+1)t-1]\right]\Bigg\}. \tag{4.17}$$

On the other hand, we also have

$$I\equiv\langle (N-n)(n+1)\rangle=(N-1)\langle n\rangle-\langle n^2\rangle+N, \tag{4.18}$$

which implies

$$\langle n^2\rangle=(N-1)\langle n\rangle-I+N. \tag{4.19}$$

This relation can be used to calculate the fluctuation in photon number as follows:

$$\sigma^2\equiv\langle N_p^2\rangle-\langle N_p\rangle^2=\langle (n-l)^2\rangle-\langle (n-l)\rangle^2=\langle n^2\rangle-2\langle nl\rangle+\langle l^2\rangle-\langle n\rangle^2+2\langle n\rangle\langle l\rangle-\langle l^2\rangle, \tag{4.20}$$

where

$$\langle nl\rangle=\sum_{l=1}^{N}\langle n\rangle_l\, l P_l(0) =N\langle l\rangle-\sum_{i=0}^{r-1} e^{-(N-i)(i+1)t}\Bigg\{(-1)^i(N-2i-1)\sum_{l=i+1}^{N-i-1} P_l(0)\, l\,\frac{(N-l)!\,(l-i-1)!}{l!\,(N-l-i-1)!} + \sum_{l=1}^{i}\frac{P_l(0)(-1)^l l(N-l)!}{l!\,(N-l-i-1)!\,(i-l)!}\left(2-(N-2i-1)\sum_{k=i+1}^{N-i-1}\frac{1}{k-l}+(N-2i-1)^2t\right)\Bigg\} \tag{4.21}$$

and

$$\langle l^2\rangle=\sum_{l=1}^{N} l^2P_l(0). \tag{4.22}$$

Now, substituting Eqs. (4.15), (4.17), (4.19), (4.21), and (4.22) into Eq. (4.20), we can easily obtain the expression for σ^2.

V. SUPERRADIANT INITIAL STATE

When the initial condition $P_l(0)=\delta_{l,0}$ is used in the general formulas obtained in the previous section, all the results of I will be recovered. In this section, the general results will be applied to another specific initial distribution, namely,

$$P_l(0)=\delta_{l,r}. \tag{5.1}$$

This initial state is called superradiant state by Dicke[6] since it has the maximum initial emission rate. As will be seen, this is the simplest possible case. It has been specifically discussed by Bialynicka-Birula[7] and by Bonifacio and Preparata.[8] A broadened form (binomial distribution centered around the superradiant state) of this initial distribution has also been discussed in Ref. 3.

Using Eq. (5.1) in Eq. (2.4), we obtain the probability distribution

$$P_n(t)=V_{n,r}(t)=\begin{cases}0, & \text{for } n<r\\ \displaystyle\sum_{i=N-n-1}^{r-1}\frac{(-1)^{N+n+i+1}(N-2i-1)n!}{(N-n)!\,(n-i)!\,(i-N+n+1)!}e^{-(N-i)(i+1)t}, & \text{for } r\le n<N\\ \displaystyle 1-\sum_{i=0}^{r-1}(-1)^i\frac{(N-2i-1)N!}{(N-i)!\,(i+1)!}e^{-(N-i)(i+1)t}, & \text{for } n=N,\end{cases} \tag{5.2}$$

where we have used Eqs. (2.14)–(2.16) and (2.20).

Substitution of Eq. (5.1) into Eqs. (4.15) and (4.16) gives the expectation value of the number of photons emitted as

$$\langle N_p\rangle = r - \sum_{i=0}^{r-1} (-1)^i (N-2i-1) e^{-(N-i)(i+1)t}. \tag{5.3}$$

This is an exact expression which can be used directly to evaluate $\langle N_p\rangle$ for small values of r, say, $\lesssim 10^4$. For large r, it is more convenient to introduce the following approximation.

Let $j=i+1$, then Eq. (5.3) can be rewritten as

$$\begin{aligned}
\langle N_p\rangle &= r + \sum_{j=1}^{r} (-1)^j [(N+1)-2j] e^{-(N+1)jt} e^{j^2 t} \\
&= r + \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=1}^{r} (-1)^j [(N+1)-2j] j^{2k} e^{-j(N+1)t} \\
&= r + \sum_{k=0}^{\infty} \frac{\tau^k}{k!} (2r+1)^{-k} \left\{ (N+1)\frac{d^{2k}}{d\tau^{2k}} + 2\frac{d^{2k+1}}{d\tau^{2k+1}} \right\} \sum_{j=1}^{r} (-1)^j e^{-j\tau} \\
&= r\tanh(\tfrac{1}{2}\tau) + \left\{ \frac{e^\tau - 1}{e^\tau + 1} - \sum_{k=1}^{\infty} \frac{\tau^k}{k!} (N+1)^{-k} \frac{d^{2k}}{d\tau^{2k}} \left((N+1) + 2\frac{d}{d\tau} \right) \right\} \frac{1-(-1)^r e^{-r\tau}}{e^\tau + 1} \\
&\simeq r\tanh(\tfrac{1}{2}\tau) + \frac{e^\tau - 1}{(e^\tau+1)^2} - \frac{e^{2\tau} - e^\tau}{(e^\tau+1)^3}\tau - \sum_{k=1}^{\infty} \frac{\tau^k}{k!} (N+1)^{-k} \frac{d^{2k}}{d\tau^{2k}} \left\{ 2\frac{d}{d\tau} + \frac{\tau}{k+1}\frac{d^2}{d\tau^2} \right\} \frac{1}{e^\tau + 1},
\end{aligned} \tag{5.4}$$

we have defined $\tau \equiv (N+1)t$ and where the fourth line of Eq. (5.4) has been written in such a way so we can see clearly that the term $(-1)^r e^{-r\tau}$ can be dropped without getting into trouble even when τ approaches 0. Otherwise, our problem would be sensitive to whether r is even or odd, which would be quite unreasonable.

For all practical purpose, it is sufficient to write

$$\langle N_p \rangle \simeq r\tanh(\tfrac{1}{2}\tau) + \frac{e^\tau - 1}{(e^\tau+1)^2} - \tau\frac{e^{2\tau} - e^\tau}{(e^\tau+1)^3}. \tag{5.5}$$

However, we can always improve the accuracy by taking more terms of the series in Eq. (5.4) without much difficulty.

The exact expression and approximation for the emission rate, in terms of photon number, can be easily obtained by taking the derivatives of Eqs. (5.3) and (5.5), respectively, as follows:

$$\begin{aligned}
I &\equiv \frac{d}{dt}\langle N_p\rangle \\
&= \sum_{i=0}^{r-1} (-1)^i (N-2i-1)(N-i)(i+1)\, e^{-(N-i)(i+1)t} \\
&\simeq (2r+1)\left\{ \frac{r}{2}\operatorname{sech}^2(\tfrac{1}{2}\tau) - \frac{2e^{2\tau} - 4e^\tau}{(e^\tau+1)^3} + \tau\frac{e^{3\tau} - 4e^{2\tau} + e^\tau}{(e^\tau+1)^4} \right\}.
\end{aligned} \tag{5.6}$$

If we just want to calculate the expectation values of the photon number and the intensity themselves, then the first terms in both Eqs. (5.5) and (5.6), which are well-known results of the very first semiclassical calculation by Dicke,[6] are accurate enough. However, as will be seen soon, only the second terms contribute to the fluctuations in both photon number and intensity which are completely quantum mechanical in origin.

An alternative expression for the intensity is $I \equiv \langle (r-N_p)(r+N_p+1)\rangle = r^2 + r - \langle N_p^2\rangle - \langle N_p\rangle$ which implies

$$\begin{aligned}
\langle N_p^2 \rangle &\equiv r^2 + r - I - \langle N_p\rangle \\
&= r^2 - \sum_{i=0}^{r-1} (-1)^i (N-2i-1)\, [(N-i)(i+1)-1] e^{-(N-i)(i+1)t} \\
&\simeq r^2\tanh^2(\tfrac{1}{2}\tau) + r\left\{ \frac{4e^{2\tau} - 6e^\tau + 2}{(e^\tau+1)^3} - 2\tau\frac{e^{3\tau} - 4e^{2\tau} + e^\tau}{(e^\tau+1)^4} \right\}.
\end{aligned} \tag{5.7}$$

Then, using Eqs. (5.5) and (5.7), we can easily calculate the photon number dispersion as

$$\langle N_p^2\rangle - \langle N_p\rangle^2 \simeq \tfrac{1}{2}r\,[\tanh(\tfrac{1}{2}\tau) + \tfrac{1}{2}\tau\operatorname{sech}^2(\tfrac{1}{2}\tau)]\operatorname{sech}^2(\tfrac{1}{2}\tau), \tag{5.8}$$

which equals $\langle N_p\rangle$ for short times, as is well known.

To calculate the normally ordered intensity fluctuation, which can be measured in a photon-count experiment, we first find the expectation value

$$\langle R^+R^+R^-R^-\rangle \equiv \langle (N-n)(n+1)(N-n-1)(n+2)\rangle$$

$$=\sum_{i=1}^{r-1}(-1)^{i+1}(N-2i-1)\frac{(i+2)!}{(i-1)!}\binom{N-i+1}{3}e^{-(N-i)(i+1)t}$$

$$\simeq (2r+1)^2\left\{\frac{r^2}{4}\operatorname{sech}^4(\tfrac{1}{2}\tau)-8r\frac{2e^{3\tau}-3e^{2\tau}}{(e^\tau+1)^5}+8r\tau\frac{e^{4\tau}-3e^{3\tau}+e^{2\tau}}{(e^\tau+1)^6}\right\}. \quad (5.9)$$

Using Eqs. (5.6) and (5.9) we can now calculate the intensity fluctuation as

$$\sigma_n^2(I)\equiv\frac{\langle a^\dagger a^\dagger aa\rangle-\langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle^2}$$

$$\equiv\frac{\langle R^+R^+R^-R^-\rangle-I^2}{I^2}$$

$$\simeq(1/r)[\tau\tanh(\tfrac{1}{2}\tau)-2]\tanh(\tfrac{1}{2}\tau)\,. \quad (5.10)$$

The analytic expression in Eq. (5.10) is, to our knowledge, derived here for the first time. It is important in determining the coherent properties of the superradiant emission since we know that $\sigma_n^2(I)$ equals 0 for Poisson statistics (coherent) and 1 for Bose statistics (chaotic). From this expression we can see that, as long as $r\gg1$ and $\tau\ll r$, $\sigma_n^2(I)$ is very close to 0; [it is exactly 0 at $\tau=0$ and at $\tau\tanh(\frac{1}{2}\tau)=2$]; hence, we conclude that the emitted field behaves essentially classically. We have plotted Eq. (5.10) in Fig. 2 with $r=100$ and have reproduced in the same figure the plot of Fig. 6 of Ref. 3 for comparison. We can see that the agreement is qualitative but not quantitative. The discrepancy is, we believe, due to the fact that the initial state used in Ref. 3 is a binomial distribution sharply peaked around $|r,0\rangle$ while we have a δ distribution. The appearance of negative values for short times might look strange but is easy to understand since $\langle (N-n)(n+1)(N-n-1)\times(n+2)\rangle=\langle (N-n)^2(n+1)^2\rangle-\langle (N-n)(n+1)(2n-N+2)\rangle$ which is approximately $I^2-(2n-N+2)I$ for extremely narrow distributions.

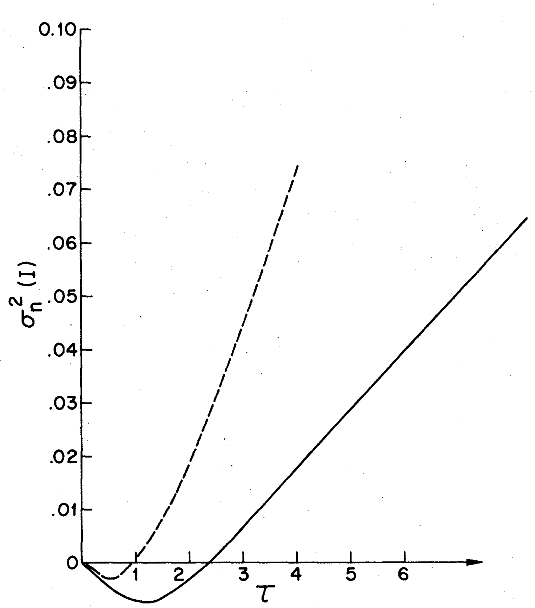

FIG. 2. Plot of Eq. (5.10) for the normally ordered intensity fluctuation $\sigma_n^2(I)$ with $r=100$ (solid line) in comparison with the result of direct numerical solution by Bonifacio, Schwendimann, and Haake (dotted line) reproduced here with proper change of scale.

VI. SUMMARY

In this article, we have presented an exact analytic solution of the superradiance master equation (SME) which describes the cooperative spontaneous emission from a system of N arbitrarily excited two-level atoms. The solution is expressed in the form of a time evolution matrix (TEM); when applied to any given initial probability distribution arranged as a column matrix, it will yield the probability distribution as a function of time. Since the SME is Markovian, the TEM is expected to satisfy the normalization condition and the Chapman-Kolmogorov equation which have been proven independently. Some exact general formulas for the expectation values of various physical observables have been derived in terms of an arbitrary initial distribution. These results have been applied to a specific example, the superradiant initial state. In particular, an analytic expression for the normally ordered intensity fluctuation have been obtained and compared with the direct numerical calculation by Bonifacio, Schwendimann, and Haake.

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APPENDIX

In this Appendix, we will prove the identity

$$\sum_{\lambda=0}^{i}(-1)^{\lambda}\binom{i}{\lambda}\binom{N-\lambda}{i-1}\frac{1}{j-\lambda}=(-1)^{i}\binom{N-j}{i-1}B(j-i,i+1)\quad\text{for } N>i+j-2 \text{ and } j>i\ , \tag{A1}$$

where we have used the notations for binomial co-efficient and beta function.

By observation we can see that the left-hand side (LHS) of Eq. (A1) is just the coefficient of x^{N-i+1} in the expansion of the integral

$$\begin{aligned}\int_0^1 t^{j-i-1}\frac{(t-x)^i}{(1-x)^i}dt&=\int_0^1 t^{j-i-1}\left(1-\frac{1-t}{1-x}\right)^i dt\\&=\sum_{k=0}^{i}(-1)^k\binom{i}{k}(1-x)^{-k}\int_0^1 t^{j-i-1}(1-t)^k dt\\&=\sum_{k=0}^{i}(-1)^k\binom{i}{k}(1-x)^{-k}B(j-i,k+1)\ .\end{aligned} \tag{A2}$$

So we have

$$\begin{aligned}\text{LHS}&=\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{N-i+k}{k-1}B(j-i,k+1)\\&=(-1)^i i\,\frac{(N-j)!\,(j-i-1)!}{(N-i+1)!}\sum_{\mu=0}^{i-1}(-1)^{\mu}\binom{i-1}{\mu}\binom{N-\mu}{N-j}\\&=(-1)^i i\,\frac{(N-j)!\,(j-i-1)!}{(N-i+1)!}\binom{N-i+1}{j}\\&=\text{RHS}\ .\end{aligned} \tag{A3}$$

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